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DEVELOPMENTS IN WATER SCIENCE

7

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V. HÁLEK AND J. ŠVEC

GROUNDWATER HYDRAULICS

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GROUNDWATER HYDRAULICS

Czechoslovak Academy of Sciences

DEVELOPMENTS IN WATER SCIENCE, 7

advisory editor

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GROUNDWATER HYDRAULICS

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PREFACE

Groundwater hydraulics is a branch of general hydraulics which by gradual development has become an independent scientific discipline with its own methodology and range of application. Its significance stems from the fact that it forms a foundation for the framework of other scientific fields and nearly all of the specialized areas of water conservation.

In 1973, we addressed the book "Groundwater Hydraulics" published in Czech by Academia, Publishing House of the Czechoslovak Academy of Sciences, to engineers engaged in studies of water conservation problems. The objective of the book was to present a comprehensive review of the fundamentals of the theory of groundwater flow, and to outline some methods for the analytical solution of practical problems. The book was intended neither as a collection of ready-made formulae, nor as a purely theoretical treatise. Its mathematical apparatus corresponded to the level of knowledge of university graduates.

The English version of the book has similar aims and proceeds from similar principles. The difference lies in that it includes recent discoveries, and that it acknowledges advances in computer techniques which have considerably enhanced the applicability of the theory. This necessitated an extension of some of the sections devoted to the fundamental theory, the basis for numerical solutions.

The book is divided into seven chapters. The first chapter deals with the laws governing flow and their formulations. The second chapter is devoted to one-dimensional motion. The third chapter, written by J. Švec, explains the theory of two-dimensional flow of groundwater. The fourth chapter presents some approximate methods applicable to two-dimensional flow problems. The fifth chapter discusses some practical problems of plane flow. The sixth chapter is concerned with groundwater motion in space, and the seventh, with unsteady phenomena and processes.

The limited scope of the book makes it necessary to curtail the discussion of many of the results so far achieved in groundwater hydraulics. It is hoped that the list of references will go some way to alleviating the consequences of the unavoidable selection without which the book could hardly have been written within the prescribed limits and in a finite amount of time.

V. Hálek, J. Švec

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List of Symbols

This is a list of basic symbols denoting the physical and geometrical quantities used throughout the book. The symbols will be explained in detail and furnished with appropriate subscripts as they occur in the text.

<i>a</i>	a constant
<i>b</i>	a constant; width of fissure
<i>c</i>	physical permeability; a constant
<i>d</i>	diameter of grain; length
<i>e</i>	parameter
<i>f</i>	function
<i>g</i>	acceleration of gravity
<i>h</i>	piezometric head; thickness of saturated layer
<i>i</i>	unit vector
<i>j</i>	unit vector
<i>k</i>	coefficient of permeability
<i>l</i>	length
<i>m</i>	porosity; number
<i>n</i>	number
<i>o</i>	volume
<i>p</i>	pressure
<i>q</i>	quantity of seepage
<i>r</i>	radius vector
<i>s</i>	length
<i>t</i>	time
<i>u</i>	velocity; function
<i>v</i>	velocity
<i>w</i>	moisture defined as part of soil volume; complex velocity
<i>x</i>	coordinate
<i>y</i>	coordinate
<i>z</i>	coordinate; geodetic head
<i>A</i>	a constant; length; designation of point
<i>B</i>	a constant; parameter; length; designation of point

C	a constant; parameter; coefficient of saturated conductivity
D	diameter of pipe; a constant
E	modulus of elasticity

$$\text{Erf } x = \frac{2}{\sqrt{\pi}} \int_0^x e^{-\zeta^2} d\zeta$$

$$\text{Exp } x = e^x$$

F	force; function
Fr	Froude number
G	Girinsky potential
H	thickness of saturated layer with free surface
I	integral; Bessel function; volume flow
J	gradient of piezometric head (inclination, slope)
K	elliptic integral; Bessel function
L	length
M	mass, designation of point
N	number; designation of point
P	specific surface
Q	volume of water flowing through area per unit time (discharge)
R	hydraulic radius; force
Re	Reynolds number
S	area; degree of saturation (ratio of volume of water to volume of pores)
T	temperature; thickness of saturated layer
U	function; complex quantity of flow
V	volume
W	saturated velocity potential
X	coordinate
Y	coordinate; Bessel function
Z	coordinate
α	angle
β	a constant; angle
γ	specific weight; parameter
δ	a constant; length
ε	coefficient of turbulent tortuosity; parameter
ζ	coordinate
η	coordinate
ϑ	angle
κ	porosity number; parameter
λ	coefficient of friction; scale; length; parameter
μ	active porosity
ν	coefficient of kinematic viscosity

ξ	coordinate
ϱ	specific mass (density); length; probability
σ	modulus
τ	angle; shear stress
φ	potential
χ	a constant
ψ	stream function
ω	angle; complex potential
Γ	coefficient of uniformity
Δ	definition; coefficient of roughness; height of projection
Θ	a constant, complex potential
Λ	coefficient of anisotropy
Σ	probability flux
Φ	modulus of resistance
Ψ	stream function
$\mathcal{F}$	Green's function

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INTRODUCTION

In the course of its existence, groundwater hydraulics has undergone intensive development, especially within recent years.

The history of groundwater hydraulics as a science can be traced back to the middle of the last century when H. Darcy established the fundamentals of flow in porous media and thus opened the way to the age of rational analysis. The empirical approach used almost exclusively until then, was gradually replaced by theoretical descriptions of phenomena and processes presented, for example, by F. Forchheimer, N. E. Zhukovsky, K. E. Lembke, N. N. Pavlovsky and others. This stage of development was characterized by the refinement of laboratory methods, a tool that has been found to provide insight into problems whose general theoretical descriptions were already available. The advent of automatic computers brought about a renewal of activities in the sphere of theoretical methods designed so as to exploit to the full the advantages of computing techniques.

The broad field of the present-day application of groundwater hydraulics extends from chemical engineering and petrochemistry to water conservation and management, all disciplines which involve the natural circulation of water with the aim of mastering it to the benefit of mankind.

In natural porous media, the flow of water is a time-dependent process characterized, on the one hand, by changes of the flow field, and on the other hand, by changes in the medium resulting from the action of flowing water. The latter phenomena and their interactions are so complicated that, for the most part, authors and practising engineers alike consider only the simple seepage, ignoring the retroactive effects.

The basic prerequisite for the application, dissemination and exploitation of accumulated facts and findings in practice, is the formulation of the problem. In subsequent discussions, essential modifications and solutions one can employ various methods, all based on a common foundation — a mathematical description of the flow of water under the given boundary and initial conditions.

Numerical methods start from the principle of discretization of time and space and from the laws of mathematical statistics. They naturally possess certain features of approximation; these, however, do not impose undue restrictions on their applicability.

Experimental methods are based essentially on the findings of theoretical and experimental physics. Their great practical significance extends far beyond the confines of groundwater hydraulics.

The decision as to which of the methods to use in an actual solution, depends first of all on the nature of the problem being considered. But neither numerical nor experimental procedures should be regarded as a universal method guaranteed to provide the required solution under all circumstances.

Since experimental methods have already been fully covered in the literature, and numerical methods will no doubt soon be ripe for a comprehensive review, we shall concentrate in this book on analytical methods and their allied mathematical techniques.

Analytical methods have been and still are the subject of worldwide attention. Their applicability has been greatly enhanced by the modification proposed by N. K. Girinsky and G. N. Kamensky who outlined the basic ideas for the solution of so-called quasi-one-dimensional problems. The advantages of this method were later put to good use in the treatment of problems of plane and quasi-plane flows.

The corner stone of the theory of two-dimensional flow is to be found in the studies of N. N. Pavlovsky who refined the method of conformal mapping, pointed out the possibility of using Zhukovsky's function and together with V. V. Vedernikov worked out the method of the velocity hodograph.

Great interest was evoked by the work of P. J. Polubarina-Kochina who published solutions to a number of special problems, achieved with the help of the analytical theory of differential equations. Exploiting the possibilities of the application of the Hilbert-Riemann problem, S. N. Numerov evolved the so-called method of boundary value problems.

Privileged positions in the development of the theory of flow in porous media should be reserved for the studies of M. Muskat dealing with three-dimensional problems, for the work of J. Boussinesq devoted to unsteady flow, for certain treatises of G. I. Barenblatt concerning the theory of elasto-plastic seepage, and for investigations of K. Terzaghi who worked out the fundamentals of the theory of consolidation.

Equally worthy of notice are the contributions of outstanding hydraulic engineers and mathematicians, such as V. I. Aravin, S. F. Averyanov, I. A. Charny, P. F. Filchakov, M. S. Hantush, C. E. Jacob, D. Kirkham, J. Kozeny, N. N. Verigin and others, whose earlier as well as more recent work is of great theoretical and practical importance.

In selecting the analytical methods to be discussed in the book, the authors attempted to stress the physical interpretation as opposed to the mathematical aspects of the formulation and the subsequent solution, which accounts for the occasional omission of proofs, theorems and criteria which the mathematician might feel are missing. By appending a list of references, an attempt has been made to provide the reader with material from which he may obtain further information.

CHAPTER 1

FUNDAMENTALS OF THE THEORY OF WATER FLOW IN SOILS AND FRACTURED ROCKS

1.1. General concepts

Groundwater flow is a special case of liquid flow in porous media. *Groundwater flow or seepage* is a movement of water in the voids of the earth's crust. The materials of interest are soils and fractured rocks.

Groundwater flow depends on the properties of the medium in which it occurs, on the properties of the liquid and on the hydraulic gradient.

The *domain of groundwater flow* is that part of space, in which the motion takes place.

Groundwater hydraulics is an independent branch of general hydraulics. All equations which define groundwater flow can be derived from general equations of hydrodynamics provided that all the peculiarities and effects which distinguish groundwater flow from normal liquid flow are duly considered and clearly described. As will be shown in the subsequent sections, it is simpler, however, to derive these equations directly, due regard being given to the distinctive properties of groundwater flow.

1.2. Mechanical characteristics of soils

All natural soils are mixtures of elementary particles whose origin may be traced in part to the disintegration of rocks by the action of water, ice, wind, temperature and live organisms, and in part to partially decomposed vegetable and animal matter. A soil is not a compact mass but a system of particles of different sizes and shapes, interconnected by voids.

Soils may be characterized from various aspects. Groundwater hydraulics essentially adopts the criteria of soil mechanics. Soil mechanics classifies soils according to the grain size and the representation of individual fractions in a given volume, and examines the behaviour of soils under the action of external forces. All soils contain water in one form or another, which influences their mechanical properties. Soil mechanics is, therefore, also concerned with other properties, e.g. moisture, consistency, compressibility, coefficient of internal friction, etc. Groundwater hydraulics extends the soil mechanics classification to embrace quantities, such as the effective porosity, the active porosity, etc., which are used in mathematical descriptions of water motion in soils.

Since mechanical characteristics and engineering properties of soils are the subjects of specialized disciplines, we shall present here only a brief review of some of the basic concepts which are likely to come up in the subsequent discussion.

1.2.1. Mechanical analysis of soils

Soils are composed of particles of diverse sizes and shapes. The size of soil particles varies from a few thousandths of a millimeter for clay particles, to decimeter dimensions for coarse gravel and boulders.

The shape of soil particles is also very varied: they can be rounded, spherical, angular, as well as very flat, like plates or scales.

Since the shape and size of particles have a direct effect on the seepage properties of soils, it is necessary to know the composition of soils in which groundwater flow occurs. An important index is the proportion, as a percentage, of grains of a given size. This is determined by a grading analysis in the course of which the soil is separated into individual fractions according to the grain size. The laboratory methods most frequently used for grading analysis are

- sieve analysis — progressive screening of soils on calibrated sieves,
- flotation — separation of soils into fractions according to their settling velocities,
- and a
- combination of both methods.

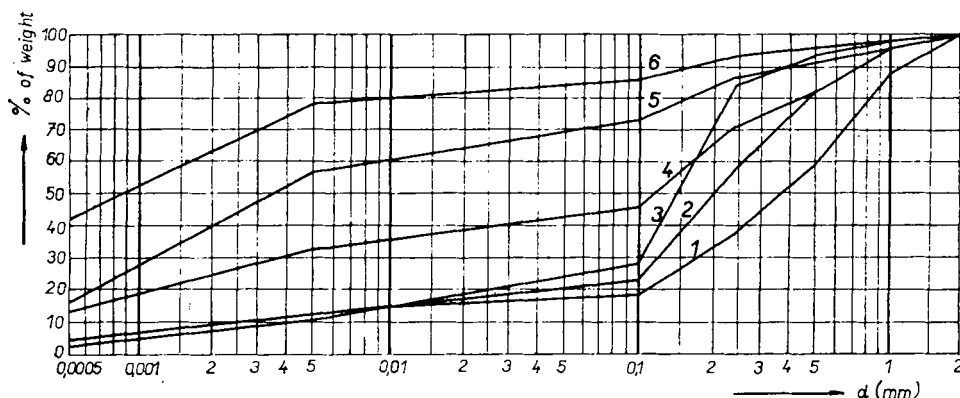


Fig. 1.1. Graphical representation of the results of a grading analysis of soils; 1 — coarse sand, 2 — sand, 3 — fine sand, 4 — clayey sand, 5 — clayey loam, 6 — clay.

The results of a grading analysis are represented by graphs. The grain diameters are shown on the horizontal axis, usually on a logarithmic scale, and the quantity by weight (in per cent of total weight) of grains smaller than, or equal to, the considered grain size are shown on the vertical axis. The resulting curve is called the grading curve. The grading curves for several kinds of soil are shown in Fig. 1.1.

The grain size is defined by the grain diameter. The diameter d denotes the size of the side (or the diameter) of the sieve opening which will pass a given grain under vibration. In grading analyses, the grain shape is of secondary importance.

The grading curve provides information about the degree of soil homogeneity and the regularity of the representation of the various fractions in a given volume. Since work with grading curves is rather tedious, engineering practice employs the so-called coefficient of uniformity Γ . The term is usually understood to mean the ratio

$$\Gamma = \frac{d_{60}}{d_{10}} \quad (1.2.1)$$

where

d_{60} — size of sieve opening passing 60% of the weight of the total volume of soil,
 d_{10} — size of sieve opening passing 10% of the weight.

Another quantity of interest is the so-called effective grain size d_e . Different authors assign it different values but $d_e = d_{10}$ at $\Gamma < 5$ is that employed most frequently.

The classification of soils according to particle size comprises several categories. Other systems of classification based on other criteria are also available but what is important to bear in mind when dealing with groundwater flow, is that a soil is the less permeable the greater is the quantity of very fine, i.e. clayey, particles it contains. Sands which are highly permeable and completely free of clayey particles are one extreme, and clays, impervious or nearly so, containing exclusively clayey particles smaller than 0.002 mm, are the other. Depending on the relative proportion of the various categories of particles, natural soils generally approximate to one or the other extreme.

1.2.2. Porosity, effective porosity and active porosity

Another important characteristic of soils is their porosity. Pores are voids linking soil particles.

The porosity of a soil, m , is defined by the ratio

$$m = \frac{V_p}{V} \quad (1.2.2)$$

where V_p is the volume of pores contained in the examined volume of soil, V . Clearly, $0 < m < 1$.

When dealing with compressible soils, the porosity number α which expresses the ratio between the volume of pores, V_p , and the volume of soil grains, is preferable. Substituting the above symbols and rearranging, the porosity number becomes

$$\alpha = \frac{m}{1 - m} \quad (1.2.3)$$

Porosity depends not only on the grain composition of the soil but also on the way it is formed, pressures to which it is exposed, etc. Its range is taken to be

$$m = 0.23 \text{ to } 0.8$$

Reference values for the porosity of some natural soils are shown in Table 1.1.

Table 1.1

Soil	Porosity
coarse sand, gravel ($2 \text{ mm} \leq d \leq 20 \text{ mm}$)	$m = 0.30 \text{ to } 0.40$
sands ($0.05 \leq d \leq 2 \text{ mm}$)	$m = 0.30 \text{ to } 0.45$
clayey sand	$m = 0.35 \text{ to } 0.45$
sandy loam	$m = 0.35 \text{ to } 0.50$
clayey loam	$m = 0.40 \text{ to } 0.55$
peat loam	$m = 0.60 \text{ to } 0.80$

Pores of natural soils always contain water bonded to soil grains. The strength of these bonds depends on the form in which water is present in the soil.

Crystalline and structural water are characterized by bonds which are so strong that the water thus bonded cannot be expelled even by heating the soil to 105°C . In grading analyses performed with dry samples of soil, such water is considered to be a part of the grains. Drying removes hygroscopically bonded water which condenses on the surface of grains from water vapours in the pores.

Drying also removes so-called film water bonded to grains, like hygroscopic water, by considerably weaker molecular bonds. The thickness of the film water layer is 0.0005 mm . Soil moisture corresponding to the maximum quantity of film water, is called the maximum molecular moisture of the soil.

Another form of water which can be removed by drying, is water bonded by capillary forces. Capillarity is essentially caused by the hydration of ions and molecules on the interface between the solid and the liquid phase. Capillary phenomena produce menisci which form the interface between the liquid and the gaseous phase. Menisci are formed inside the pores (in the region of contact of the grains). The remaining volume of the pores is filled with air.

The gaseous phase in the form of bubbles is retained in the soil even after its whole volume has been filled with water (the quantity of air is estimated at 5% of the volume of pores). A major part of the volume of the pores is then filled with water which is free to move. When this water begins to move under the action of external forces (especially gravitational forces in nature), it moves through only a part of the pore cross-section. Hygroscopic and film water adhere to the grains and some of the capillary-bonded water remains sticking to the corners between grains. Air bubbles

reduce the pore cross-section. The process depends on the pressure and velocity of the water: at higher pressures and higher velocities, a larger quantity of water contained in the pores is set in motion. In this connection we introduce the concept of the dynamically effective or effective porosity

$$m_e = \frac{V_{pe}}{V} \quad (1.2.4)$$

where V_{pe} is the volume of water moving in the pores of a volume of soil V . Obviously, $V_{pe} < V_p$. This effective porosity is the quantity considered in (side) groundwater flows.

Water present in soils due mostly to the effect of gravitational forces (so-called gravitational water) is readily removed by being allowed to flow out prior to the drying of soil samples. What remains in the soil is the hygroscopic and film water, and a considerable quantity of capillary water. From a given volume of soil, V , there will flow out a certain volume of water, $V_{p\mu_1}$, whose ratio to the total volume of soil defines the so-called coefficient of water yield

$$\mu_1 = \frac{V_{p\mu_1}}{V} \quad (1.2.5)$$

If we saturate a given sample of soil containing hygroscopic and film water as well as capillary-bonded water, with an additional charge of water, we find that the soil absorbs a certain volume $V_{p\mu_2}$ whose ratio to the volume of the whole sample gives the coefficient of saturation deficiency

$$\mu_2 = \frac{V_{p\mu_2}}{V} \quad (1.2.6)$$

Generally, $\mu_1 \neq \mu_2$. However, to simplify the mathematical description, we usually set

$$\mu = \mu_1 = \mu_2$$

The value of μ defines the so-called active porosity of the soil which is not the same as the dynamically effective porosity m_e . The difference between them lies in that the dynamically effective porosity is used in calculations relating to the interior of groundwater flow, while the active porosity is considered only in connection with the free surface of groundwater, provided both water and surface are in motion, i.e. change their shape and position with time.

The foregoing discussion plainly shows the considerable significance of soil moisture in the theoretical solution of groundwater motion. Evaluation of soil moisture may be made in two ways: one defines moisture as the ratio between the weight of water contained in the soil, and the weight of the dry matter. The other determines the degree of saturation of the soil with water, i.e. the ratio between the volume of pores filled with water, and the volume of pores in the dry matter.

1.2.3. Capillary phenomena

In the preceding exposition we touched on the subject of various forms of water contained in soils. From the point of view of groundwater flow theory, it is necessary to consider separately the flow in the zone below the free surface, where the degree of saturation of soil with water is high, and the flow above the free surface of groundwater, in the so-called capillary zone where the degree of saturation is less than the maximum. The course of the free surface which separates the region where capillary forces are the most marked, from the region in which gravitational forces predominate, is determined by means of piezometers. In a non-capillary observation tube driven below the free surface, water rises just to the level of the free surface.

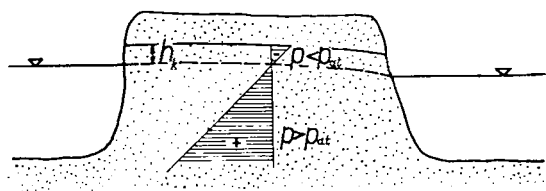


Fig. 1.2. Distribution of water pressure in soil above and below the free surface of gravitational water.

In the zone below the free surface, water is elevated by the action of capillary forces to the capillary height h_k above the continuous groundwater level (Fig. 1.2). This is the so-called capillary water which fills, partially or completely, the pores of the soil; its free surface is bounded by hollow menisci. Unlike in gravitational water where the hydrostatic pressure at rest exceeds atmospheric pressure p_{at} (the atmospheric pressure acts on the free surface of water), the hydrostatic pressure in the capillary zone is less than atmospheric. The hydrostatic pressure in the capillary zone at rest falls off linearly with height from $p = 0$ on the free surface to $p = -\gamma h_k$ on the upper boundary of the capillary zone below the menisci (γ = specific weight of water).

The capillary height h_k depends on the kind and size of pores, i.e. on the mechanical composition of the soil. Informative values of h_k for various soils are given in Table 1.2.

Table 1.2

Soil	Capillary height
sands	$h_k = 0.03$ to 0.1 m
fine sands	$h_k = 0.1$ to 0.5 m
clayey sands	$h_k = 0.5$ to 2.0 m
loess	$h_k = 2.0$ to 5.0 m
clay-loam soils	$h_k =$ to 10 m

The degree of saturation varies along the height of the capillary zone because the soil pores contain an amount of air which increases with the height. The increased

quantity of air in the pores of the capillary zone causes a considerable reduction in the quantity of water flowing. For both qualitative and quantitative reasons, distinction should be made between the flow of water in a saturated soil (in pores filled mostly with gravitational water) and that in an unsaturated soil.

1.3. Seepage characteristics of soils

The quantities of interest in hydraulics are the pressure p and the velocity v of flowing water at a general point of space. Both quantities are functions of position and time

$$v = v(x, y, z, t) \quad p = p(x, y, z, t)$$

where x, y, z , are Cartesian coordinates, and t is the time.

A detailed analysis of groundwater flow is very difficult because the seepage motion is incomparably more complicated than the usual kinds of motion of surface water. The region in which seepage motion takes place, is made up of a large number of irregularly distributed, irregularly shaped interconnected voids. A very complicated liquid motion is seen to take place even in the simplest case of seepage — for example, in a porous medium consisting of identical spheres. Hydrodynamics can provide solutions to very straightforward cases of flow past spheres or ellipsoids; however, no exact solution is at present available for flows in porous media.

This is the reason why calculations relating to seepage problems are made using the mean values of the pertinent quantities. A determination, for example, of the velocity at a given point of a void, would be of no use for practical purpose. A solution is required to yield the mean values of the characteristic quantities of groundwater flow, i.e. the velocities, pressures, quantity of flow, independently of the size and shape of the individual pores, and the values of interest to engineers are those applicable throughout areas inside a porous medium which — though fairly small — are much larger than the dimensions of the pores.

1.3.1. Mean velocity of flow in pores of saturated soils. Fictitious flow velocity

The random distribution of pores in soils makes it necessary to ignore the shape and size of the individual pores and to apply statistical methods.

The velocity of flow in soils at the various points of a pore cross-section varies from zero to a maximum value. The direction of the flow velocity in a pore cross-section is also variable. Within a given area, small but considerably larger than the dimensions of the pore cross-section, we can consider a certain mean velocity in the pores, v_* (regarded as a vector), defined by

$$v_* = \frac{\Delta Q}{\Delta S_p} \quad (1.3.1)$$

where

ΔQ — volume of water flowing through an area ΔS per unit time (discharge, quantity of seepage),

ΔS_p — area of pore cross-sections in the total area ΔS of the cross-section.

This definition is evidently identical with that of the mean velocity in general liquid flow.

Engineers studying seepage flows find it convenient to use a quantity which is independent of the shape and size of the pores, the so-called seepage velocity. This is a fictitious quantity defined as follows:

Assume that the moving liquid continuously fills the entire space in which the flow occurs, i.e. the space taken up by the soil pores as well as the space occupied by solid particles. This means that the liquid flows not just across the area ΔS_p of the pore cross-sections but across the whole area ΔS . The quantity of flow across this area is equal to the actual quantity of flow ΔQ across this area. The seepage velocity is then defined by

$$v = \frac{\Delta Q}{\Delta S} \quad (1.3.2)$$

In this way we have replaced the actual seepage flow by a fictitious flow in which the liquid continuously fills the entire space, i.e. the space occupied by the pores as well as that taken up by the solid particles.

Comparing eqn. (1.3.1) with eqn. (1.3.2) gives

$$\frac{v}{v_*} = \frac{\Delta Q / \Delta S}{\Delta Q / \Delta S_p} = m \quad (1.3.3)$$

where $m = \Delta S_p / \Delta S$ is the so-called two-dimensional porosity of the soil equal, according to Terzaghi, to the three-dimensional porosity discussed in Section 1.2.2.

It was noted in Section 1.2.2 that the quantity of flow ΔQ does not pass across the entire area ΔS_p but only across the dynamically effective area ΔS_{pe} . For this reason the porosity m in eqn. (1.3.3) is replaced by the dynamically effective porosity m_e , and eqn. (1.3.3) is written as follows

$$\frac{v}{v_*} = m_e$$

or

$$v = m_e v_* \quad (1.3.4)$$

The fictitious seepage velocity v is obtained by multiplying the mean velocity in the pores, v_* , by the effective porosity m_e . Since m_e is always less than unity,

$$v < v_*$$

The introduction of the fictitious seepage flow enabled us to get rid of the geometrical effect of the shape and size of the pores, i.e. of the geometrical effect of the solid particles in soils. What cannot be neglected, however, are the forces which these particles exert on flowing liquid. Flowing liquid acts on particles by forces which generally result in the force of normal pressure on the surface of the particles, and the force of skin friction. According to Newton's third law, particles act on liquid by equal forces of opposite sense.

Assuming that a certain small volume specified in the region of seepage, contains a sufficiently large number of solid particles, we may expect the resistance forces acting against the motion to be distributed uniformly throughout the whole volume. These forces are proportional to the size of the volume, and may be regarded as volume forces in calculations.

The above exposition leads to the following:

- a) the quantity of flow across any area (considerably larger than the area of pores), in the region of seepage, is the same in fictitious as in actual flow,
- b) the pressure on such an area is the same in fictitious as in actual flow,
- c) the resistance forces acting on a certain volume of the region of seepage flow, are the same as the resistance forces acting on this volume in actual flow,
- d) the velocities of fictitious flow are distributed continuously over the whole region of seepage, and are related to the mean velocities in the pores by eqn. (1.3.4).

Under these conditions, the seepage velocity v as well as the pressure p are continuous functions of position and time.

1.3.2. Linear (Darcy's) law

In the years 1852 to 1855, H. Darcy made experimental studies of the laws of flow, using a tube of circular cross-section S (Fig. 1.3) filled with sand. He found that a linear relation exists between the quantity of flow Q and the piezometric head, viz.

$$Q = kS \frac{h_1 - h_2}{\Delta s} \quad (1.3.5)$$

and since $Q/S = v$, according to eqn. (1.3.2),

$$v = k \frac{h_1 - h_2}{\Delta s} = k \frac{\Delta h}{\Delta s} \quad (1.3.6)$$

In eqn. (1.3.6)

$$h_1 = z_1 + \frac{p_1}{\gamma} \quad \text{and} \quad h_2 = z_2 + \frac{p_2}{\gamma}$$

where the values with subscripts 1, 2 refer, respectively, to the profiles 1, 2 where they were measured,

z — geodetic head of the centroid of the profile above the chosen reference plane,

p/γ — pressure head at profile centroid (γ — specific weight of water),

h — piezometric head

$$h = \frac{p}{\gamma} + z \quad (1.3.7)$$

Δs — distance between centroids of profiles 1 and 2 in the direction of flow (Fig. 1.3),

k — a constant, termed the coefficient of permeability of the linear law of flow.

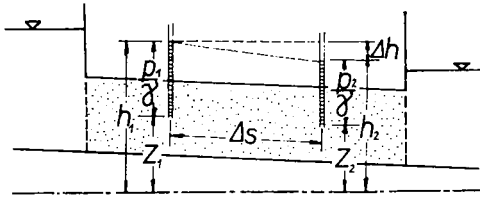


Fig. 1.3. Flow in a cylinder filled with sand.

As noted in Section 1.3.1, $p = p(x, y, z, t)$. Hence the piezometric head h is also a continuous function of position and time:

$$h = h(x, y, z, t) \quad (1.3.8)$$

In eqn. (1.3.6) Δh denotes the losses of the piezometric head along the path Δs , due to the resistance force which the medium presents to water flow. The ratio $\Delta h/\Delta s$ defines the mean gradient of the line of the piezometric heads along the path Δs

$$\frac{\Delta h}{\Delta s} = J_{\text{avg}}$$

Passing to the limit as $\Delta s \rightarrow 0$, we obtain

$$\lim_{\Delta s \rightarrow 0} \left(- \frac{h_2 - h_1}{\Delta s} \right) = - \frac{dh}{ds} = J \quad (1.3.9)$$

Eqn. (1.3.6) then takes the form

$$v = -k \frac{dh}{ds} = kJ \quad (1.3.10)$$

where J is the gradient of the piezometric head.

Eqn. (1.3.10) is a mathematical expression of Darcy's law. For $J = 1$, the coefficient of permeability, k , is defined as a velocity.

Darcy's scheme shown in Fig. 1.3 enables us to determine the magnitude of the force R_v exerted by the soil on unit mass of a real liquid. Assume that the block of sand is fictitiously filled with liquid which is continuously distributed through its volume. Hence an above-atmospheric pressure $\gamma S \Delta h$ exists on the outside. In uniform motion, the pressure $\gamma S \Delta h$ must be equal to the sum of the actions of the internal forces, viz.

$$\gamma S \Delta h = m R_v S \Delta s$$

Hence

$$R_v = \frac{\gamma}{m} \frac{\Delta h}{\Delta s} = - \frac{\gamma}{m} J$$

If the linear law of flow applies, $J = v/k$ and

$$R_v = - \frac{\gamma}{m} \frac{v}{k} \quad (1.3.11)$$

where R_v is the force acting on unit mass of liquid.

The coefficient of permeability is a quantity which depends on the properties of both the porous media and the liquids. The motions of different liquids are distinguished by different values of k . Since we are concerned with the motion of water, we shall use the symbol k to denote the coefficient of permeability to water.

The value of k may be determined in various ways, for example:

- a) in the laboratory — by testing a soil sample in special apparatus,
- b) by field experiments,
- c) from eqn. (1.3.10), after the hydraulic gradient J and the velocity v have been established by field measurements,
- d) from formulae taking into account the effect of the geometrical properties of the permeable medium and temperature.

Table 1.3 gives some useful informative values of the coefficient of permeability of several kinds of soil.

Table 1.3

Soil	Coefficient of permeability k [cm/s]
clay	0.000 001 and smaller
sandy clay	0.000 1 and smaller
consolidated clayey sand	0.000 1 to 0.000 5
sands with clay particles	0.000 1 to 0.000 2
fine sand and loose clayey sand	0.001 to 0.005
coarse sand	0.01 to 0.05
gravel sand	0.02 to 0.1 and greater

1.3.3. Kármán-Kozeny equation

Numerical values of the coefficient of permeability are obtained from relations based on certain assumptions and ideas. Many theories illustrating the subject were developed, and some of them are reviewed by Scheidegger. The best known procedure is based on a so-called capillary model, in which the flow of water in soil is compared to an equivalent flow in a system of capillary tubes.

In soils, water particles move along zig-zag paths, and for this reason the system of capillaries is considered to be full of twists and turns. On their way from one profile to another, the particles move along a curvilinear path of length αs rather than on a rectilinear path s . The value of the coefficient of tortuosity ranges between 1.1 and 1.6. According to our opinion which is borne out by the results of experiments described in the literature, as well as by our own experience, the coefficient of tortuosity

$$\alpha = m^{-1/4} \quad (1.3.12)$$

The flow in the capillaries is assumed to be laminar; by Poisseuille's theoretical description of such a flow in a straight capillary

$$v_* = \frac{g D^2}{32\nu} J \quad (1.3.13)$$

where v_* — mean profile velocity in the capillary,

D — diameter of the capillary,

g — acceleration of gravity.

The value of the kinematic coefficient of viscosity ν depends on temperature. According to Poisseuille

$$\nu = \frac{\nu_0}{1 + 0.033T + 0.000221T^2} \quad (1.3.14)$$

where ν_0 is the kinematic coefficient of viscosity of water at 0°C and T is the temperature.

In passing from the capillary model to the soil we make use of the hydraulic radius R , i.e. the ratio of the area to the wetted circumference of the cross-section. For circular pipes $R = D/4$, and hence $D = 4R$. For other capillary cross-sections, the relation is more general, i.e. $D = aR$.

In soils, the hydraulic radius is defined as the ratio of the volume of pores in a unit volume of soil, to the wetted surface of the grains in a unit volume of soil

$$R = \frac{m}{P} \quad (1.3.15)$$

P is the surface area of the grains in a unit volume of soil [m^{-1}].

Noting that in eqn. (1.3.13)

$$D = aR = a \frac{m}{P} \quad (1.3.16)$$

and, taking account of tortuosity, using eqn. (1.3.13) gives

$$v = \frac{ga^2m^3}{32\nu P^2} \frac{\Delta h}{\alpha \Delta s} = A \frac{gm^3}{\alpha \nu P^2} J \quad (1.3.17)$$

$$A = \frac{a^2}{32} \quad (1.3.18)$$

where A is Kozeny's constant.

Since a is found to depend only slightly on the shape of the capillary cross-section, we can take $a = 4$, so that $A = 1/2$.

In his verification of Kozeny's equation (1.3.17), Kármán arrived at the conclusion that the zig-zag shape of the tube has an effect on the choice of the plane of the section in which the mean profile velocity is to be calculated. Kozeny's calculations were made for other than the appropriate cross-sectional area. The corrected equation, known as the Kármán-Kozeny equation, is of the form

$$v = \frac{Agm^3}{\alpha^2 \nu P^2} J \quad (1.3.19)$$

The value of the coefficient of permeability k

$$k = \frac{Agm^3}{\alpha^2 \nu P^2} = \frac{cg}{\nu} \quad (1.3.20)$$

depends on the properties of the soil, characterized by the coefficient c

$$c = \frac{Am^3}{\alpha^2 P^2} \quad (1.3.21)$$

termed the physical permeability [m^2].

Another important factor is the viscosity of the liquid. As eqn. (1.3.14) implies, the value of k grows with increasing temperature.

Eqn. (1.3.19) contains the symbol P denoting the specific surface. This quantity is generally difficult to determine; a method for its determination was worked out by Zunker. It can be proved that in a simple case of soil consisting of randomly arranged identical spheres

$$P = \frac{6}{d} (1 - m) \quad (1.3.22)$$

where d is the diameter of the spherical grain.

For $A = 0.5$, eqns. (1.3.20) and (1.3.21), and eqn. (1.3.12), (1.3.22) imply that

$$c = \frac{m^{3/2} \kappa^2 d^2}{72} \quad (1.3.23)$$

$$k = \frac{g m^{3/2} \kappa^2 d^2}{72\nu} \quad (1.3.24)$$

where κ is the porosity number.

Eqns. (1.3.23) and (1.3.24) can also be used to estimate the seepage activity of soils made up of non-uniform grains. In that case we set $d = d_e$, and choose

$$d_e = d_{10} \quad \text{to} \quad d_{20}$$

depending on what we wish to emphasize. The lower limit stresses the importance of fine soil fractions, and is used more frequently than the upper one.

1.4. Mathematical description of motion of water flowing according to the linear law through saturated soils

The fundamental equations of motion will be derived on the assumption that the porous medium undergoes no distortion during the flow, and that the liquid density, ϱ , is constant.

1.4.1. Equation of motion and equation of continuity

The equation of motion is derived from the condition for the equilibrium of all forces acting on the liquid. These forces are:

- forces due to the hydrodynamic pressure of liquid
- forces due to the acceleration of gravity
- resistance forces in seepage flow, and
- forces needed to accelerate the flowing liquid.

In the region of seepage, consider a small body of volume V (e.g. a parallelepiped with dimensions $\Delta x, \Delta y, \Delta z$, and edges parallel to the coordinate axes); the body is fairly large compared to the soil grains, and may, therefore, be assumed to contain a number of grains distributed uniformly throughout the volume V . dV is an element of the volume V . The area of the surface enclosing the chosen volume V is S . An element of this surface is dS , its corresponding surface vector is $d\mathbf{S}$ (the positive direction of this vector is outwards from the enclosed space, i.e. in the direction of the positive normal).

We now determine the forces which act on the liquid contained in the volume V . The volume of this liquid is mV , and there is $m dV$ of liquid contained in an element dV . We carry out the derivation in vector form.

In Section 1.3.2 we found the magnitude of the resistance forces in seepage flow. The resistance force acting on a unit mass of liquid is

$$\mathbf{R}_v = - \frac{\varrho g}{mk} \mathbf{v}$$

and on the liquid in the volume V

$$\bar{\mathbf{R}}_V = - \int_V \frac{\varrho g}{mk} \mathbf{v} m dV = - \frac{\varrho g}{k} \int_V \mathbf{v} dV \quad (1.4.1)$$

Forces produced by the hydrodynamic pressure p have two components. One component includes the forces which the liquid outside the volume exerts on the liquid inside that volume. These forces act only on the surface mS of pores cut by the surface S . Their resultant (pressure acting against the surface vector $d\mathbf{S}$) is

$$\mathbf{P}_1 = -m \oint_S p d\mathbf{S}$$

where $p d\mathbf{S}$ is the resultant of the pressure on an element of surface $d\mathbf{S}$. By Gauss' theorem, the surface integral may be transformed to a volume integral, as the condition that p be a scalar function of position in the ambient space is satisfied, viz.

$$\mathbf{P}_1 = -m \oint_S p d\mathbf{S} = -m \int_V \text{grad } p dV.$$

The second component comprises forces which are produced — according to the law of action and reaction — by grains of soil enclosed in the volume V pressing on the liquid inside that volume. Assuming the grains to be very small compared with the volume V and to be distributed uniformly throughout the volume, and the pressure p to remain constant within the compass of a grain, the resulting pressure on the soil grains is given by the resulting pressure on the grains cut by the surface S of the chosen volume. Assuming that $p = \text{const.}$, the resultant acting on the grains inside the volume, is zero. This resultant does not include the uplift force which is included among the forces produced by the acceleration of gravity. The resulting pressure on the cut grain is given by the magnitude of the pressure on the area of the section, and is directed towards the outside of the volume. The grains press against the liquid by an equal force in the opposite direction. If the selected volume is observed from the outside, the magnitude of this force is

$$\mathbf{P}_2 = -(1 - m) \oint_S p d\mathbf{S} = -(1 - m) \int_V \text{grad } p dV.$$

The force produced by the hydrostatic pressure is

$$\mathbf{P} = \mathbf{P}_1 + \mathbf{P}_2 = - \int_V \text{grad } p \, dV \quad (1.4.2)$$

The forces produced by the acceleration of gravity are body forces, and can also be divided into two parts. If g is the acceleration of gravity, the force on the liquid enclosed in the volume V acting vertically downward is

$$\mathbf{G}_1 = \int_V \mathbf{g} m \varrho \, dV = \varrho \mathbf{g} m \int_V dV$$

Grains immersed in the liquid are buoyed up with a force equal to the weight of the amount of liquid that the grains displace. According to the law of action and reaction, the grains act with an equal force on the liquid, i.e. the force exerted (vertically downward) by the grains on the liquid enclosed in the volume V , is

$$\mathbf{G}_2 = \int_V \mathbf{g}(1 - m) \varrho \, dV = \varrho \mathbf{g}(1 - m) \int_V dV$$

Hence the force produced by the acceleration of gravity is

$$\mathbf{G} = \mathbf{G}_1 + \mathbf{G}_2 = \varrho \mathbf{g} \int_V dV \quad (1.4.3)$$

Forces required to accelerate the flowing liquid are also body forces, and have two components. The acceleration of the liquid is given by the rate of change of the actual velocity $\mathbf{v}_*$ in a statistically average pore. The force acting on the liquid enclosed in the volume V (in the direction contrary to the motion) is

$$\mathbf{F}_1 = - \int \frac{d\mathbf{v}_*}{dt} m \varrho \, dV.$$

Body forces produced by the acceleration of inertia, act not only on the liquid but also on the grains in the volume V — just as do the body forces produced by the acceleration of gravity. The grains act on the liquid by the force in the opposite direction

$$\mathbf{F}_2 = - \int_V \frac{d\mathbf{v}_*}{dt} (1 - m) \varrho \, dV$$

The resulting force on the liquid, produced by the acceleration of gravity, is

$$\mathbf{F} = \mathbf{F}_1 + \mathbf{F}_2 = - \int_V \frac{d\mathbf{v}_*}{dt} \varrho \, dV = - \frac{\varrho}{m} \int_V \frac{d\mathbf{v}}{dt} dV \quad (1.4.4)$$

The condition of equilibrium requires that

$$\mathbf{P} + \mathbf{G} + \bar{\mathbf{R}}_V + \mathbf{F} = 0$$

After substituting in accordance with eqns. (1.4.1) to (1.4.4), differentiating with respect to V and some manipulation, we obtain the equation of motion of three-dimensional seepage flow of an incompressible liquid in a non-deforming medium:

$$-\int \text{grad } p \, dV + \varrho \mathbf{g} \int_V dV - \frac{\varrho g}{k} \int_V \mathbf{v} \, dV - \frac{\varrho}{m} \int \frac{d\mathbf{v}}{dt} \, dV = 0$$

$$\frac{\varrho}{m} \frac{d\mathbf{v}}{dt} + \text{grad } p - \varrho \mathbf{g} + \frac{\varrho g}{k} \mathbf{v} = 0 \quad (1.4.5)$$

Writing the expression $(\text{grad } p - \varrho \mathbf{g})$ in component form (the $+z$ axis points upwards, the acceleration of gravity $\mathbf{g}$ downwards), and assuming the validity of eqn. (1.3.7) which we now introduce in the more general form

$$h = \frac{p}{\gamma} + z + Z \quad (1.3.7')$$

where the length constant Z characterizes the position of the reference plane chosen for the determination of the magnitude of h , we may write

$$\text{grad } p - \varrho \mathbf{g} = \varrho g \, \text{grad } h$$

and the equation of motion takes the form

$$\frac{1}{mg} \frac{d\mathbf{v}}{dt} + \text{grad } h + \frac{\mathbf{v}}{k} = 0 \quad (1.4.6)$$

In the next step we write out the term $(1/mg)(d\mathbf{v}/dt)$ of eqn. (1.4.6). By elementary vector analysis we have

$$\frac{d\mathbf{v}}{dt} = \frac{dv_x}{dt} \mathbf{i} + \frac{dv_y}{dt} \mathbf{j} + \frac{dv_z}{dt} \mathbf{k}$$

where $\mathbf{i}, \mathbf{j}, \mathbf{k}$ are unit vectors in the directions of the coordinate axes. The expression dv_x/dt , and similarly the expressions dv_y/dt and dv_z/dt , can be written as follows, recalling that $v_x = v_x(x, y, z, t)$:

$$\begin{aligned} \frac{dv_x}{dt} &= \frac{\partial v_x}{\partial x} \frac{dx}{dt} + \frac{\partial v_x}{\partial y} \frac{dy}{dt} + \frac{\partial v_x}{\partial z} \frac{dz}{dt} + \frac{\partial v_x}{\partial t} \frac{dt}{dt} = \\ &= v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y} + v_z \frac{\partial v_x}{\partial z} + \frac{\partial v_x}{\partial t} \end{aligned}$$

Since the seepage velocity and its derivative in the directions of the coordinate axes are comparatively small, terms such as $v_x(\partial v_x/\partial x)\mathbf{i}$, $v_y(\partial v_x/\partial y)\mathbf{j}$, etc. may be neglected in eqn. (1.4.6). Therefore, we may write $(1/mg)(\partial \mathbf{v}/\partial t)$ instead of $(1/mg) \cdot (d\mathbf{v}/dt)$ to obtain eqn. (1.4.6) in its final form:

$$\frac{1}{mg} \frac{\partial \mathbf{v}}{\partial t} + \text{grad } h + \frac{\mathbf{v}}{k} = 0 \quad (1.4.7)$$

Since it contains two unknowns, the velocity $\mathbf{v}$ and the piezometric head h , the equation (1.4.7) alone is not enough to give the solution. We also need an equation which analyzes the balance relations.

The quantity of liquid flowing in time dt through an element dS of the surface S which encloses the specified volume V , is

$$dQ = dt \mathbf{v} dS$$

where $\mathbf{v} dS$ is the scalar product of the velocity vector $\mathbf{v}$ and the surface vector dS . The quantity of liquid flowing through the whole surface S in the time dt is

$$Q = dt \oint_S \mathbf{v} dS$$

Since the liquid is incompressible, the condition of continuity requires that the quantity of liquid which flows into the specified volume V of the porous medium in the time dt , should in the same period flow out of the volume V . Hence

$$Q = dt \oint_S \mathbf{v} dS = 0$$

By Gauss' theorem, the surface integral $\oint_S \mathbf{v} dS$ may be transformed into the volume integral

$$\oint_S \mathbf{v} dS = \int_V \text{div } \mathbf{v} dV$$

so that the resulting form of the equation of continuity is

$$\int_V \text{div } \mathbf{v} dV = 0$$

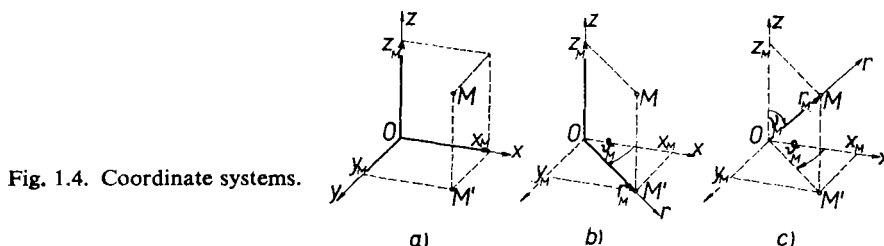
which after differentiation with respect to V yields

$$\text{div } \mathbf{v} = 0 \quad (1.4.8)$$

Eqns. (1.4.7) and (1.4.8) form a system of equations of unsteady seepage flow of an incompressible liquid, from which we can calculate the required quantities $\mathbf{v}$ and h ,

or possibly with the help of eqn. (1.3.7') the quantities $\mathbf{v}$ and p , if we know the boundary and the initial conditions of the solution.

Practical problems of seepage flow are usually solved using rectangular Cartesian coordinates. However, in some cases it is expedient to use cylindrical coordinates (e.g. cases of axi-symmetrical flows towards wells) or spherical coordinates (e.g. discharge through the bottom into an incomplete well).



The various systems of coordinates are shown in Fig. 1.4. The formulae for transforming cylindrical and spherical coordinates into rectangular coordinates, and vice versa, are given below.

a) Rectangular coordinates: x, y, z

b) Cylindrical coordinates: r, ϑ, z

Transformation formulae:

$$x = r \cos \vartheta \quad r = + \sqrt{(x^2 + y^2)}$$

$$y = r \sin \vartheta \quad \vartheta = \operatorname{arctg} \frac{y}{x}$$

$$z = z \quad z = z$$

c) Spherical coordinates: r, ϑ, σ

Transformation formulae:

$$x = r \sin \sigma \cos \vartheta \quad r = + \sqrt{(x^2 + y^2 + z^2)}$$

$$y = r \sin \sigma \sin \vartheta \quad \vartheta = \operatorname{arctg} \frac{y}{x}$$

$$z = r \cos \sigma \quad \sigma = \operatorname{arctg} \frac{\sqrt{(x^2 + y^2)}}{z}$$

We now write out eqns. (1.4.7) and (1.4.8) for each system of coordinates. In keeping with the vector language, we write the vector expressions in terms of the components of the directions of the coordinate axes so that a vector equation yields three equations for the vector components in the directions considered.

a) Rectangular coordinates

The equations of motion:

$$\begin{aligned}
\frac{1}{mg} \frac{\partial v_x}{\partial t} + \frac{\partial h}{\partial x} + \frac{v_x}{k} &= 0 \\
\frac{1}{mg} \frac{\partial v_y}{\partial t} + \frac{\partial h}{\partial y} + \frac{v_y}{k} &= 0 \\
\frac{1}{mg} \frac{\partial v_z}{\partial t} + \frac{\partial h}{\partial z} + \frac{v_z}{k} &= 0
\end{aligned} \tag{1.4.9}$$

The equation of continuity:

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} = 0 \tag{1.4.10}$$

b) Cylindrical coordinates:

The equations of motion:

$$\begin{aligned}
\frac{1}{mg} \frac{\partial v_r}{\partial t} + \frac{\partial h}{\partial r} + \frac{v_r}{k} &= 0 \\
\frac{1}{mg} \frac{\partial v_\vartheta}{\partial t} + \frac{\partial h}{r \partial \vartheta} + \frac{v_\vartheta}{k} &= 0 \\
\frac{1}{mg} \frac{\partial v_z}{\partial t} + \frac{\partial h}{\partial z} + \frac{v_z}{k} &= 0
\end{aligned} \tag{1.4.11}$$

The equation of continuity:

$$\frac{1}{r} \frac{\partial(rv_r)}{\partial r} + \frac{1}{r} \frac{\partial v_\vartheta}{\partial \vartheta} + \frac{\partial v_z}{\partial z} = 0 \tag{1.4.12}$$

c) Spherical coordinates:

The equations of motion:

$$\begin{aligned}
\frac{1}{mg} \frac{\partial v_r}{\partial t} + \frac{\partial h}{\partial r} + \frac{v_r}{k} &= 0 \\
\frac{1}{mg} \frac{\partial v_\vartheta}{\partial t} + \frac{1}{r \sin \vartheta} \frac{\partial h}{\partial \vartheta} + \frac{v_\vartheta}{k} &= 0 \\
\frac{1}{mg} \frac{\partial v_\sigma}{\partial t} + \frac{\partial h}{r \partial \vartheta} + \frac{v_\sigma}{k} &= 0
\end{aligned} \tag{1.4.13}$$

The equation of continuity:

$$\frac{1}{r^2} \frac{\partial(r^2 v_r)}{\partial r} + \frac{1}{r \sin \nu} \frac{\partial v_\theta}{\partial \theta} + \frac{1}{r \sin \nu} \frac{\partial(\sin \sigma v_\nu)}{\partial \nu} \quad (1.4.14)$$

We now have in each coordinate system a set of four equations in four unknowns, i.e. the three components of the velocity v and the piezometric head h . We can calculate these quantities if we know the initial and the boundary conditions. The initial conditions refer to the state which prevails at the initial instant of the solution, i.e. at the time $t = 0$. The boundary conditions will be discussed in Section 1.4.4.

The equations derived above apply generally to any liquid whose coefficient of permeability for a given porous medium is k . As mentioned in Section 1.3.2, k usually denotes the coefficient of permeability of water, which is what we shall assume it to be in the sequel.

1.4.2. Steady flow of incompressible liquid

In steady-state seepage flow, the velocity does not vary with time. Hence the first term in eqns. (1.4.6) or (1.4.7), and similarly, in eqns. (1.4.9), (1.4.11) and (1.4.13) is zero.

Eqn. (1.4.6) thus simplified immediately yields the relation between the seepage velocity $\mathbf{v}$ and the piezometric head h , viz.

$$\mathbf{v} = -k \text{ grad } h \quad (1.4.15)$$

The motion of water in soil is described with the aid of the theory of the potential flow of an ideal liquid, which brings in the concept of the velocity potential φ , where φ is a function of position in the region of seepage. The velocity potential is defined as

$$\varphi = -kh \quad (1.4.16)$$

Substituting into eqn. (1.4.15)

$$\mathbf{v} = \text{grad } \varphi \quad (1.4.17)$$

we see that the velocity potential is evidently a function whose partial derivatives in the direction of the selected coordinates uniquely determine the components of seepage velocity in the direction of those coordinates.

Substituting from eqn. (1.4.15) or eqn. (1.4.17) in the equation of continuity, (1.4.8), we obtain

$$\text{div } \mathbf{v} = -k \text{ div grad } h = \text{div grad } \varphi = 0$$

The introduction of the Hamilton operator nabla and the Laplace operator delta makes it possible to write the equation in the form

$$\operatorname{div} \mathbf{v} = -k \nabla^2 h = -k \Delta h = 0 \quad (1.4.18a)$$

$$\operatorname{div} \mathbf{v} = \nabla^2 \varphi = \Delta \varphi = 0 \quad (1.4.18b)$$

Both the velocity potential φ and the piezometric head h satisfy the Laplace differential equation $\Delta \varphi = 0$, or $\Delta h = 0$, and are, therefore, harmonic functions in the region of seepage. Writing eqns. (1.4.15), (1.4.17), (1.4.18) in component form in the various systems of coordinates, we obtain:

a) Rectangular coordinates:

The components of seepage velocity:

$$\begin{aligned} v_x &= \frac{\partial \varphi}{\partial x} = -k \frac{\partial h}{\partial x} \\ v_y &= \frac{\partial \varphi}{\partial y} = -k \frac{\partial h}{\partial y} \\ v_z &= \frac{\partial \varphi}{\partial z} = -k \frac{\partial h}{\partial z} \end{aligned} \quad (1.4.19)$$

The equation of continuity

$$\Delta \varphi = \frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} + \frac{\partial^2 \varphi}{\partial z^2} = 0 \quad (1.4.20a)$$

$$\Delta h = \frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2} + \frac{\partial^2 h}{\partial z^2} = 0 \quad (1.4.20b)$$

and according to (1.3.7')

$$\Delta p = \frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} + \frac{\partial^2 p}{\partial z^2} = 0 \quad (1.4.20c)$$

b) Cylindrical coordinates:

The components of seepage velocity:

$$\begin{aligned} v_r &= \frac{\partial \varphi}{\partial r} = -k \frac{\partial h}{\partial r} \\ v_\theta &= \frac{1}{r} \frac{\partial \varphi}{\partial \theta} = -k \frac{1}{r} \frac{\partial h}{\partial \theta} \\ v_z &= \frac{\partial \varphi}{\partial z} = -k \frac{\partial h}{\partial z} \end{aligned} \quad (1.4.21)$$

The equation of continuity

$$\Delta\varphi = \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial\varphi}{\partial r} \right) + \frac{\partial^2\varphi}{r^2 \partial\vartheta^2} + \frac{\partial^2\varphi}{\partial z^2} = 0 \quad (1.4.22a)$$

$$\Delta h = \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial h}{\partial r} \right) + \frac{\partial^2 h}{r^2 \partial\vartheta^2} + \frac{\partial^2 h}{\partial z^2} = 0 \quad (1.4.22b)$$

$$\Delta p = \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial p}{\partial r} \right) + \frac{\partial^2 p}{r^2 \partial\vartheta^2} + \frac{\partial^2 p}{\partial z^2} = 0 \quad (1.4.22c)$$

c) Spherical coordinates:

The components of seepage velocity:

$$\begin{aligned} v_r &= \frac{\partial\varphi}{\partial r} = -k \frac{\partial h}{\partial r} \\ v_\vartheta &= \frac{\partial\varphi}{r \sin \vartheta \partial\vartheta} = -k \frac{\partial h}{r \sin \vartheta \partial\vartheta} \\ v_\sigma &= \frac{\partial\varphi}{r \partial\vartheta} = -k \frac{\partial h}{r \partial\vartheta} \end{aligned} \quad (1.4.23)$$

The equation of continuity

$$\Delta\varphi = \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial\varphi}{\partial r} \right) + \frac{1}{r^2 \sin^2 \vartheta} \frac{\partial^2\varphi}{\partial\vartheta^2} + \frac{1}{r^2 \sin \vartheta} \frac{\partial}{\partial \vartheta} \left(\sin \vartheta \frac{\partial\varphi}{\partial \vartheta} \right) = 0 \quad (1.4.24a)$$

$$\Delta h = \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial h}{\partial r} \right) + \frac{1}{r^2 \sin^2 \vartheta} \frac{\partial^2 h}{\partial\vartheta^2} + \frac{1}{r^2 \sin \vartheta} \frac{\partial}{\partial \vartheta} \left(\sin \vartheta \frac{\partial h}{\partial \vartheta} \right) = 0 \quad (1.4.24b)$$

$$\Delta p = \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial p}{\partial r} \right) + \frac{1}{r^2 \sin^2 \vartheta} \frac{\partial^2 p}{\partial\vartheta^2} + \frac{1}{r^2 \sin \vartheta} \frac{\partial}{\partial \vartheta} \left(\sin \vartheta \frac{\partial p}{\partial \vartheta} \right) = 0 \quad (1.4.24c)$$

Eqn. (1.4. 20) (or eqn. (1.4.22), or eqn. (1.4.24)) describes the case of steady-state seepage flow of an incompressible liquid. It is a differential equation in one unknown (φ , h or p) which may be set up for particular boundary conditions and solved to give all the relevant quantities.

Eqs. (1.4.20), (1.4.22) and (1.4.24) also enable us to obtain a solution for an instantaneous state of unsteady seepage flow or for cases of unsteady seepage flow in which the effect of inertia forces may be assumed to be negligible. This initial condition applies to the great majority of practical problems.

1.4.3. Equipotential surfaces. Streamlines

By solving the resulting differential equations derived in the foregoing sections, we can obtain the values of the velocity potential φ or of the piezometric head h at various points of the region of flow (the piezometric head was defined in Section 1.3.2, the velocity potential in Section 1.4.2):

$$\varphi = \varphi(x, y, z) \quad (1.4.25a)$$

$$h = h(x, y, z) \quad (1.4.25b)$$

If a constant value is assigned to the quantity φ , eq. (1.4.25a) becomes the equation of an equipotential surface (i.e. a surface on which the potential φ has a constant value), and by eqn. (1.4.16), also the equation of equal piezometric heads, and vice versa.

In seepage flow, a streamline is defined in the same way as in general liquid flow. It is a line whose tangent at a given point determines the direction of the velocity vector at that point.

At any point of the equipotential surface (the surface of equal piezometric heads), the velocity vector and hence also the streamlines are normal to that surface.

1.4.4. Boundary conditions

The differential equations derived in Section 1.4.2 yield the values of the potential or of the piezometric head as functions of the coordinates of points in the region of seepage, and of time.

If a flow is to be considered as a whole, the following two cases must be distinguished:

- flow under pressure, i.e. flow without a free surface
- flow with a free surface.

a) Flow under pressure is a steady flow in most instances. Even if it is unsteady, we can assume the time variations to be small (as is usually the case), and neglect the first term in eqn. (1.4.7) and hence also in eqns. (1.4.9), (1.4.11) and (1.4.13). Cases in which pressure impacts occur at the boundary of the flow region (e.g. a sudden rise of the level at the boundary of an earth massif) are exceptions to this rule.

b) Flow with a free surface has two different forms:

- the free surface is at rest,
- the free surface is in motion.

If the free surface is at rest, the flow is steady.

The case of a moving free surface is more important, for the velocity at a general point of the whole region varies with time. However, these time variations are usually

small, and the first term in eqns. (1.4.7), (1.4.9), (1.4.11) and (1.4.13) may again be neglected. Despite this fact, the flow is unsteady because the free surface is in motion. This circumstance must be expressed by the boundary conditions.

The solution of the differential equations describing seepage flow depends on the geometrical shape of the region of seepage as well as on the hydrodynamic properties of the segments forming the boundary of the region of seepage flow.

The four types of boundaries encountered in groundwater flow include: permeable segments, impervious segments, free surface and surface of seepage (Fig. 1.5). We consider the characteristics of each of these boundaries separately.

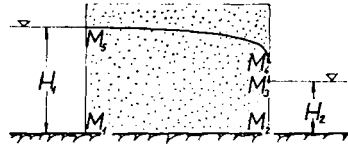


Fig. 1.5. Boundaries of the region of seepage.

a) Impervious segments — Fig. 1.5, the segment M_1M_2 :

At impervious boundaries the velocity vector can only have a direction parallel to the direction of the boundary since the velocity component normal to the boundary must be zero. Denoting by v_n the velocity component normal to the boundary, we can write, for an impervious segment,

$$v_n = 0 \quad (1.4.26)$$

b) Permeable segments — Fig. 1.5, the segments M_1M_5 and M_2M_3 :

At these boundaries water enters or leaves the region of seepage. The pressure distribution there may be taken to be specified. In locations where the seepage region adjoins a water reservoir and the boundary is under water, the piezometric head h and the potential φ are constant

$$h = \frac{p}{\gamma} + z = h_x = \text{const.} \quad \varphi = -kh_x = \text{const.} \quad (1.4.27)$$

where h_x is the depth of water in the reservoir. The pressure p is a linear function of the coordinate z , viz.

$$\frac{p}{\gamma} = h_x - z \quad (1.4.28)$$

Sometimes the flow field is determined on the assumption that the flow velocity normal to the boundary

$$v_n = -k \frac{\partial h}{\partial n} = \frac{\partial \varphi}{\partial n} = \text{const.} \quad (1.4.29)$$

is prescribed along the segment.

c) Free surface (line of seepage, depression curve).

At this boundary the pressure is equal to atmospheric, so that

$$\frac{p}{\gamma} = 0 \quad (1.4.30)$$

and since $h = z + p/\gamma$ (cf. eqn. (1.3.7)), on the free surface

$$h = z, \quad \varphi = -kz \quad (1.4.31)$$

Assuming that a capillary zone (Section 1.2.3) exists above the free surface, the pressure immediately below its upper boundary is less than atmospheric.

In steady-state groundwater flow without infiltration or evaporation, the velocity component normal to the surface is zero and eqn. (1.4.26) applies. In the case of infiltration (or evaporation) defined by the velocity v_0 we have on the free surface

$$v_z = -v_0 \quad (\text{infiltration}) \quad (1.4.32)$$

$$v_z = v_0 \quad (\text{evaporation}) \quad (1.4.33)$$

d) Surface of seepage*) — Fig. 1.5, the segment M_3M_4 :

At this boundary, groundwater enters a zone free of both liquid and soil. Since the pressure there is atmospheric, eqn. (1.4.31) applies again.

e) For unsteady groundwater flow, the boundary conditions at the free surface are more complicated.

Assume that the groundwater level which moves in unsteady flow, forms a continuous surface described by the equation

$$F(x, y, z, t) = 0 \quad (1.4.34)$$

The free surface moves and the components of seepage velocity at the points of the free surface are

$$v_x = \mu \frac{dx}{dt}, \quad v_y = \mu \frac{dy}{dt}, \quad v_z = \mu \frac{dz}{dt} \quad (1.4.35)$$

where μ is the active porosity.

*) Whenever the free surface of groundwater in a permeable layer lies above the free surface in the adjoining water reservoir (river, canal, ditch or well, etc.), groundwater seeps into the reservoir. The free surface of groundwater falls in the direction of the reservoir; on the wall bounding the permeable layer, it does not terminate precisely at the height of the water level in the reservoir, but somewhat higher. The segment on the wall where the groundwater seeps freely to the surface and flows into the reservoir, is called the surface of seepage. The problem of the surface of seepage will be taken up in Section 6.1.

Differentiating eqn. (1.4.34) with respect to time t and recalling eqn. (1.4.35) we obtain

$$\frac{\partial F}{\partial t} + \frac{1}{\mu} \frac{\partial F}{\partial x} v_x + \frac{1}{\mu} \frac{\partial F}{\partial y} v_y + \frac{1}{\mu} \frac{\partial F}{\partial z} v_z = 0. \quad (1.4.36)$$

Infiltration into the free surface is given by the infiltration velocity v_0 . The infiltration flow may simply be regarded as a kind of potential flow with the potential function

$$\varphi = -kh_0 \quad (1.4.37)$$

where h_0 is some height, generally a function of z, t : $h_0 = f(z, t)$.

This infiltration flow takes place immediately above as well as immediately below the free surface. The fictitious velocity of the infiltration flow immediately above the free surface is

$$v_0 = -\mu \frac{\partial h_0}{\partial t} \quad (1.4.38)$$

(the surface rises, the velocity v_0 points downwards).

In the layer immediately below the free surface

$$v_0 = -k \frac{\partial h_0}{\partial z} \quad (1.4.39)$$

The potential function applying in the narrow region above and below the free surface is given by the sum of the two potentials defined by eqs. (1.4.31) and (1.4.37):

$$\varphi = -kz - kh_0 = -k(z + h_0) \quad (1.4.40)$$

where z is the height of the free surface.

Eqn. (1.4.40) yields

$$\varphi + k(z + h_0) = 0 \quad (1.4.41)$$

According to eqns. (1.4.34) and (1.4.41)

$$F(x, y, z, t) = \varphi(x, y, z, t) + k[z + h_0(z, t)]$$

Substitution from eqn. (1.4.38) leads to

$$\begin{aligned} \frac{\partial F}{\partial t} &= \frac{\partial \varphi}{\partial t} + k \frac{\partial h_0}{\partial t} = \frac{\partial \varphi}{\partial t} - \frac{k}{\mu} v_0 \\ \frac{\partial F}{\partial x} &= \frac{\partial \varphi}{\partial x} \\ \frac{\partial F}{\partial y} &= \frac{\partial \varphi}{\partial y} \\ \frac{\partial F}{\partial z} &= \frac{\partial \varphi}{\partial z} + k + k \frac{\partial h_0}{\partial z} = \frac{\partial \varphi}{\partial z} + k - v_0 \end{aligned}$$

Recalling eqn. (1.4.19), substitution in eqn. (1.4.36) gives

$$\frac{\partial \phi}{\partial t} - \frac{k}{\mu} v_0 + \frac{1}{\mu} \left(\frac{\partial \phi}{\partial x} \right)^2 + \frac{1}{\mu} \left(\frac{\partial \phi}{\partial y} \right)^2 + \frac{1}{\mu} \frac{\partial \phi}{\partial z} \left(\frac{\partial \phi}{\partial z} + k - v_0 \right) = 0$$

or

$$\mu \frac{\partial \phi}{\partial t} + \left(\frac{\partial \phi}{\partial x} \right)^2 + \left(\frac{\partial \phi}{\partial y} \right)^2 + \left(\frac{\partial \phi}{\partial z} \right)^2 + (k - v_0) \frac{\partial \phi}{\partial z} - k v_0 = 0 \quad (1.4.42)$$

Eqn. (1.4.42) expresses the condition for the free surface. The term v_0 describes the effect of positive or negative external discharge into the free surface (infiltration, evaporation).

Substituting the piezometric head defined by eqn. (1.4.16) in eqn. (1.4.42) followed by some manipulations yields

$$-\frac{\mu}{k} \frac{\partial h}{\partial t} + \left(\frac{\partial h}{\partial x} \right)^2 + \left(\frac{\partial h}{\partial y} \right)^2 + \left(\frac{\partial h}{\partial z} \right)^2 - \frac{\partial h}{\partial z} + \frac{v_0}{k} \frac{\partial h}{\partial z} - \frac{v_0}{k} = 0 \quad (1.4.43)$$

and assuming that $v_0 = 0$

$$\mu \frac{\partial \phi}{\partial t} + \left(\frac{\partial \phi}{\partial x} \right)^2 + \left(\frac{\partial \phi}{\partial y} \right)^2 + \left(\frac{\partial \phi}{\partial z} \right)^2 + k \frac{\partial \phi}{\partial z} = 0 \quad (1.4.44)$$

$$-\frac{\mu}{k} \frac{\partial h}{\partial t} + \left(\frac{\partial h}{\partial x} \right)^2 + \left(\frac{\partial h}{\partial y} \right)^2 + \left(\frac{\partial h}{\partial z} \right)^2 - \frac{\partial h}{\partial z} = 0 \quad (1.4.45)$$

1.4.5. Effect of anisotropy of permeable media on steady flow

The alternate layers of natural soils are sometimes so thin that the stratification effect vanishes when viewed macroscopically. The non-homogeneity takes on a special character.

The description of the flow is simplified if the flow is studied in a continuous medium whose properties depend on the direction of the seeping water. Assume that the layers are parallel and occur alternately in a definite order (Fig. 1.6) The extreme values of k are calculated from the equations

$$k_{\max} = \frac{k_1 \Delta T_1 + k_2 \Delta T_2 + \dots + k_n \Delta T_n}{\Delta T_1 + \Delta T_2 + \dots + \Delta T_n} \quad (1.4.46)$$

$$k_{\min} = \frac{\frac{\Delta T_1}{k_1} + \frac{\Delta T_2}{k_2} + \dots + \frac{\Delta T_n}{k_n}}{\Delta T_1 + \Delta T_2 + \dots + \Delta T_n} \quad (1.4.47)$$

where ΔT_i is the height, and k_i the coefficient of permeability of the i -th layer.

In most cases the schematized flow obeys the linear law. The velocity components are given by

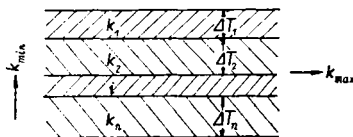
$$v_x = -k_x \frac{\partial h}{\partial x}, \quad v_y = -k_y \frac{\partial h}{\partial y}, \quad v_z = -k_z \frac{\partial h}{\partial z} \quad (1.4.48)$$

Eqns. (1.4.48) are substituted in the equation of continuity, (1.4.8) and also in the condition (1.4.36) for the free surface. For constant values of k_x, k_y, k_z we obtain

$$k_x \frac{\partial^2 h}{\partial x^2} + k_y \frac{\partial^2 h}{\partial y^2} + k_z \frac{\partial^2 h}{\partial z^2} = 0 \quad (1.4.49)$$

$$-\mu \frac{\partial h}{\partial t} + k_x \left(\frac{\partial h}{\partial x} \right)^2 + k_y \left(\frac{\partial h}{\partial y} \right)^2 + k_z \left(\frac{\partial h}{\partial z} \right)^2 - k_z \frac{\partial h}{\partial z} \left(1 - \frac{v_0}{k_z} \right) - v_0 = 0 \quad (1.4.50)$$

Fig. 1.6. Parallel stratified layers with different values of the coefficient of permeability.



The first equation is independently applicable to the description of pressure flow. It is simplified by the substitution

$$\bar{x} = \frac{x}{\sqrt{(k_x/k_0)}}, \quad \bar{y} = \frac{y}{\sqrt{(k_y/k_0)}}, \quad \bar{z} = \frac{z}{\sqrt{(k_z/k_0)}} \quad (1.4.51)$$

where k_0 is an arbitrary constant with the dimension of a velocity.

Substituting in eqn. (1.4.49) we obtain the Laplace equation in the new coordinates $\bar{x}, \bar{y}, \bar{z}$. This equation formally describes the flow in a homogeneous medium

$$\frac{\partial^2 h}{\partial \bar{x}^2} + \frac{\partial^2 h}{\partial \bar{y}^2} + \frac{\partial^2 h}{\partial \bar{z}^2} = 0. \quad (1.4.52)$$

For the z axis oriented vertically, it is frequently found in fluvial deposits or artificial fills that

$$k_x = k_y = k_{\max}, \quad k_z = k_{\min};$$

setting $k_0 = k_z$ we have

$$\bar{x} = x, \quad \bar{y} = y, \quad \bar{z} = \frac{z}{\sqrt{(k_{\min}/k_{\max})}} = z \sqrt{\Lambda} \quad (1.4.53)$$

where Λ is the coefficient of anisotropy defined as the ratio of the coefficient of permeability in the principal directions of seepage (directions x, y) to that in the secondary direction (z).

As the transformation functions indicate, operations in the $(\bar{x}, \bar{y}, \bar{z})$ system are carried out for an elevated image of the original region x, y, z . The quantity of seepage is calculated from the distribution of the values of h in the transformed image using the mean value of the coefficient of permeability

$$k = \sqrt{(k_{\max} k_{\min})} \quad (1.4.54)$$

Consider now a flow with a free surface. For the z axis oriented vertically, eqns. (1.4.49) and simultaneously (1.4.50) are applicable to this flow. The two equations are simplified by means of the substitutions (1.4.53) and the substitution

$$\bar{t} = t \sqrt{A}, \quad \bar{h} = h \sqrt{A} \quad (1.4.55)$$

where $\bar{h}$ — elevated value of the piezometric head,
 $\bar{t}$ — transformed time.

The substitutions lead to the following equations:

$$\frac{\partial^2 \bar{h}}{\partial \bar{x}^2} + \frac{\partial^2 \bar{h}}{\partial \bar{y}^2} + \frac{\partial^2 \bar{h}}{\partial \bar{z}^2} = 0 \quad (1.4.56)$$

$$-\frac{\mu}{k_{\min}} \frac{\partial \bar{h}}{\partial \bar{t}} + \left(\frac{\partial \bar{h}}{\partial \bar{x}} \right)^2 + \left(\frac{\partial \bar{h}}{\partial \bar{y}} \right)^2 + \left(\frac{\partial \bar{h}}{\partial \bar{z}} \right)^2 - \frac{\partial \bar{h}}{\partial \bar{z}} + \frac{\partial \bar{h}}{\partial \bar{z}} \frac{v_0}{k_{\min}} - \frac{v_0}{k_{\min}} = 0 \quad (1.4.57)$$

Eqns. (1.4.56) and (1.4.57) describe the flow in a homogeneous region with a coefficient of permeability $k_{\min}$. In the transformed region we find the distribution of the values of $\bar{h}(\bar{x}, \bar{y}, \bar{z}, \bar{t})$ and calculate the quantity of seepage using the coefficient of permeability $k = k_{\min}$.

The two methods of transformation outlined above make it possible to carry out calculations with fairly simple equations. The actual pattern of flow is obtained by inverse transformation into the original region where, of course, the streamlines will no longer be at right angles to the equipotential surfaces.

1.4.6. Hydraulic theory of groundwater flow

In the case where a horizontal permeable layer in which the flow occurs is underlain by an impervious subsoil, the appropriate calculations can follow from equations which are less elaborate than those derived in Section 1.4.2.

Substituting eqn. (1.4.48) in eqn. (1.4.8) we obtain

$$\frac{\partial}{\partial x} \left(k_x \frac{\partial h}{\partial x} \right) + \frac{\partial}{\partial y} \left(k_y \frac{\partial h}{\partial y} \right) + \frac{\partial}{\partial z} \left(k_z \frac{\partial h}{\partial z} \right) = 0 \quad (1.4.58)$$

Selecting a coordinate system with the z axis vertically upward and the origin of coordinates in the reference plane for the determination of the piezometric heads, we have for the points of the free surface

$$z = h = f(x, y) \quad (1.4.59)$$

Assume that the coefficient k_z is infinitely large compared with k_x and k_y , i.e. $k_z \rightarrow \infty$.

In groundwater flow, an infinite velocity can theoretically appear in only very few places (so-called singular points — cf. Section 3.1.3). Thus, if eqn. (1.4.48) is to apply to the whole region of seepage, the expression $\partial h / \partial z$ in its third term must tend to zero, and hence

$$h(z) \rightarrow \text{const.} \quad (1.4.60)$$

The piezometric head is practically independent of the coordinate z , and the lines of $h = \text{const.}$ and hence also the lines of $\varphi = \text{const.}$ are vertical. Consequently, the soil offers no resistance to the flow in the z direction.

A constant value of the piezometric head, and hence also of the potential in the z direction, is the main assumption from which stems the so-called hydraulic theory of groundwater flow (the Dupuit theorem). The theory finds application in cases of small inclination of the free surface, or — as in the case being considered — of small piezometric gradient. In such cases the streamlines can be regarded as nearly horizontal lines.

Assume further that there exists a moving free surface of groundwater. For the vertical velocity component,

$$\frac{\partial v_z}{\partial z} \neq 0$$

Let the velocity in the z direction vary continuously. Then

$$\int_{z_0}^{z_1} \frac{\partial}{\partial z} \left(k_z \frac{\partial h}{\partial z} \right) dz = - \int_{z_0}^{z_1} \frac{\partial v_z}{\partial z} dz = -(v_{z_1} - v_{z_0})$$

where z_0 and z_1 are the coordinates of two points on the same vertical.

Hence eqn. (1.4.58) may be integrated with respect to z

$$\int_{z_0}^{z_1} \frac{\partial}{\partial z} \left(k_z \frac{\partial h}{\partial z} \right) dz = - \int_{z_0}^{z_1} \left[\frac{\partial}{\partial x} \left(k_x \frac{\partial h}{\partial x} \right) + \frac{\partial}{\partial y} \left(k_y \frac{\partial h}{\partial y} \right) \right] dz$$

and the integration yields

$$-(v_{z_1} - v_{z_0}) = - \left\{ \frac{\partial}{\partial x} \left[k_x (z_1 - z_0) \frac{\partial h}{\partial x} \right] + \frac{\partial}{\partial y} \left[k_y (z_1 - z_0) \frac{\partial h}{\partial y} \right] \right\} \quad (1.4.61)$$

Assume now that the point $z = z_1$ lies on the free surface which is in motion, and the point $z = z_0$ on the surface of the impervious subsoil. Writing

$$H = z_1 - z_0 \quad (1.4.62)$$

$$v_0 = v_{z_0} \quad (1.4.63)$$

$$q_x = -(z_1 - z_0) k_x \frac{\partial h}{\partial x} = -k_x H \frac{\partial h}{\partial x}$$

$$q_y = -(z_1 - z_0) k_y \frac{\partial h}{\partial y} = -k_y H \frac{\partial h}{\partial y} \quad (1.4.64)$$

we note that $H = H(x, y, t)$ is the height of the free surface above the impervious subsoil, $v_x = -k_x(\partial h/\partial x)$ and $v_y = -k_y(\partial h/\partial y)$ are the velocity components in the x, y directions, and q_x and q_y are the components of the specific quantity of seepage (defined per unit width of flow) in the same directions.

The velocity v_0 expresses the effect of an external discharge. Because of the assumption that $k_z \rightarrow \infty$, this discharge can take place at any point of the vertical line being considered (e.g. directly into the free surface — infiltration, evaporation).

The velocity v_{z_1} at a point of the free surface is given by the relation

$$v_{z_1} = \mu \frac{\partial h}{\partial t} \quad (1.4.65)$$

where t is the time and μ the active porosity.

Making use of the above, we can rearrange eqn. (1.4.61) in the form

$$\frac{\partial h}{\partial t} = \frac{1}{\mu} \left[\frac{\partial}{\partial x} \left(k_x H \frac{\partial h}{\partial x} \right) + \frac{\partial}{\partial y} \left(k_y H \frac{\partial h}{\partial y} \right) + v_0 \right] \quad (1.4.66)$$

or

$$\frac{\partial h}{\partial t} = \frac{1}{\mu} \left(-\frac{\partial q_x}{\partial x} - \frac{\partial q_y}{\partial y} + v_0 \right) \quad (1.4.67)$$

Eqn. (1.4.66) (or eqn. (1.4.67)) is the equation of unsteady seepage flow with a free surface in a strongly anisotropic porous medium.

Frequently, we can set $k_x = k_y = k$. Then

$$\frac{\partial h}{\partial t} = \frac{k}{\mu} \left[\frac{\partial}{\partial x} \left(H \frac{\partial h}{\partial x} \right) + \frac{\partial}{\partial y} \left(H \frac{\partial h}{\partial y} \right) + \frac{v_0}{k} \right] \quad (1.4.68)$$

In the literature, eqn. (1.4.68) is known as the Boussinesq equation.

Sometimes it is expedient to use cylindrical coordinates. In the derivation we shall start from the equation of continuity (1.4.12) which — for $k_r \neq k_\theta \neq k_z$ —

takes the form

$$\frac{1}{r} \frac{\partial}{\partial r} \left(k_r r \frac{\partial h}{\partial r} \right) + \frac{1}{r^2} \frac{\partial}{\partial \vartheta} \left(k_\vartheta \frac{\partial h}{\partial \vartheta} \right) + \frac{\partial}{\partial z} \left(k_z \frac{\partial h}{\partial z} \right) = 0 \quad (1.4.69)$$

Assuming that $k_r = k_\vartheta = k$, a similar procedure leads to an equation analogous to eqn. (1.4.68), viz.

$$\frac{\partial h}{\partial t} = \frac{k}{\mu} \left[\frac{1}{r} \frac{\partial}{\partial r} \left(H r \frac{\partial h}{\partial r} \right) + \frac{1}{r^2} \frac{\partial}{\partial \vartheta} \left(H \frac{\partial h}{\partial \vartheta} \right) + \frac{v_0}{k} \right] \quad (1.4.70)$$

The equation of steady flow with a free surface is:
in rectangular coordinates

$$\frac{\partial}{\partial x} \left(H \frac{\partial h}{\partial x} \right) + \frac{\partial}{\partial y} \left(H \frac{\partial h}{\partial y} \right) = 0 \quad (1.4.71)$$

in cylindrical coordinates

$$\frac{1}{r} \frac{\partial}{\partial r} \left(H r \frac{\partial h}{\partial r} \right) + \frac{1}{r^2} \frac{\partial}{\partial \vartheta} \left(H \frac{\partial h}{\partial \vartheta} \right) + \frac{v_0}{k} = 0 \quad (1.4.72)$$

If the impervious subsoil of the flow region can be taken to be horizontal, eqns. (1.4.66) to (1.4.72) may be further rearranged by making the surface of the impervious subsoil coincide with the plane of the x, y axes. The height of the free surface above the impervious subsoil, H , may then be equal to the piezometric head h ,

$$H = h$$

On the strength of this, eqn. (1.4.71) (in rectangular coordinates) becomes

$$\begin{aligned} \frac{\partial}{\partial x} \left(h \frac{\partial h}{\partial x} \right) + \frac{\partial}{\partial y} \left(h \frac{\partial h}{\partial y} \right) + \frac{v_0}{k} &= \\ = \frac{\partial^2}{\partial x^2} \left(\frac{h^2}{2} \right) + \frac{\partial^2}{\partial y^2} \left(\frac{h^2}{2} \right) + \frac{v_0}{k} &= 0 \end{aligned} \quad (1.4.73)$$

and eqn. (1.4.72) (in cylindrical coordinates) changes to

$$\frac{1}{r} \frac{\partial}{\partial r} \left(r h \frac{\partial h}{\partial r} \right) + \frac{1}{r^2} \frac{\partial}{\partial \vartheta} \left(h \frac{\partial h}{\partial \vartheta} \right) + \frac{v_0}{k} \quad (1.4.74)$$

A case of importance from the point of view of certain problems of seepage flow, is that arising in the absence of infiltration or evaporation. Then $v_0 = 0$,

$$\frac{\partial^2 h^2}{\partial x^2} + \frac{\partial^2 h^2}{\partial y^2} = 0 \quad (1.4.75)$$

Referring to eqn. (1.4.64) we find that

$$\begin{aligned} q_x &= -kh \frac{\partial h}{\partial x} = -k \frac{\partial}{\partial x} \left(\frac{h^2}{2} \right) \\ q_y &= -kh \frac{\partial h}{\partial y} = -k \frac{\partial}{\partial y} \left(\frac{h^2}{2} \right) \end{aligned} \quad (1.4.76)$$

and after substituting $v_0 = 0$ in eqn. (1.4.73), we obtain

$$\frac{\partial q_x}{\partial y} + \frac{\partial q_y}{\partial x} = 0 \quad (1.4.77)$$

Eqn. (1.4.75) is the Laplace equation for the function $h^2 = f(x, y)$. The solution of this equation for given boundary conditions yields the values of $h = h(x, y)$. Its application will be illustrated in the section dealing with plane seepage flow.

Eqn. (1.4.77) is the equation of continuity of steady seepage flow in which the components of the specific quantity of seepage q defined by eqn. (1.4.76) appear.

1.4.7. Girinsky potential

The components of the specific quantity of seepage q appearing in the equation of continuity (1.4.77) are defined by the function $G(x, y)$ whose derivatives in the x, y directions give uniquely the components of the quantity of seepage q (regarded as a vector) in those directions:

$$q_x = \frac{\partial G}{\partial x}, \quad q_y = \frac{\partial G}{\partial y} \quad (1.4.78)$$

We shall call the function $G(x, y)$ the Girinsky potential after N. K. Girinsky who was the first to use it in the solution of certain groundwater problems [12]. On substitution of eqn. (1.4.78) the equation of continuity (1.4.77) becomes

$$\frac{\partial^2 G}{\partial x^2} + \frac{\partial^2 G}{\partial y^2} = 0 \quad (1.4.79)$$

Assume that in a homogeneous soil, there occurs a flow with a free surface, a point of which has an ordinate $z = h$ on the vertical line considered, and that the permeable medium is bounded below by the surface of an impervious subsoil. Define the Girinsky potential as

$$G(x, y) = -k \int_0^h z \, dz = -k \frac{h^2}{2} \quad (1.4.80)$$

For $h = H$, $k_x = k_y = k$, differentiating with respect to x and y leads to eqn. (1.4.64); hence the potential G satisfies eqn. (1.4.78).

The Girinsky potential is an integral of all values of the potential at all possible positions of the free surface on the vertical line considered ($0 \leq z \leq h$). Geometrically, the Girinsky potential may be thought of as the sum of two areas: the rectangle (Fig. 1.7a) marked with the $-$ sign, and the triangle marked with the $+$ sign. The resulting area shown hatched in Fig. 1.7b, lies to the left of the vertical line considered. We assign to it the $-$ sign.

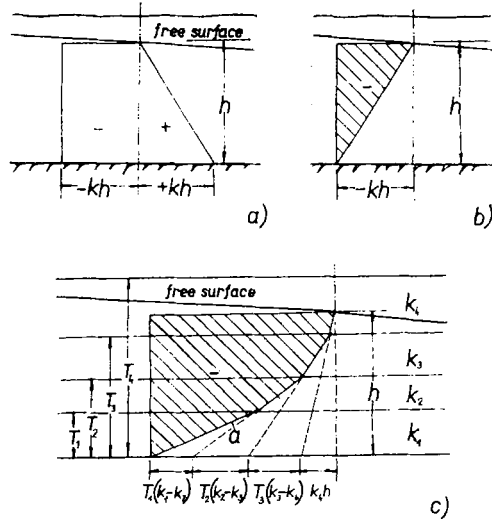


Fig. 1.7. Graphical representation of the Girinsky potential.

Consider the case of several independent layers in a saturated stratum, whose coefficients of permeability are $k_1, k_2, \dots, k_n, k_{n+1}$. Assume that on a given vertical line a point lying on the free surface is in the $(n + 1)$ th layer so that the n -th layer is completely under water.

The value of the Girinsky potential is represented by the hatched area shown in Fig. 1.7c for $n = 3$. The construction is clear from the figure. The segments $T_i(k_i - k_{i+1})$ are plotted to the left or to the right, depending on whether the expression is positive or negative.

The physical nature and geometry of the problem make possible another formulation of the Girinsky potential in a modified form

$$G(x, y) = \int_0^h k(z - h) dz \quad (1.4.81)$$

In the configuration given by n fully saturated layers assumed to be homogeneous

$$G(x, y) = -k_{n+1} \left[\frac{h^2}{2} - T_n \left(h - \frac{T_n}{2} \right) \right] - \sum_{i=1}^n k_i \left[T_i \left(h - \frac{T_i}{2} \right) - T_{i-1} \left(h - \frac{T_{i-1}}{2} \right) \right] \quad (1.4.82)$$

After a simple rearrangement

$$G(x, y) = -h \left[k_{n+1} \frac{h}{2} + \sum_{i=1}^n T_i(k_i - k_{i+1}) \right] + \frac{1}{2} \sum_{i=1}^n T_i^2(k_i - k_{i+1}) \quad (1.4.83)$$

Further

$$\begin{aligned} q_x &= \frac{\partial G}{\partial x} = - \left[k_{n+1} \frac{\partial}{\partial x} \left(\frac{h^2}{2} \right) + \frac{\partial h}{\partial x} \sum_{i=1}^n T_i(k_i - k_{i+1}) \right] \\ q_y &= \frac{\partial G}{\partial y} = - \left[k_{n+1} \frac{\partial}{\partial y} \left(\frac{h^2}{2} \right) + \frac{\partial h}{\partial y} \sum_{i=1}^n T_i(k_i - k_{i+1}) \right] \end{aligned} \quad (1.4.84)$$

The validity of the result can readily be checked by expressing q_x or q_y with the help of eqn. (1.4.64), taking account of the stratification of the medium.

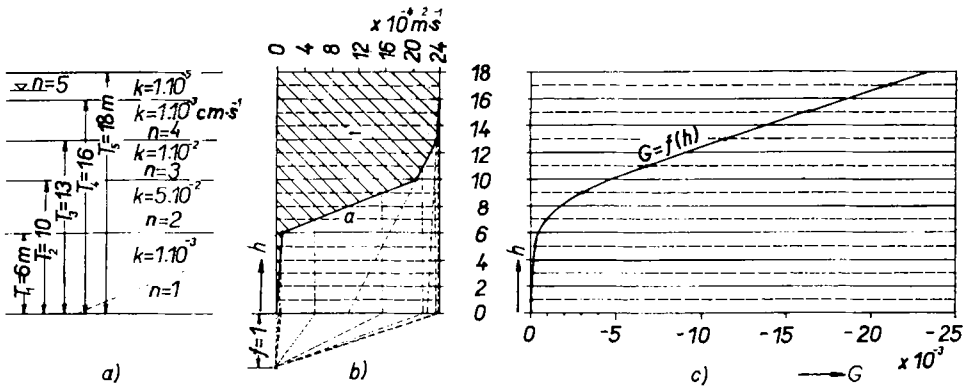


Fig. 1.8. Example of the graphical construction of the function $G = f(h)$.

In the solution of practical problems we need to know how the value of the Girinsky potential depends on the height of the free surface above the reference plane. We can either calculate it point by point or depict it graphically. An example is indicated in Fig. 1.8c. The graph represents the function $G = f(h)$, i.e. we can easily read from it the value of G corresponding to a given value of h , and vice versa.

1.5. Non-linear laws of groundwater flow in saturated soils

As the results of numerous experiments published in the literature suggest, Darcy's linear law of flow applies within certain limits. Outside those limits, the law of motion is non-linear.

1.5.1. Post-linear regime of flow

The capillary model can be imagined as a bundle of tubes with very rough walls. The laminar regime of flow can be maintained to a certain limit. Deviations are observed as the mean profile velocity is being increased: the effect of inertia forces becomes stronger and the effect of turbulence starts to play a major role. In very rough tubes the transition to turbulent flow is not sudden. Changes in cross-sectional area create conditions conducive to rhythmical acceleration and deceleration of particle motion. In the corners of widening passages the liquid is in turbulent agitation. At the entrance to narrower sections, the flow is seen to undergo stabilization. In emerging from the turbulent regions, the flow is enabled to absorb particles which it tears off the walls and carries towards the axis of the piping. All these phenomena manifest themselves in the velocity distribution across pipe sections: in narrowing sections the velocity profile is more "rounded" than in wide passages.

In soils, the deviations from the linear law are of a similar character. They are, therefore, estimated with the help of formulae constructed along lines similar to those in piping hydraulics

$$J = \lambda \frac{1}{d} \frac{v^2}{2g} \quad (1.5.1)$$

where

- J — hydraulic gradient,
- d — grain diameter,
- v — fictitious velocity of flow,
- g — acceleration of gravity.

The coefficient of friction λ generally has a variable value. According to the view of most authors, a flow may be described by the linear law in the region where $Re < 5$. The effect of inertia forces comes into play at $5 \leq Re \leq 20$ to 50. Turbulence begins to predominate at 20 to $50 \leq Re \leq 200$. For $Re > 200$, the effect of turbulence is already decisive. For $Re > 2000$ the law of motion no longer depends on Re (the coefficient of friction has a nearly constant value).

The value of λ in the linear flow regime is determined by an analysis of the co-operating factors using, for example, the Kármán-Kozeny equation. By eqns. (1.3.24) and (1.5.1) we have for identical spherical grains

$$\frac{v^2}{2gd} = \frac{kv}{2gd} J = \frac{gm^{3/2}\kappa^2vd^2}{2 \cdot 72vgd} J = \frac{m^{3/2}\kappa^2}{144} \frac{vd}{v} J \quad (1.5.2)$$

The Reynolds number is defined by

$$Re = \frac{vd}{v} \quad (1.5.3)$$

Hence

$$\lambda = \frac{144}{m^{3/2} \kappa^2 Re} \quad (1.5.4)$$

For a general soil with an irregular size distribution and grains of various shapes, the coefficient of friction

$$\lambda = \frac{C}{Re} \quad (1.5.5)$$

where C is a constant.

The linear law of flow assumes the coefficient of friction to be inversely proportional to the value of Re . If this inverse proportionality is absent, we speak of a post-linear regime. The term embraces all the phenomena which begin to predominate as soon as water flows with a velocity higher than that corresponding to the linear law. This does not mean, however, that the flow might not be laminar.

Worthy of special notice is the case of $\lambda = \text{const.}$ in the postlinear regime, i.e. the case whose analogy can be found in the effect of turbulence in a rough pipe. The approach to this problem can be based on the experiments of Nikuradse [30] from which we can deduce — with the help of the concept of hydraulic radius — a formula applicable to randomly arranged particles.

The question is how to determine the coefficient of tortuosity. Next, as far as the tendency to zig-zag flow is concerned, due consideration must be given to the mean value of velocity fluctuations. We propose that the coefficient of tortuosity ε be expressed by the relation

$$\varepsilon = m^{-5/4} \sqrt{\delta} \quad (1.5.6)$$

where δ is the shape factor.

The value of ε^2 covers the effect of tortuosity as well as the effect of a lack of agreement between the fictitious straight direction and the actual direction of flow. The factor δ characterizes the grain shape. We do not intend it to convey the idea that the shape of the grains has a direct effect on the character of the turbulence. We propose to use it only to express the fact that turbulent regions are apt to occur more frequently in soils with angular grains. This is taken care of by substituting $\delta = 2$ for angular, and $\delta = 1$ for spherical grains in eqn. (1.5.6).

The resulting relation takes the form

$$\lambda_t = \frac{\delta}{8m^{5/2} \log^2 \left(\frac{29.6m}{Pd} \right)} \quad (1.5.7)$$

where

- P — specific surface of the grains (surface of the particles in unit volume of soil),
- d — grain diameter,
- m — porosity,
- λ_t — coefficient of friction under fully developed turbulence — a constant.

For a set of identical spheres arranged at random, P is defined by eqn. (1.3.22) so that

$$\lambda_t = \frac{1}{8m^{5/2} \log^2 (4.933\kappa)} \quad (1.5.8)$$

where κ is the porosity number.

The fact that no grain size figures in eqn. (1.5.8) demonstrates the universal character of turbulent flow. Naturally, the value of d will appear in the relation which defines the coefficient of turbulent permeability, k_t :

$$k_t = \sqrt{\frac{2gd}{\lambda_t}} = 4 \sqrt{\frac{gd}{\delta}} m^{5/4} \log \left(\frac{29.6m}{Pd} \right) \quad (1.5.9)$$

For a set of identical spheres this becomes

$$k_t = 4 \sqrt{(gd)} m^{5/4} \log (4.933\kappa). \quad (1.5.10)$$

Experimental assessment of k_t is very difficult because fully developed turbulence occurs at high Re . The respective test devices must, therefore, be operated at comparatively high gradients.

The value of k_t is defined as the velocity which could be attained at the gradient $J = 1$. In reality, however, turbulence does not arise at low gradients. We are devising an imaginary situation which conceals the fact that a general and concurrent action of turbulence is theoretically assumed for the post-linear regime. We thus theoretically also replace the effect of inertia forces. At high Re our abstract ideas agree with reality. As stated earlier, a mathematical description of the flow stems from the assumption that the actual space filled with soil and water, is replaced by a space filled with liquid. If we admit the existence of turbulence, we must also admit the existence of the velocity component normal to the particle's path. In explaining the consequences of this, we shall refer to Fig. 1.9a showing a scheme of a two-dimensional flow.

In the x, z plane, consider a curvilinear isoline $h = \text{const.}$ and a point thereon. Project the velocity vector $\mathbf{v}$ onto the tangent to the isoline. The projection has two components, v_x, v_z . Similarly, decompose an element of the line $h = \text{const.}$, whose length is dn , into the components dx, dz . Denote by τ the angle made by the velocity vector $\mathbf{v}$ with the tangent to the line $h = \text{const.}$ The angle made by the tangent with the x axis is τ_1 , that with the z axis, τ_2 , and we can thus write

$$\cos \tau_1 = \frac{dx}{dn}, \quad \cos \tau_2 = \frac{dz}{dn} \quad (1.5.11)$$

$$v \cos \tau = v_x \cos \tau_1 + v_z \cos \tau_2 = v_x \frac{dx}{dn} + v_z \frac{dz}{dn} \quad (1.5.12)$$

Assume that the non-linear law of flow applies in the form

$$v_x = -k_\beta \left(\frac{\partial h}{\partial x} \right)^{1/\beta}, \quad v_z = -k_\beta \left(\frac{\partial h}{\partial y} \right)^{1/\beta} \quad (1.5.13)$$

where

β — a number,

k_β — coefficient of permeability whose value depends on β and hence also on the gradient, or possibly, on the velocity

Substituting from eqn. (1.5.13) in eqn. (1.5.12) we find

$$-\frac{v}{k_\beta} dn \cos \tau = \left(\frac{\partial h}{\partial x} \right)^{1/\beta} dx + \left(\frac{\partial h}{\partial z} \right)^{1/\beta} dz \quad (1.5.14)$$

In the special case of $\beta = 1$, the right-hand side of eqn. (1.5.14) will contain the total differential dh whose value along the line $h = \text{const.}$ is zero. In this special case, eqn. (1.5.14) will be satisfied for $v \neq 0$, provided that $\tau = \pi/2$. The vector $\mathbf{v}$ is normal to the line $h = \text{const.}$ only in the case when the linear law of flow with $\beta = 1$, $k_\beta = k$ applies. Otherwise, $\tau \neq \pi/2$.

In the post-linear regime, the directions of the velocity vector are not collinear with the normals to the line $h = \text{const.}$

1.5.2. Pre-linear regime of flow

The physico-chemical bonds acting on film water are strong in the vicinity of the grain surface. Their effect grows weaker in the direction of the interior of the pores. The strength of the bonds is affected by static water pressure as well as by hydrodynamic action. Neglecting the former we find that the effective porosity is a quantity which depends on the hydraulic gradient and is essentially a non-linear function.

Changes in effective porosity play a major role in fine-grained soils which at low gradients display deviations from Darcy's law to the so-called pre-linear law.

A typical example of the $v = f(J)$ dependence is shown in Fig. 1.9b. Curve 1 corresponds to sandy loam. The pre-linear law manifests itself at low gradients. Curve 2 illustrates the situation in medium-heavy loams. In this case, the $v = f(J)$ dependence tends tangentially to the axis of the abscissae. Curve 3 characterizes heavy loams. At gradients lower than J_0 , water was found not to flow at all. Adopting Puzyrevski's term, we call J_0 the initial gradient.

As demonstrated by numerous tests made by various investigators, the value of J_0 depends on the quality of the grain material, the shape and size of the pores, the chemical composition of the water, temperature and pressure. The strength of the physico-chemical bonds also depends on the period of time during which water

flows through the medium. Thus, for example, curve 3 refers to the state ascertained during the initial stages of the test; the dash-line portion represents the changes recorded after several weeks. The examples shown in Fig. 1.9b are intended to serve as initial information. It is quite possible that — due to the diversity of actual conditions — J_0 will tend to zero in clays, and have a definite value in very fine sands.*)

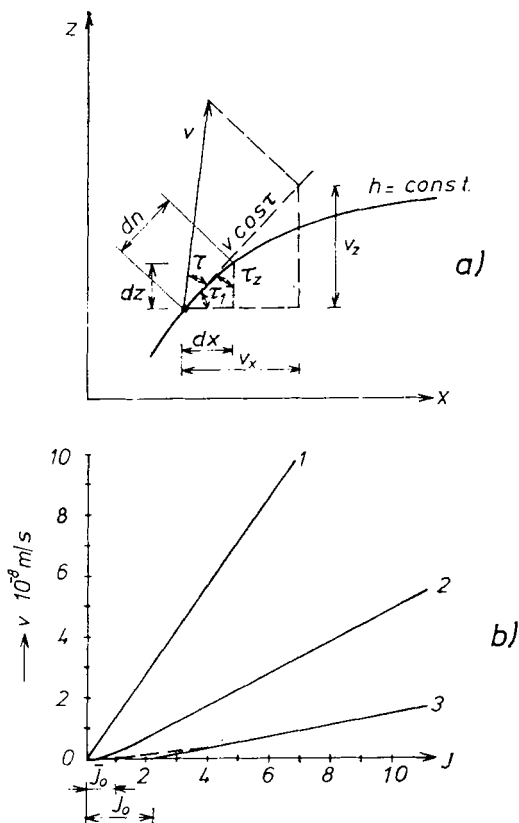


Fig. 1.9. Schemes of flow in post-linear and pre-linear regimes.

Attempts to define mathematically the pre-linear law of flow resulted in several different formulations. Thus, e.g. Swartzendruber [39] presents the relation

$$v = k[J - J_0(1 - e^{J/J_0})] \quad (1.5.15)$$

*) Sometimes circumstances of apparently secondary importance, such as the effect of plant roots in the upper sections of the sediments of valley meadows, are apt to become a predominant factor under natural conditions. Investigations reveal that the initial gradient is practically non-existent and that the value of the coefficient of permeability is virtually independent of the soil itself. Seams of more permeable materials sometimes present in natural soils, lead water more readily along so-called preferred paths which have a substantial influence on the over-all pattern of flow.

It is evident from the structure of eqn. (1.5.15) that its graphical representation is the curve 2 in Fig. 1.9b. Kutílek [23] constructed a dependence which also contains an exponential function. Its graphical representation resembles the curve 2 except that the tangent to Kutílek's curve has a non-zero slope at the origin.

Elaborate theoretical justifications of the pre-linear law are based on the fact that the action of physico-chemical bonds in the layer of film water manifests itself in a change in the viscosity of the water, compared with the viscosity in the region occupied by gravitational water. This effect is interpreted differently by different authors, all of whom start out from simplified notions, including the capillary models of soils.

According to the theory of non-Newtonian liquids, the mean profile velocity v_* in a capillary of diameter D is given by the relation

$$v_* = \frac{nD}{2(3n+1)} \left(\frac{Dg}{2K/\varrho} \right)^{1/n} J^{1/n}$$

where

K — measure of liquid consistency,

n — a number smaller than, or at most, equal to unity, characterizing the degree of non-Newtonian behaviour of the liquid,

ϱ — liquid density,

g — acceleration of gravity.

For a soil and a given liquid we could consider analogously the relation

$$v = k_n J^{1/n} \quad (1.5.16)$$

which is only acceptable in the pre-linear region. Eqn. (1.5.16) assumes the independence of the measure of consistency and of the parameter n on the magnitude of the shear stress. Taking these two effects into account would lead to the conclusion that the value k_n is not a constant.

Using the theory of visco-plastic materials, Bondarenko [4] suggests an equation of the type

$$v = kJ \left[1 - B \frac{\bar{J}_0}{J} + (B-1) \left(\frac{\bar{J}_0}{J} \right)^4 \right] \quad (1.5.17)$$

where B is a constant, and k is the classical Darcy's coefficient of permeability.

Eqn. (1.5.17) shows that water motion is possible once the gradient exceeds a certain basic threshold value $\bar{J}_0$. The graphical representation of eqn. (1.5.17) corresponds to the dash-line curve 3 in Fig. 1.9b, and is in agreement with notions stemming from Schwedoff's concept of a rheological process.

A connection with the linear law of seepage follows from an approximate expression for $J > 2\bar{J}_0$. Bondarenko's equation then changes to

$$v = k(J - B\bar{J}_0)$$

The expression obtained for $B = 1$ corresponds to Puzyrevski's formula at $\bar{J}_0 = J_0$. His conception agrees with Bingham's ideas about the course of a rheological process (solid line 3 in Fig. 1.9b).

It is clear, on the other hand, that for small values

$$\frac{J - J_0}{J_0} \ll 1 \quad (1.5.18)$$

eqn. (1.5.17) may be approximated by the equation

$$v = k_n(J - J_0)^2 \quad (1.5.19)$$

where k_n is the coefficient of permeability in the pre-linear regime.

For $J_0 = 0$ this becomes

$$v = k_n J^2 \quad (1.5.20)$$

Eqn. (1.5.20) corresponds to eqn. (1.5.16) for $n = 0.5$. The approximation assumes the value k_n to be a constant.

Eqn. (1.5.20) which corresponds to the theory of non-Newtonian liquids (lower part of the solid-line curve 2 in Fig. 1.9b) seems to be acceptable for long-term processes, within the range of low gradients.

An idea about the transition of the pre-linear to the linear regime may be gained from an abstract notion according to which the two kinds of flow exist concurrently. At high Re , the effect of pre-linearity is relatively so small as to be hardly discernible.

1.5.3. Combined regime of flow of water in soils

In Sections 1.5.1 and 1.5.2 we mentioned the hypothetical idea of several regimes of flow existing simultaneously. Mathematically, the idea may be expressed, e.g. by the so-called two-term formula

$$J = c_1 v + c_2 v^2 \quad (1.5.21)$$

where c_1, c_2 are constants.

Less clearly, the connection between the regimes is described by the relation

$$v = k_\alpha J^{1/\beta} \quad (1.5.22)$$

where

k_α – the (varying) value of the coefficient of permeability,

β – a number in the range $1/2 \leq \beta \leq 2$.

Eqn. (1.5.22) defines the state at an Re , on the value of which depend both k_α and β . Exceptions are the linear regime of flow where $k_\alpha = k$, $\beta = 1$, and the flow with fully developed turbulence, where $k_\alpha = k_t$, $\beta = 2$. In these two cases k_α has a constant value.

We consider it possible to evolve a comprehensive formulation embracing all the regimes of flow which exist concurrently. However, parallel superposition is feasible only in the case that all the phenomena to be superposed are expressed in a form based on sufficiently universal laws of hydraulics. One of these laws is the relation defined by the Bernoulli equation, essentially a mathematical expression of the law of conservation of energy. We shall, therefore, postulate the state of fully developed turbulence and re-formulate all the laws. We shall simultaneously respect the pre-linear and the linear regime characterized by the coefficients of permeability k_n, k . Since a combined regime is a fictitious process, we shall introduce — for the purpose of differentiation and quantitative appraisal — the new symbols k_p, k_t which describe the effect of pre-linearity and linearity in the combined regime.

In fully developed turbulence

$$v = k_t \sqrt{J}, \quad \frac{v^2}{2gd} = \frac{J}{\lambda_t}, \quad \lambda_t = \frac{2gd}{k_t^2} \quad (1.5.23)$$

In the linear regime

$$v = kJ, \quad \frac{v^2}{2gd} = \frac{v}{2gd} kJ = \frac{J}{\lambda_1}, \quad \lambda_1 = \frac{2gd}{vk}$$

or in other form

$$v = k_1 \sqrt{J}, \quad k_1 = (vk)^{1/2}, \quad \lambda_1 = \frac{2gd}{k_1^2} \quad (1.5.24)$$

In the pre-linear regime

$$v = k_n J^2, \quad J \sqrt{\frac{k_n}{v}} = 1, \quad \frac{v^2}{2gd} = \frac{v^2}{2gd} \sqrt{\frac{k_n}{v}} J = \frac{v^{3/2} k_n^{1/2}}{2gd} J, \quad \lambda_p = \frac{2gd}{v^{3/2} k_n^{1/2}}$$

or in other form

$$v = k_p \sqrt{J}, \quad k_p = v^{3/4} k_n^{1/4}, \quad \lambda_p = \frac{2gd}{k_p^2} \quad (1.5.25)$$

where

k_n, k — coefficients of permeability for the pre-linear and the linear regime, respectively,

$\lambda_p, \lambda_1, \lambda_t$ — coefficients of friction for the pre-linear and the linear regime and for fully developed turbulence, respectively,

k_p, k_1, k_t — fictitious values of the coefficients of permeability for the pre-linear and the linear regime, and the actual value of the coefficient of permeability for fully developed turbulence.

The combined behaviour of the three kinds of flow can be described by a single, representative, value of the coefficient of friction, λ_c , which depends on Re as well as on v :

$$\lambda_c = \lambda_1 + \lambda_t + \lambda_p \quad (1.5.26)$$

$$\lambda_c = \frac{2gd}{vk} \left[1 + \frac{vk}{k_t^2} + k(vk_n)^{-1/2} \right] \quad (1.5.27)$$

or

$$\lambda_c = \frac{2gd^2}{kvRe} \left\{ 1 + k \left[\frac{vRe}{dk_t^2} + \left(\frac{vk_n Re}{d} \right)^{-1/2} \right] \right\} \quad (1.5.28)$$

The fictitious coefficient of permeability, k_c , corresponding to the superposed regime, depends generally on v and hence also on Re :

$$k_c = \sqrt{\frac{2gd}{\lambda_c}} \quad (1.5.29)$$

Since simultaneously

$$v = k_c \sqrt{J} \quad (1.5.30)$$

we set $\beta = 2$, $k_s = k_c$ in eqn. (1.5.22).

As to eqn. (1.5.28): at high values of Re , the effect of the first term will predominate, while at low Re it will be that of the second term in the square brackets.

Superposition automatically defines the region of Re in which we may expect a definite, predominant, regime of flow. At the same time it indicates the character of continuous transition from one regime into another.

Consider, for example, a system of identical but randomly arranged spherical grains, and assume that linear and post-linear flow exist concurrently. In this particular case

$$\lambda_c = \frac{144}{m^{3/2} \kappa^2 Re} \left[1 + \frac{\kappa^2 Re}{1,152m \log^2 (4.933\kappa)} \right] \quad (1.5.31)$$

It can readily be shown that for $Re < 7$, the linear law applies with an accuracy better than 5%, while for $Re > 1800$, fully developed turbulence may be predicted with an accuracy better than 5%.

1.5.4. Mathematical description of flow in the combined regime through saturated soils

In Section 1.3.2 we derived an equation describing the magnitude of the resistance force R_v exerted by the soil on unit mass of liquid. The derivation — based on an analysis of the forces at work — omitted to consider the prescribed law of flow.

Hence

$$R_v = - \frac{\gamma}{m} J \quad (1.5.32)$$

in any regime of flow. Since in the concurrent regime of flow

$$J = \left(\frac{v}{k_c} \right)^2 \quad (1.5.33)$$

eqn. (1.5.32) becomes

$$R_v = - \frac{\gamma}{m} \left(\frac{v}{k_c} \right)^2 \quad (1.5.34)$$

By assuming the flow to take place in a cylinder, we have automatically ensured the collinearity of the velocity vector and the gradient vector. However, such collinearity does not generally happen because parallel flow in space is a special case. A more general mathematical description relies on the use of the projection of the gradient vector in the direction of the velocity vector, or conversely, on the projection of the velocity vector in the direction of the gradient vector.

By application of the second method, we can determine in space the total force $\bar{R}_v$ acting on the liquid inside the volume V

$$\bar{R}_v = - \frac{\gamma}{m} \int_v \left(\frac{v}{k_c} \right)^2 m \, dV = -\gamma \int_v \left(\frac{v}{k_c} \right)^2 \, dV \quad (1.5.35)$$

Proceeding as indicated in Section 1.4.1 we find that the equation of motion is of the form

$$\frac{1}{mg} \frac{dv}{dt} + \text{grad } h + \left(\frac{v}{k_c} \right)^2 = 0 \quad (1.5.36)$$

The introduction of nonlinear functions sometimes leads to further formal difficulties. It follows from the nature of these formulae (1.5.30), (1.5.36) that they respect a certain physical point of view. However their mathematical formulations frequently preclude their exact general use or further numerical operations.

The formulae automatically assume a positive value of J . Their applications in a three-dimensional space defined in a specified coordinate system call for the use of absolute values of J ; only after the operation the relevant sign must be put to the corresponding expressions (for example $\sqrt{J} = J|J|^{-1/2}$). This approach is not always necessary but is inevitable in some interesting cases with complicated boundary and initial conditions.

In the following text we shall discuss a directly defined but approximately described nonlinear flow.

For $dv/dt = 0$ we presuppose

$$\begin{aligned} v_x &= -k_{cx} \frac{\partial h / \partial x}{\sqrt{|\partial h / \partial x|}} \\ v_y &= -k_{cy} \frac{\partial h / \partial y}{\sqrt{|\partial h / \partial y|}} \\ v_z &= -k_{cz} \frac{\partial h / \partial z}{\sqrt{|\partial h / \partial z|}} \end{aligned} \quad (1.5.37)$$

where k_{cx} , k_{cy} , k_{cz} are the values of k_c in the x , y , z directions.

Substituting eqns. (1.5.37) in the equation of continuity (1.4.8) we obtain

$$\frac{\partial}{\partial x} \left(k_{cx} \frac{\partial h}{\partial x} \left| \frac{\partial h}{\partial x} \right|^{-1/2} \right) + \frac{\partial}{\partial y} \left(k_{cy} \frac{\partial h}{\partial y} \left| \frac{\partial h}{\partial y} \right|^{-1/2} \right) + \frac{\partial}{\partial z} \left(k_{cz} \frac{\partial h}{\partial z} \left| \frac{\partial h}{\partial z} \right|^{-1/2} \right) = 0 \quad (1.5.38)$$

Similarly, substituting eqns. (1.5.37) in eqn. (1.4.36) we have for the free surface

$$\begin{aligned} -\mu \frac{\partial h}{\partial t} + k_{cx} \frac{\partial h}{\partial x} \left| \frac{\partial h}{\partial x} \right|^{1/2} + k_{cy} \frac{\partial h}{\partial y} \left| \frac{\partial h}{\partial y} \right|^{1/2} + k_{cz} \frac{\partial h}{\partial z} \left| \frac{\partial h}{\partial z} \right|^{1/2} - \\ - k_{cz} \frac{\partial h}{\partial z} \left| \frac{\partial h}{\partial z} \right|^{-1/2} \left[1 + \left(\frac{v_0^2}{k_{cz}} \right) \right] - v_0 = 0 \end{aligned} \quad (1.5.39)$$

where v_0 is the discharge into or from the free surface. Its dimension is that of velocity because it can be expressed in meters of water column per unit time.

Eqns. (1.5.38) and (1.5.39) describe the flow in the concurrent regime under the conditions stated at the beginning. A practical solution is easier to obtain in cases when k_{cx} , k_{cy} , k_{cz} may be taken as constants applicable throughout the whole flow region. This means that the process is assumed to be one of fully developed turbulence in an anisotropically permeable medium.

Assume that $k_{cx} = k_{cy} = k_{\max}$, $k_{cz} = k_{\min}$

$$\frac{k_{\max}}{k_{\min}} = A^{3/4} \quad (1.5.40)$$

$$A = \left(\frac{k_{\max}}{k_{\min}} \right)^{4/3} \quad (1.5.41)$$

A is the coefficient of anisotropy in the region of turbulent flow.

Use the notation

$$\bar{x} = x, \quad \bar{y} = y, \quad \bar{z} = z \sqrt{A}, \quad \bar{h} = h \sqrt{A}, \quad \bar{t} = t \sqrt{A} \quad (1.5.42)$$

Substituting the above in eqn. (1.5.39) we obtain the equation

$$-\frac{\mu}{k_{\min}} \frac{\partial \bar{h}}{\partial \bar{t}} + \frac{\partial \bar{h}}{\partial \bar{x}} \left| \frac{\partial \bar{h}}{\partial \bar{x}} \right|^{1/2} + \frac{\partial \bar{h}}{\partial \bar{y}} \left| \frac{\partial \bar{h}}{\partial \bar{y}} \right|^{1/2} + \frac{\partial \bar{h}}{\partial \bar{z}} \left| \frac{\partial \bar{h}}{\partial \bar{z}} \right|^{1/2} -$$

$$-\frac{\partial \bar{h}}{\partial \bar{z}} \left| \frac{\partial \bar{h}}{\partial \bar{z}} \right|^{-1/2} \left[1 + \left(\frac{v_0}{k_{\min}} \right)^2 \right] - \frac{v_0}{k_{\min}} = 0 \quad (1.5.43)$$

for the free surface and the equation

$$\frac{\partial}{\partial \bar{x}} \left(\frac{\partial \bar{h}}{\partial \bar{x}} \left| \frac{\partial \bar{h}}{\partial \bar{x}} \right|^{-1/2} \right) + \frac{\partial}{\partial \bar{y}} \left(\frac{\partial \bar{h}}{\partial \bar{y}} \left| \frac{\partial \bar{h}}{\partial \bar{y}} \right|^{-1/2} \right) + \frac{\partial}{\partial \bar{z}} \left(\frac{\partial \bar{h}}{\partial \bar{z}} \left| \frac{\partial \bar{h}}{\partial \bar{z}} \right|^{-1/2} \right) = 0 \quad (1.5.44)$$

for the zone below the free surface. In either case the transformed region will be homogeneous.

The description of the plane flow of groundwater is based on the assumption that $k_{cx} \rightarrow \infty$ in the z -direction.

Consider the function $G_t(x, y)$ whose derivatives in the directions of the coordinate axes give the square of the specific quantity of flow in the directions of these axes:

$$q_x^2 = \frac{\partial G_t}{\partial x}, \quad q_x = \frac{\partial G_t}{\partial x} \left| \frac{\partial G_t}{\partial x} \right|^{-1/2}$$

$$q_y^2 = \frac{\partial G_t}{\partial y}, \quad q_y = \frac{\partial G_t}{\partial y} \left| \frac{\partial G_t}{\partial y} \right|^{-1/2} \quad (1.5.45)$$

For $k_c = k_{cx} = k_{cy} = \text{const.}$, the function $G_t(x, y)$ is defined as follows:

$$G_t = -2k_c \int_0^h Z \, dz \quad (1.5.46)$$

with

$$Z = -k_c \int_0^z z \, dz \quad (1.5.47)$$

The function $G_t(x, y)$ is essentially the integral of the Girinsky potential. In soils lying on the horizontal surface of a relatively impermeable subsoil

$$G_t(x, y) = k_c^2 \frac{h^3}{3} \quad (1.5.48)$$

$$q_x = \frac{\partial G_t}{\partial x} \left| \frac{\partial G_t}{\partial x} \right|^{-1/2} = k_c h \frac{\partial h}{\partial x} \left| \frac{\partial h}{\partial x} \right|^{-1/2} \quad (1.5.49)$$

$$q_y = k_c h \frac{\partial h}{\partial y} \left| \frac{\partial h}{\partial y} \right|^{-1/2} \quad (1.5.50)$$

To illustrate the practical application of the method, consider the case of steady-state flow to which applies the equation of continuity (1.4.7). Substituting eqns. (1.5.49) and (1.5.50) in eqn. (1.4.77) we obtain

$$\frac{\partial}{\partial x} \left(h \frac{\partial h}{\partial x} \left| \frac{\partial h}{\partial x} \right|^{-1/2} \right) + \frac{\partial}{\partial y} \left(h \frac{\partial h}{\partial y} \left| \frac{\partial h}{\partial y} \right|^{-1/2} \right) = 0 \quad (1.5.51)$$

Direct substitution of eqn. (1.5.45) leads to a formulation in terms of the function G_t , viz.

$$\frac{\partial}{\partial x} \left(\frac{\partial G_t}{\partial x} \left| \frac{\partial G_t}{\partial x} \right|^{-1/2} \right) + \frac{\partial}{\partial y} \left(\frac{\partial G_t}{\partial y} \left| \frac{\partial G_t}{\partial y} \right|^{-1/2} \right) = 0 \quad (1.5.52)$$

Just as in Section 1.4.7, we could set up equations for a stratum containing different soils. The problem would reduce to formal rearrangement and application of procedures whose principles we have already explained.

1.6. Flow of water in unsaturated soils

Unsaturated soils contain gases which are free to move in the interconnected system of pores. The content of gravitational water in pores partially filled with liquid is fairly small and consequently, the physico-chemical bonds assume major importance.

1.6.1. Suction pressure (hydrotension)

We shall make this general statement concerning the significance of physico-chemical bonds more precise by pointing out the various partial effects, e.g. the effect of osmotic phenomena. Osmosis plays an important part in many processes; controlled by an externally applied action, e.g. the effect of an electric field (electro-osmosis) it forms the basis for a variety of technological procedures of practical value.

The complexity and interaction of the various partial bonds manifest themselves in the resulting state. In unsaturated soils the effects of capillary and osmotic forces predominate, and so the literature often refers to the total suction pressure of water. This is a negative pressure which must be applied to the free surface in order to attain equilibrium between the free surface and the water in pores separated by a semi-permeable membrane. The unit of suction pressure is a bar or $\text{Nm}^{-2} = \text{Pa}$.

The basic (capillary) pressure is easier to identify in cases when the osmotic phenomena are not predominant (a small content of dissolved salts).

Fig. 1.10a shows a typical curve of moisture distribution along the height of a profile in the capillary zone, above the continuous free surface of groundwater. The moisture w as a part of the total volume of soil is shown on the horizontal axis, and

the height h_k measured from the free surface, on the vertical axis. The hatched section of the diagram represents the zone of capillary water which almost completely saturates the soil, while above this lies the zone of soil unsaturated with capillary water. The maximum height of rise is defined at the point where the moisture $w = w_0$ cor-

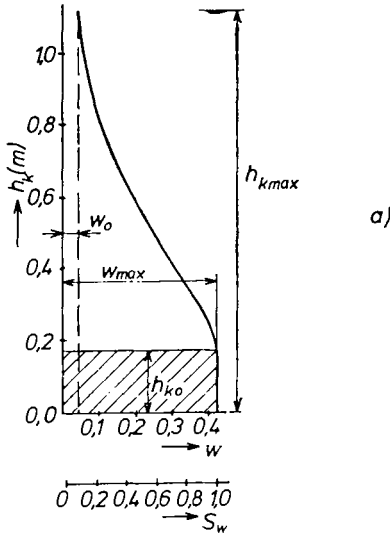
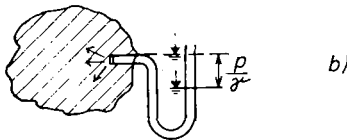


Fig. 1.10. Moisture distribution along the height of the capillary zone.



responds to the content of hygroscopically bonded water (the initial threshold moisture). The pressure difference at the point defined by the ordinate h_k is found to be

$$p = p_k - p_0 = \gamma(h_k - h_{kmax}) \quad (1.6.1)$$

where

p_0 — atmospheric pressure,

p_k — pressure at the point with the ordinate h_k ; in the case being considered, h_k has the meaning of geodetic altitude.

Assume that the moisture distribution shown in Fig. 1.10a developed gradually. At the very beginning there existed only the free surface of gravitational water, and immediately above it, only the initial moisture. As the subsequent development has shown, the possibility of water rising to the level of h_{kmax} was already present at the beginning of the process.

This possibility is characterized by the suction pressure p , defined in the case in question relative to the initial moisture. The capillary zone started to come into existence later on. At a certain instant there was noted at a certain point above the

free surface, a moisture w to which corresponded a certain instantaneous capillary elevation above that point. The possibility of further rise persisted, however. Simultaneously, the moisture at the point under observation also increased during the subsequent development.

At a given instant, the possibility of water rise is characterized by the suction pressure p corresponding to the instantaneous moisture.

The problem may be considered from another angle. We note that at a given instant there was water at the point with moisture w . Its motion was made possible by the action of a force. We regard the suction pressure as tension required to set water in motion at a given moisture.

The example discussed served to illustrate certain basic concepts. However, the suction pressure is not related to the capillary zone alone. Its significance is much broader, as will be seen from the following: Consider a block of soil with moisture w . Insert in the block a U -tube open at one end and provided with a semi-permeable element at the other (Fig. 1.10b). At the beginning, the tube was filled with water up to the dash-line level. Subsequently, the water in the straight arm of the tube dropped a height equal to p/γ . This height characterizes the suction pressure (provided that the water contained in the tube proper has no effect). Naturally, p/γ will vary in time depending on the gradual saturation of the soil, or possibly, on the instantaneous moisture of the soil near the semi-permeable element and its vicinity. To each value of moisture corresponds a certain suction pressure, and vice versa, to each suction pressure corresponds a certain value of moisture. In a saturated soil, the suction pressure is zero.

Knowledge of the dependence $p = f(w)$ or $h_k = f(w)$ may provide a clue to the theoretical solution of the problem. Of the many attempts at an analysis, undertaken in the past and going on at present, we mention the semi-empirical approach of Leverett who based his considerations on a simple capillary model. He started from the fact that the maximum height of capillary elevation in a system of capillaries intended to model the hydraulic effects in a soil, is defined by

$$h_{k\max} = \frac{4cP}{\gamma am} \quad (1.6.2)$$

where

P — surface of particles in unit volume of soil,

c — capillary constant,

a — multiplying constant (cf. Section 1.3.3),

m — porosity.

Calculating the values of P/am from the Kármán-Kozeny equation

$$\frac{P}{am} = \frac{1}{4\alpha} \sqrt{\frac{gm}{2kv}} \quad (1.6.3)$$

we obtain

$$h_{k\max} = \frac{c}{\alpha\gamma} \sqrt{\frac{gm}{2kv}} \quad (1.6.4)$$

where

α — coefficient of tortuosity,

k — Darcy's coefficient of permeability.

Leverett considered the above relation to be insufficient for an adequate description of natural conditions. He introduced the dimensionless number Le implied by eqn. (1.6.4), which enables us to characterize in general the pressure-moisture dependence:

$$Le = \left[\frac{p}{c} \sqrt{\frac{kv}{mg}} \right] \quad (1.6.5)$$

A series of experiments made with various materials with different degrees of saturation $S_w = w/m$ have shown the results to group about two curves of $Le = f(S_w)$. One pertains to experiments during which the material was saturated gradually, the other to experiments employing the drain effect (capillary hysteresis). Generally, the following relation can be applied in either case:

$$p_w = B \sqrt{\frac{mg}{kv}} f(S_w) \quad (1.6.6)$$

where

$f(S_w)$ — a universal, continuous, function,

p_w — suction pressure corresponding to moisture w ,

B — a constant [N/m].

This conclusion is very useful, for it enables us to derive from the function the derivative $\partial p_w / \partial w$. Assuming w to be continuous, we find that in a given plane along the curve s the following holds

$$\frac{\partial p_w}{\partial s} = \frac{\partial p_w}{\partial w} \frac{\partial w}{\partial s} \quad (1.6.7)$$

Consequently, if we know the distribution of moisture, we can employ the Leverett function to estimate the distribution of the pressure gradient along a particular curve.

As research studies suggest, the concept of suction pressure has a fairly broad validity, and its connection with moisture is apparent even in cases when allowance is made for quantitative changes in the water content near the surface of soil grains. Particles held to grains by weaker bonds are removed, e.g. in a high-speed centrifuge by the action of a large external force. The corresponding suction pressures are, of course, quite large, and the best way of representing the function $p = f(w)$ graphically, is to plot it on a semi-logarithmic chart. No doubt, it was these practical aspects

that led to the introduction of the unit (pF) to serve in the definition of the suction pressure. According to Schofield, (pF) is equal to the common logarithm of suction pressure expressed in cm of water column.

Since the methodology of (pF) determination is treated at length in textbooks on pedology, we shall not deal with it here but merely note that the graphical representation of the function $pF = f(w)$, i.e. the (pF) curve, makes it possible to determine the relation $p = f(w)$ over a range of moistures that is of practical interest.

To solve real problems, we must know the laws of motion in unsaturated media. Since the motion to be examined is generally that of water and gases, the theory starts from an analysis of the flow of a two-phase medium. Each phase is of a different physical nature; hence the analysis makes use of the concept of physical permeability [m^2] (Section 1.3.3). The total volume of pores is partially filled with water, and the permeability for water is less than the permeability of saturated soil. Pores are also partially filled with gases. The literature establishes the concept of relative phase permeability, which is defined as the ratio of the actual permeability of a phase to the permeability obtained in the case when the phase fills the soil pores completely. The relative phase permeabilities are functions of the degree of saturation. If the two phases are water and air, we may assume that the resistance to the flow of the gaseous phase will be comparatively small, and that it is sufficient to examine only the flow of water and use the common definition of the coefficient of permeability. This was the approach adopted by Averyanov [2] who based his considerations on the idea of a capillary model. He assumed that each capillary contained a concentric air tube which reduced the flow area for water moving in the annular space. He arrived at the conclusion that to a laminar flow of water, there applies an analogy of Darcy's law

$$v = k_w J \quad (1.6.8)$$

where

$$k_w = k \left(\frac{w - w_0}{m - w_0} \right)^n = k \left(\frac{S_w - S_{w_0}}{1 - S_{w_0}} \right)^n \quad (1.6.9)$$

v — velocity referred to unit area of soil section (fictitious velocity),

k_w — coefficient of permeability at a given moisture,

S_w — degree of saturation,

S_{w_0} — degree of saturation at a moisture corresponding to the volume of hygroscopically bonded water,

J — gradient,

k — coefficient of permeability of completely saturated soil when $w_{\max} = m$, $S_w = 1$.

For $S_{w_0} \ll 1$, it is approximately the case that

$$k_w = k(S_w - S_{w_0})^n \quad (1.6.10)$$

For the exponent of eqn. (1.6.9), Averyanov obtained the value $n = 4$. On the basis of experiments, other authors recommend the use of $3 \leq n \leq 4$.

What is of utmost importance is the fact that the model embraces the abstract idea which we encountered in the description of water motion in a saturated medium. We imagine that in an unsaturated medium the whole region is continuously filled with water. The effect of the variability of w is implicit in the value of k_w ; indeed, the coefficient of permeability k_w is a continuous function and in an unsaturated soil its value varies from one place to another. In a saturated soil, $w \rightarrow m$ so that the equality $k_w = k$ holds good everywhere. A saturated soil is just a special case of unsaturated soil.

1.6.2. Force potential. Laws of water motion

A water particle is acted upon by forces defined in space by a continuous force field. The accelerations in the directions of the coordinate axes can then be determined as the partial derivatives of a certain continuous function (force potential) in the directions of those axes. Work is performed as a particle is transported from a point of higher potential to a point of lower potential. Hence the force potential is defined as energy per unit mass of liquid.

In the gravitational field where the partial derivatives are considered only in the vertical direction z , the force potential is defined by the product

$$\varphi_g = -gZ \quad (1.6.11)$$

where

φ_g — force potential in the gravitational field (work required to displace unit mass through distance Z),

g — acceleration of gravity.

Hence

$$g = -\frac{\partial \varphi_g}{\partial z} \quad (1.6.12)$$

Assume that a pressure p acts at a point with ordinate z . The energy per unit mass of liquid is symbolized with the help of the relation

$$Z = h = \frac{p}{\gamma} + z \quad (1.6.13)$$

where

p — pressure,

γ — specific weight of liquid,

z — geodetic head measured from a certain reference plane

$$\varphi_g = -gh \quad (1.6.14)$$

Complying with eqn. (1.3.10) we can write the classical Darcy's law of flow as follows:

$$v = \frac{k}{g} \frac{\partial \varphi_g}{\partial s} \quad (1.6.15)$$

If the soil is assumed to contain gas, then according to eqn. (1.6.9) we can write analogously for the remaining water that

$$v = \frac{k_w}{g} \frac{\partial \varphi_g}{\partial s} \quad (1.6.16)$$

where k_w is the coefficient of permeability whose value also depends on moisture expressed as a part of the volume of soil.

The functional relation which exists between the coefficient k_w and the coefficient of permeability of completely saturated soil, k , is

$$k_w = k f_w(w) \quad (1.6.17)$$

and hence

$$v = \frac{k f_w(w)}{g} \frac{\partial \varphi_g}{\partial s} \quad (1.6.18)$$

The signs on the right-hand sides of eqns. (1.6.15), (1.6.16) and (1.6.17) are chosen with respect to the orientation of the direction of the applied force relative to the chosen direction. The choice is a matter of convention and has no essential significance.

To account for the numerous force fields of varying natures (capillary forces, osmotic forces, etc.) coupled in complicated ways, which are at work in unsaturated soils, the concept of the total potential of water in pores of an unsaturated medium was introduced. This potential is the sum of all the different potentials present in the soil, and the free surface of gravitational water is the reference plane with respect to which the sum is taken. We are dealing with a quantity of work which is required to transport a unit mass of water from the plane free surface to a given point at a given moisture and pressure. This transport is regarded as a reversible and isothermal process.

Although the capillary (basic) potential expresses mainly the effect of capillary phenomena, it also includes the effect of adsorption phenomena. The effect of capillarity predominates, especially at low moisture. This being so, we can usually manage with the simple definition of the capillary potential φ_k

$$\varphi_k = -gh_k \quad (1.6.19)$$

Several authors have investigated the relation between the value of the potential φ_k and the moisture w (expressed as a part of the volume of soil). They obtained functions of the form

$$\varphi_k = \varphi_{k0} f(w) \quad (1.6.20)$$

where φ_{k_0} is some constant value of the potential, e.g. the value corresponding to the moisture of soil containing only hygroscopically bonded water.

With the function (1.6.20) are associated the corresponding relations

$$h_k = h_{k_0} f(w) \quad p_w = p_{w_0} f(w) \quad (1.6.20a)$$

where

p_w — suction pressure,

h_k — piezometric head in terms of which one can describe the suction pressure.

The function $f(w)$ can be constructed on the basis of laboratory tests, for example, with the help of the (pF) curve. In another method use is made of a well-known and mathematically defined relationship. Thus, for example, Gardner [10] sets up the relation

$$\varphi_k = \frac{c_1}{w} + c_2 \quad (1.6.21)$$

where c_1, c_2 are constants.

Assume that $\varphi = \varphi_{km}$ for $w = w_0$, and $\varphi = \varphi_{k_0}$ for $w = w_{max} = w_m$. After calculating the constants we obtain

$$\varphi_k = (\varphi_{km} - \varphi_{k_0}) \frac{w_0}{w} \frac{w_m - w}{w_m - w_0} + \varphi_{k_0} \quad (1.6.22)$$

Since in a given space the function φ_k is continuous, there exists its derivative

$$\frac{\partial \varphi_k}{\partial w} = f'(w) \quad (1.6.23)$$

which defines the slope of the capillary potential-moisture dependence and indirectly characterizes the soil in question.

The assumed continuity of the capillary potential enables us to define the velocity of capillary motion, v_k . In accordance with eqn. (1.6.16) we can write

$$v_k = \frac{k_k}{g} \frac{\partial \varphi_k}{\partial s} \quad (1.6.24)$$

where

v_k — velocity of capillary motion relative to the whole cross-sectional area,

k_k — coefficient of capillary motion (with a meaning similar to that of the coefficient of permeability).

We have mentioned the possibility of expressing k_k as a function of the coefficient of permeability k and moisture. If we know the pressure-moisture dependence, we can easily establish a relation of the type

$$k_k = k f(p_w) = k f_k(w) \quad (1.6.25)$$

Ready-made equations also exist: such equations include the Gardner formula

$$k_k = k \frac{p_{ws}^\zeta}{p_w^\zeta + p_{w0}^\zeta} \quad (1.6.26)$$

where

p_{ws} — suction pressure corresponding to the value $k_k = k/2$,

ζ — a coefficient; for fine materials $\zeta = 1$, for sands $\zeta = 10$.

Equally applicable is the Averyanov equation (1.6.9). Since this equation has a fairly broad validity, we shall distinguish — for the sake of clarity — between k_w and k_k .

It was established in the course of development of the matter under discussion that the motion of water in an unsaturated medium is affected by the temperature T . Therefore, the capillary potential is a composite function and the equation for the velocity of capillary motion becomes

$$v_k = \frac{k_k}{g} \left(\frac{\partial \varphi_k}{\partial w} \frac{\partial w}{\partial s} + \frac{\partial \varphi_k}{\partial T} \frac{\partial T}{\partial s} \right) \quad (1.6.27)$$

Using the relation set up in the previous paragraph, we can rearrange eqn. (1.6.27) to obtain

$$k_k \frac{\partial \varphi_k}{\partial w} = k_k f'(w) = k f_k(w) f'(w) \quad (1.6.28)$$

Assuming further that $\partial \varphi_k / \partial T$ is also a function of moisture:

$$\frac{\partial \varphi_k}{\partial T} = f_T(w) \quad (1.6.29)$$

we then have

$$v_k = \frac{k}{g} f_k(w) \left[f'(w) \frac{\partial w}{\partial s} + f_T(w) \frac{\partial T}{\partial s} \right] \quad (1.6.30)$$

The temperature (and its spatial distribution) affects the course of physico-chemical processes as well as the properties of bonds produced by those processes. Other phenomena are also apt to come into play.

As an example consider the osmotic motion which in its results may be thought of as the sliding of molecules in shells of solid particles. Its quantitative appraisal rests on the hypothetical concept of the velocity w_ω relative to the whole cross-section of the soil

$$v_\omega = \frac{k_\omega}{g} \frac{\partial \varphi_\omega}{\partial T} \frac{\partial T}{\partial s} \quad (1.6.31)$$

where

φ_ω — osmotic potential,

k_ω — coefficient of osmotic motion, whose meaning is similar to that of the coefficient of permeability; numerically, it can be expressed as a function of moisture

$$k_\omega = k \tilde{f}_k(w) \quad (1.6.32)$$

Since the osmotic potential is assumed to be a continuous function, we may set

$$\frac{\partial \varphi_\omega}{\partial T} = f_\omega(w) \quad (1.6.33)$$

and substituting eqns. (1.6.32), (1.6.33) in eqn. (1.6.31) we obtain

$$v_\omega = \frac{k}{g} \tilde{f}_k(w) f_\omega(w) \frac{\partial T}{\partial s} \quad (1.6.34)$$

A soil may be regarded as a continuum in which the various processes and phenomena occurring in its pores, manifest themselves globally. One such phenomenon is the effect of air whose volume varies with pressure and temperature. Its quantitative analysis is based on the pneumatic potential φ_p .

Another remarkable phenomenon is diffusion. Its effect plays a role whenever in a given space the concentration of substances dissolved in water departs from normal conditions. A tool for a quantitative analysis of this effect is the diffusion potential φ_d .

It is clear from the above that without a thermodynamic analysis on a macro- as well as a micro-scale, the motion of water in an unsaturated zone defies exact solution. Since the total potential is defined for isothermal conditions, the temperature potential φ_T expressing the effect of temperature, is included in the sum. In the notation introduced above, the total potential is then defined by the sum

$$\varphi_x = \varphi_p + \varphi_k + \varphi_\omega + \varphi_d + \varphi_T \quad (1.6.35)$$

If we know the potentials φ_x and φ_k , we can readily establish the effect of the sum of the remaining potentials (including those with which we have not dealt specifically so far).

The approach suggested in the foregoing enables us to define the motion under any conditions without the labour of having to find out the effects of the various component actions.

In an attempt to find a unifying criterion for water motion in both saturated and unsaturated media, Philip [32] put forward the concept of the integral total potential φ_I . He based his proposal on the fact that the complete removal of water from unit volume of soil requires a certain amount of energy. A proportion of this total energy is transferred to the unit mass of water thus obtained. In a somewhat modified version, the integral total potential is characterized as the work required in an iso-

thermal and reversible process at a given point of space to remove completely unit mass of liquid and to transport it to the level of the reference plane chosen for the definition of the gravitational potential:

$$\varphi_I = - \frac{\int_0^w \varphi_z dw}{\int_0^w dw} + \varphi_g \quad (1.6.36)$$

The first term on the right-hand side of the equation, taken negatively, expresses the effect of moisture, and we speak of the integral moisture potential φ_{Iw} . It is clear from the definition and the mathematical expression for the integral moisture potential that it represents an extraordinarily large amount of energy. In problems in which no account need be taken of the effect of hygroscopically bonded water, we shall make do with a somewhat narrower definition of the integral total potential. The removal of water at a given point is terminated at the instant when the moisture remaining in the soil is w_0

$$\varphi_I = - \int_{w_0}^w \varphi_z dw / \int_{w_0}^w dw + \varphi_g = \varphi_{Iw} + \varphi_g \quad (1.6.37)$$

This restricted concept of the integral potential expresses mainly the effect of capillary bonds, and possibly also the effect of osmotic forces.

The concept of the integral total potential enables us to separate the effect of the gravitational potential more easily. It also leads to a straightforward mathematical description of the flow provided that it occurs within the bounds of the linear law. The velocity v relative to the whole area of soil cross-section is simply given by

$$v = \frac{k_w}{g} \frac{\partial \varphi_I}{\partial s} = \frac{k_w}{g} \left(\frac{\partial \varphi_{Iw}}{\partial s} + \frac{\partial \varphi_g}{\partial s} \right) \quad (1.6.38)$$

By way of illustration we consider an example in which we can neglect the effect of the temperature, diffusion and pneumatic potentials. The functions $f(w)$ which appear in the relations defining the various kinds of potential, may be modified by integration as indicated in eqns. (1.6.36) or (1.6.37), and the effect of integration on the functional dependence will be expressed by considering the modified functions $F(w)$, viz.

$$v = \frac{1}{g} \left\{ k_w \frac{\partial \varphi_g}{\partial s} - k \left[F_k(w) F'(w) \frac{\partial w}{\partial s} + F_k(w) F_T(w) \frac{\partial T}{\partial s} + \bar{F}_k F_\omega(w) \frac{\partial T}{\partial s} \right] \right\} \quad (1.6.39)$$

When neglecting the effect of osmotic forces, setting $F(w) = f(w)$ will lead to no great error. The predominant effect will then be that of gravitation and surface

tension forces, and eqn. (1.6.39) becomes

$$v = \frac{k_w}{g} \frac{\partial \varphi_g}{\partial s} - \frac{k f_k(w)}{g} f'_k(w) \frac{\partial w}{\partial s} - \frac{k}{g} f_k(w) f_T(w) \frac{\partial T}{\partial s} \quad (1.6.40)$$

The second and third terms on the right-hand side of eqn. (1.6.40) contain the products of the functions of moisture and constants. Klute [21] incorporates these terms into independent functions and terms the resulting relations $D(w)$, the coefficient of diffusivity. We have some doubts about the aptness of this term because the function $D(w)$ is in no way connected with the description of diffusion phenomena. However, we shall respect the customary usage:

$$D(w) = \frac{k}{g} f_k(w) f'_k(w) = \frac{k_w}{g} f'_k(w) \quad (1.6.41)$$

$D(w)$ is the coefficient of diffusivity

$$D_T(w) = \frac{k}{g} f_k(w) f_T(w) = \frac{k_w}{g} f_T(w) \quad (1.6.42)$$

$D_T(w)$ is the thermodynamical coefficient of diffusivity.

Rearranging eqn. (1.6.40) we obtain the simplified formulation

$$v = -k_w \frac{\partial h}{\partial s} - D(w) \frac{\partial w}{\partial s} - D_T(w) \frac{\partial T}{\partial s} \quad (1.6.43)$$

which under isothermal conditions becomes

$$v = -k_w \frac{\partial h}{\partial s} - D(w) \frac{\partial w}{\partial s} \quad (1.6.44)$$

In his studies concerning the determination of the coefficient $D(w)$, Gardner [11] arrived at the conclusion that

$$D(w) = D_0 e^{\chi(w-w_0)}$$

where D_0 [m^2s^{-1}] is the value of D at $w = w_0$ (w_0 is the initial moisture), and χ is a constant.

The value of the coefficient of diffusivity can be determined experimentally or by calculation. In the latter case, the point of departure is Gardner's relation (1.6.21). Differentiating it with respect to w gives

$$f'_k(w) = \frac{\partial \varphi_k}{\partial w} = - \frac{\varphi_{km} - \varphi_{k0}}{\frac{1}{w_0} - \frac{1}{w_m}} \frac{1}{w^2} = - \frac{\varphi_{km}(1 - \delta) w_0 w_m}{w_m - w_0} \frac{1}{w^2} \quad (1.6.45)$$

where δ is a constant, usually taken as 0.15 to 0.25.

In the case where surface tension forces predominate, the motion may be expected to cease at a certain, sufficiently low threshold moisture. On the strength of this, eqn. (1.6.45) may be modified as follows:

$$f'(w) = - \frac{\varphi_{km}(1 - \delta) w_0 w_m}{w_m - w_0} \frac{1}{(w - w_0)^2} \quad (1.6.46)$$

According to the definition of the capillary potential

$$\varphi_{km} = -gh_{kmax} = -\frac{g}{\gamma} p_{w_0} \quad (1.6.47)$$

where p_{w_0} is the suction pressure corresponding to the moisture w_0 . Hence

$$D(w) = \frac{k_w p_{w_0}(1 - \delta) w_0 w_m}{\gamma(w_m - w_0)(w - w_0)^2} \quad (1.6.48)$$

Applying Averyanov's equation (1.6.9) we have finally

$$D(w) = \frac{k p_{w_0}(1 - \delta) w_0 w_m}{\gamma(w_m - w_0)^{n+1}} (w - w_0)^{n-2} \quad (1.6.49)$$

If we now set

$$D(w) = C(w - w_0)^{n-2} \quad (1.6.50)$$

$$C = \frac{k p_{w_0}(1 - \delta) w_0 w_m}{\gamma(w_m - w_0)^{n+1}} \quad (1.6.51)$$

The constant C has the dimensions of $[m^2/s]$ and is termed the coefficient of moisture conductivity. It can also be assumed to be of the form

$$C = C_0 \frac{p_{w_0}}{\gamma} \quad (1.6.52)$$

$$C_0 = \frac{k(1 - \delta) w_0 w_m}{(w_m - w_0)^{n+1}} \quad (1.6.53)$$

The coefficient C_0 has the dimensions of a velocity and is termed the coefficient of moisture permeability.

Eqn. (1.6.44) then yields

$$v = -k_w \frac{\partial h}{\partial s} - C(w - w_0)^{n-2} \frac{\partial w}{\partial s} \quad (1.6.54)$$

or

$$v = -k_w \frac{\partial h}{\partial s} - C_0(w - w_0)^{n-2} \frac{p_{w_0}}{\gamma} \frac{\partial w}{\partial s} \quad (1.6.55)$$

Consider now the function W which we shall call the moisture velocity potential

$$W = C \frac{(w - w_0)^{n-1}}{n-1} \quad (1.6.56)$$

The function W (Wenceslaus potential) provides the basis for calculating the velocity, since we have

$$v = -k_w \frac{\partial h}{\partial s} - \frac{\partial W}{\partial s} \quad (1.6.57)$$

It is worth mentioning that we are sometimes dealing with fine-grained materials in which we may expect the pre-linearity of flow to come into play. The coefficient of permeability k in eqn. (1.6.49) can be replaced by one of the laws of the pre-linear regime. If we can accept Puzyrevski's idea, we assume that the motion can take place so long as

$$\left| \frac{p_{w_0}}{\gamma} \frac{\partial w}{\partial s} \right| > J_{p_0}$$

Assume further that k_w is small and hence that the first term on the right-hand side of eqn. (1.6.54) is negligible. Introduce the effect of the initial gradient J_{p_0} as follows:

$$v = C_0(w - w_0)^{n-2} \left(\left| \frac{p_{w_0}}{\gamma} \frac{\partial w}{\partial s} \right| - J_{p_0} \right) \quad (1.6.58a)$$

$$v = C_0(w - w_0)^{n-2} \left| \frac{p_{w_0}}{\gamma} \frac{\partial w}{\partial s} \right| - C_0(w - w_0)^{n-2} J_{p_0} \quad (1.6.58b)$$

Provided that $J_p \rightarrow 0$ or $w \rightarrow w_0$, the second term on the right-hand side of eqn. (1.6.58b) will tend to zero. Evidently, the effect of the initial gradient diminishes in an unsaturated soil. In most cases we shall manage with equations containing no J_{p_0} .

In conclusion, a few notes concerning the constants referred to in our discussion:

a) The value of the coefficient of diffusivity $D(w)$ of a given soil, which appears in some of the relations derived above, can be determined experimentally. Once it is known, it can be used in the calculation of the value of C

$$C = \frac{D(w)}{(w - w_0)^{n-2}}$$

as an average of several known values of $D_i(w)$ for different w_i .

b) The coefficient C can also be established approximately from the (pF) curve. Because two curves are obtained for each soil — one for the case of gradual satura-

tion, the other for the case of gradual drying of the soil — we must first decide which motion we intend to follow in the subsequent solution. We must also bear in mind that all soils contain gases: approximately 5% of air remains in a saturated soil. Hence $w_m \neq m$.

In practice we first read from the curve the values of p_{w_0} corresponding to w_0 , then the values of p_{w_m} corresponding to w_m . Since the coefficient of permeability of saturated soils is assumed to be known, we can calculate the coefficient

$$C = \frac{k(p_{w_0} - p_{w_m}) w_0 w_m}{\gamma(w_m - w_0)^{n+1}}$$

The use of the coefficient of moisture conductivity is advantageous in that it enables us — in contrast to eqn. (1.6.44) — to use the linear equation (1.6.57) which greatly facilitates practical solutions.

1.6.3. Mathematical description of flow in unsaturated soils

The basic equations of motion in saturated soils derived in Sections 1.4.1 and 1.4.2 were based on the assumptions of the validity of the linear law. We concluded that steady-state motion without the effect of inertia forces is governed by Darcy's law and apparently nothing was proved except for the fact that the formulation of the linear law is simultaneously the equation of motion.

Since the motion in an unsaturated medium is very slow, the time increments of velocity can be assumed to be comparatively small. We can, therefore, state immediately that the laws of flow derived in the preceding sections, are the equations of motion, and it merely remains to set up the balance equation.

Assume that the quantity of flow through an element dS of the surface S bounding the volume V in time dt is

$$dQ = dt \, v \, dS \quad (1.6.59)$$

The total quantity which will reach the volume V is

$$Q = dt \int_S v \, dS \quad (1.6.60)$$

The quantity of flow will cause the moisture to increase

$$Q = - \int_V \frac{\partial w}{\partial t} dt \, dV \quad (1.6.61)$$

Comparing eqn. (1.6.60) with eqn. (1.6.61) we see that

$$- \int_V \frac{\partial w}{\partial t} dt \, dV = dt \int_S v \, dS = dt \int_V \operatorname{div} v \, dV \quad (1.6.62)$$

Hence

$$-\frac{\partial w}{\partial t} = \text{div } v \quad (1.6.63)$$

Substituting eqn. (1.6.43) written in the form

$$v = -[k_w \text{grad } h + D(w) \text{grad } w + D_T(w) \text{grad } T] \quad (1.6.64)$$

in eqn. (1.6.63), we obtain the resulting equation

$$\frac{\partial w}{\partial t} = \text{div} [k_w \text{grad } h + D(w) \text{grad } w + D_T(w) \text{grad } T] \quad (1.6.65)$$

In Cartesian coordinates, eqn. (1.6.65) can readily be written as follows:

$$\begin{aligned} \frac{\partial w}{\partial t} = & \frac{\partial}{\partial x} \left(k_w \frac{\partial h}{\partial x} \right) + \frac{\partial}{\partial y} \left(k_w \frac{\partial h}{\partial y} \right) + \frac{\partial}{\partial z} \left(k_w \frac{\partial h}{\partial z} \right) + \\ & + \frac{\partial}{\partial x} \left[D(w) \frac{\partial w}{\partial x} \right] + \frac{\partial}{\partial y} \left[D(w) \frac{\partial w}{\partial y} \right] + \frac{\partial}{\partial z} \left[D(w) \frac{\partial w}{\partial z} \right] + \\ & + \frac{\partial}{\partial x} \left[D_T(w) \frac{\partial T}{\partial x} \right] + \frac{\partial}{\partial y} \left[D_T(w) \frac{\partial T}{\partial y} \right] + \frac{\partial}{\partial z} \left[D_T(w) \frac{\partial T}{\partial z} \right] \end{aligned} \quad (1.6.66)$$

In isothermal conditions $\partial T/\partial x = \partial T/\partial y = \partial T/\partial z = 0$.

In the subsequent exposition we shall refer to the definitions of the coefficient of diffusivity $D(w)$ (eqn. (1.6.49)), and of the coefficient of the moisture velocity potential W (eqn. (1.6.56)):

$$\frac{\partial w}{\partial t} = \frac{\partial}{\partial x} \left(k_w \frac{\partial h}{\partial x} \right) + \frac{\partial}{\partial y} \left(k_w \frac{\partial h}{\partial y} \right) + \frac{\partial}{\partial z} \left(k_w \frac{\partial h}{\partial z} \right) + \frac{\partial^2 W}{\partial x^2} + \frac{\partial^2 W}{\partial y^2} + \frac{\partial^2 W}{\partial z^2} \quad (1.6.67)$$

As eqn. (1.6.56) implies

$$w - w_0 = \left[\frac{(n-1)W}{C} \right]^{1/(n-1)} \quad (1.6.68)$$

differentiating with respect to t yields

$$\frac{\partial w}{\partial t} = C^{-1/(n-1)} [(n-1)W]^{-[(n-2)/(n-1)]} \frac{\partial W}{\partial t}$$

and on substituting in eqn. (1.6.67) we have

$$\begin{aligned} \frac{\partial W}{\partial t} = & C^{1/(n-1)} [(n-1)W]^{(n-2)/(n-1)} \left[\frac{\partial}{\partial x} \left(k_w \frac{\partial h}{\partial x} \right) + \frac{\partial}{\partial y} \left(k_w \frac{\partial h}{\partial y} \right) + \right. \\ & \left. + \frac{\partial}{\partial z} \left(k_w \frac{\partial h}{\partial z} \right) + \frac{\partial^2 W}{\partial x^2} + \frac{\partial^2 W}{\partial y^2} + \frac{\partial^2 W}{\partial z^2} \right] \end{aligned} \quad (1.6.69)$$

All the differential equations just derived can also be written in terms of cylindrical or spherical coordinates: the appropriate manipulations are omitted here and we merely note that the theory assumes the functions to be continuous, and that this condition is not always satisfied in stratified media. The discontinuities are expected to occur at the interfaces of layers with markedly different properties. An exception is a stratum composed of relatively thin layers of not dissimilar properties, alternating in a certain order.

Consider, for example, a stratum in which a uniform threshold moisture can be assumed (or chosen) everywhere. Similarly as in Section 1.4.5, we shall determine the maximum and minimum values of the coefficient of moisture conductivity

$$C_{\max} = \frac{\sum_{j=1}^M C_j \Delta T_j}{\sum_{j=1}^M \Delta T_j} \quad (1.6.70)$$

$$C_{\min} = \frac{\sum_{j=1}^M \Delta T_j}{\sum_{j=1}^M \frac{\Delta T_j}{C_j}} \quad (1.6.71)$$

where $j = 1, 2, \dots, M$,

M — number of layers considered,

ΔT_j — thickness of the j -th layer,

C_j — coefficient of moisture conductivity of the j -th layer.

Assume that the z axis of the system x, y, z is oriented vertically, and that the effect of the moisture regime predominates in the horizontal direction. In eqn. (1.6.66)

$$\frac{\partial w}{\partial t} = \frac{\partial}{\partial x} \left[C_x (w - w_0)^{n-2} \frac{\partial w}{\partial x} \right] + \frac{\partial}{\partial y} \left[C_y (w - w_0)^{n-2} \frac{\partial w}{\partial y} \right] + \frac{\partial}{\partial z} \left[C_z (w - w_0)^{n-2} \frac{\partial w}{\partial z} \right] \quad (1.6.72)$$

for

$$C_x = C_y = C_{\max}$$

$$C_z = C_{\min}$$

use the substitution

$$\bar{x} = x, \quad \bar{y} = y, \quad \bar{z} = z \sqrt{A} \quad (1.6.73)$$

where the coefficient of anisotropy

$$A = \frac{C_x}{C_z} = \frac{C_y}{C_z} \quad (1.6.74)$$

to obtain

$$\begin{aligned} \frac{\partial w}{\partial t} = \frac{\partial}{\partial \bar{x}} \left[C_{\max} (w - w_0)^{n-2} \frac{\partial w}{\partial \bar{x}} \right] + \frac{\partial}{\partial \bar{y}} \left[C_{\max} (w - w_0)^{n-2} \frac{\partial w}{\partial \bar{y}} \right] + \\ + \frac{\partial}{\partial \bar{z}} \left[C_{\max} (w - w_0)^{n-2} \frac{\partial w}{\partial \bar{z}} \right] \end{aligned} \quad (1.6.75)$$

Consider the function

$$W = C_{\max} \frac{(w - w_0)^{n-1}}{n-1} \quad (1.6.76)$$

Substituting this function in eqn. (1.6.72) we have

$$\frac{\partial W}{\partial t} = C_{\max}^{1/(n-1)} [(n-1) W]^{(n-2)/(n-1)} \left[\frac{\partial^2 W}{\partial \bar{x}^2} + \frac{\partial^2 W}{\partial \bar{y}^2} + \frac{\partial^2 W}{\partial \bar{z}^2} \right] \quad (1.6.77)$$

Eqn. (1.6.77) applies to a homogeneous medium obtained by the transformation of the original flow region. When eqn. (1.6.77) has been solved, we return to the original system by using the inverse transformations implied by eqn. (1.6.73).

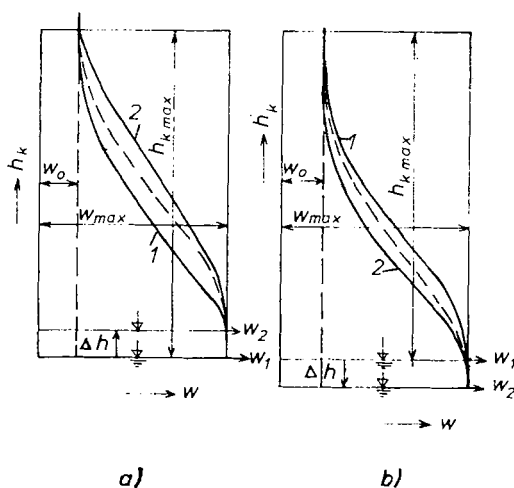
1.6.4. Distribution and re-distribution of moisture in the zone above the free surface

From the point of view of moisture, the transition from the saturated soil immediately below the free surface of gravitational water to the unsaturated soil immediately above the free surface, is neither sudden nor sharp. Yet the motion of the free surface has an effect on the development in the unsaturated zone. As an example, consider a sudden rise of the free surface in a relatively small height range, approximately in the range of the region in which the nearly saturated capillary zone originally existed. The curve 1 in Fig. 1.11a shows the initial state. To this state corresponds the axis of abscissae w_1 . After the sudden rise, the free surface will take up a new position with the corresponding axis of abscissae w_2 . The re-distribution of moisture which occurs in the subsequent stages, is most pronounced in the lower part of the saturated zone (the dash-line curve). Later on, changes will also be noted in the upper parts, and eventually the moisture will be distributed as shown by the curve 2.

In the case of a small sudden rise of the free surface (Fig. 1.11b) we find that the character of moisture distribution differs both qualitatively and quantitatively from that accompanying a sudden rise. Due to hysteresis effects, changes of moisture in the upper part of the saturated zone are smaller and slower.

A large sudden rise of the free surface, e.g. above the range of the original saturated zone produces a capillary saturated zone above the new level, which subsequently develops into an unsaturated zone. At a large sudden fall of the free surface, the capillary saturated zone formed above the new level of the free surface, is smaller than in the former case, with capillary suspended water appearing above the saturated zone. Some of this water will subsequently pass into the unsaturated zone which begins to form above the new free surface.

Fig. 1.11. Effect of free surface motion on moisture distribution in the capillary zone.



Several hypotheses explaining the origin and existence of capillary suspended water have been proposed. A hypothesis which has received wide recognition is based on water behaviour in capillary tubes whose cross-sectional area varies rhythmically — either continuously or discontinuously — along the height of the tube. The column of water is closed by menisci from above as well as from below, and the radius of curvature of the upper meniscus is larger than that of the lower meniscus. The latter always tends to take up a position in which its radius of curvature would be as large as possible while the maximum capillary elevation corresponds to the case of maximum radius of curvature. Now, if some water is added from above to the tube filled in this way, water will drop off the lower surface. This is a proof of the possibility that quanta of water may be transported through the tube, even though the particles leaving the bottom of the tube are those formerly stored in the tube rather than those just brought in.

The height of capillary elevation in soils is in direct proportion to the square root of time; as a result, re-distribution of moisture is also affected by the velocity of rise or fall of the free surface, v . At approximately $v \geq k$, we can speak (with respect to the development in the unsaturated zone) of a sudden change in position. At $v < k$ (where k is the classical Darcy's coefficient of permeability) a distinct unsaturated zone develops instantaneously in the lower part of the moisture profile.

Periodic fluctuations of the position of the free surface produce a definite, average, moisture profile. A zone of discontinuous changes of moisture is noted at large fluctuations and the range of the zone of capillary suspended water varies periodically.

If, moreover, account is also taken of the effect of the infiltration of water from the terrain surface, the moisture conditions become even more complicated. Studies, past and present, have, therefore, been directed at conditions of separated infiltration assuming the free surface of gravitational water to lie at a considerable depth below the terrain surface, and infiltration into the soil to occur with a certain, constant moisture everywhere. As an example, consider the theoretical solution of Philip which leads to the equation

$$v = \frac{\alpha}{2} t^{-1/2} + \beta \quad (1.6.78)$$

where

v — velocity of infiltration on the terrain surface,
 α, β — constants. The first denotes the so-called sorption coefficient $[m/\sqrt{s}]$, the other has the dimension of $[m/s]$.

Evidently, when the time $t \rightarrow \infty$, $v = \beta$, so that β should be equal to the coefficient k_w determined for the prescribed constant moisture on the terrain surface while to the cases of $t \ll \infty$ and $\alpha > 0$, the inequality $v > k_w$ applies.

The conditions of the proposed theoretical solution may not be identical with those of actual infiltration. Thus, for example, theory fails to consider the effect of air which escapes in the direction against the flow of percolating water and decelerates the infiltration. The many empirical and semi-empirical formulae developed for the case of free infiltration, may be classified into two groups. The first group contains equations of the type

$$v = f[q, k_w, \text{Exp}(\beta t)] \quad (1.6.79)$$

where β is a constant, q is the rain intensity.

The equations of the second group are of the form

$$v = f[q, k_w, t^{-\delta}] \quad (1.6.80)$$

where δ is a constant.

The first type of equation is represented by the equation of Horton; this equation is not too well suited for analyzing situations at $t \rightarrow 0$. Kostyakov's equation, representative of the second type, is, on the other hand, less appropriate for cases of $t \rightarrow \infty$. In either case, the total volume of water, V , which has penetrated into the ground in time t , can be estimated by integrating the infiltration velocity v near the surface, viz.

$$V = \int_0^t v \, dt \quad (1.6.81)$$

The equations cannot adequately describe the cases where the front of infiltrating water (i.e. the line along which the initial moisture begins to change in space and time under the effect of infiltration) reaches the capillary zone or the zone of capillary suspended water because the assumption of constant initial moisture becomes invalid there.

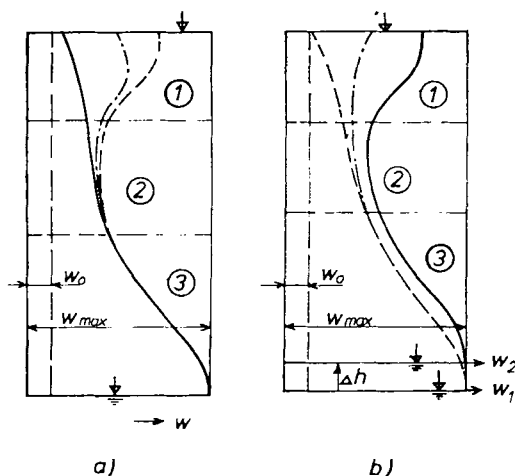


Fig. 1.12. Effect of infiltration and evaporation on moisture distribution.

An upward motion is noted in the evaporation of water from soils, produced by the effect of temperature difference. A decrease of moisture can also be caused by the biological consumption of plants (transpiration activity). A combination of these effects is called evapotranspiration. Its measure is the evaporation from the free surface of surface water, with which evapotranspiration is compared (relative evapotranspiration).

Theoretical mastery of the process of relative or absolute evapotranspiration is extremely difficult, if for no other reason, then because the partial phenomena are insufficiently described.

The various aspects of evapotranspiration in combination with time-distributed infiltration have received full coverage in soil mechanics and agricultural engineering literature. All we can add is that so far no exact procedure has been developed which enables the problem to be analyzed in its entirety. By way of an example we shall show a typical moisture profile between the terrain surface and the level of gravitational water.

Fig. 1.12a shows in dashed lines the state corresponding to the development of moisture at the instant when infiltration was interrupted and evaporation started to predominate. Along the height of the unsaturated zone lying between the surface of the terrain and the still free surface of gravitational water, moisture distribution will in due course follow the dot-and-dash line, and after a prolonged period of time, the solid line. The entire height of the saturated zone can be divided into three sections. The upper section is the zone of capillary suspended water, in which both

infiltration and evaporation play a major role. The lower section is the zone of capillary discharge from the free surface of gravitational water. The intermediate section is the zone of minimum moisture. Its extent depends on the position of the free surface of gravitational water. If this surface lies close to the terrain surface, the zone 2 (the transport zone) is absent, and the effect of climatic factors manifests itself markedly in the zone 3.

Fig. 1.12b shows the distribution of moisture under the effect of gradually predominating infiltration. The initial situation is indicated by dash lines. Infiltration causes an increase of moisture in the zone 1 (dot-and-dash lines). It is assumed that infiltration will stop after a time, at which point moisture distribution will be as shown by the solid line.

1.6.5. Supply into the free surface of gravitational water

Assume that a zone of unsaturated soil is underlaid by a zone of gravitational water performing no horizontal motion. The free surface is horizontal and capable of changing its position in the vertical direction.

In an unsaturated zone, moisture is re-distributed by the effect of infiltration or evapotranspiration. Of importance from the point of view of water balance, is the transfer of water in the zone 3. Under predominating infiltration, water starts to flow into the free surface of gravitational water. As a result, the free surface will rise (Fig. 1.12b) and the extent of the transport zone will diminish. Under predominating evapotranspiration the process is reversed, the free surface falls and the extent of the transport zone increases.

Another phenomenon worth mentioning is condensation of water vapours. Its effect is most pronounced in cases when the groundwater level lies deep below the terrain surface and moisture in the zone 2 is almost minimal. Condensation can manifest itself by the transfer of moisture into the free surface of gravitational water.

All these factors contribute to the discharge into the free surface, which we denoted in Section 1.4 by the letter v_0 (inflow or outflow defined in meters of water column per unit time). This quantity can be positive or negative, and its magnitude depends on the depth H of the groundwater level below the terrain surface. So far, the determination of v_0 in space and time has defied theoretical treatment, and the problem can only be solved by experimental methods.

Fig. 1.13 shows an example of experimental facilities developed to that end – a system of field lysimetric batteries designed at the Technical University, Brno, Research Institute for Water Management.

A cylinder of diameter D encloses a soil monolith overgrown with plants on the top, and resting on an inverted filter. The space of the coarsest section of the filter communicates with a gauge tube (to the right of the cylinder in Fig. 1.13) and with an auxiliary cylindrical vessel (to the left of the cylinder) of diameter $d \ll D$. The

vessel is equipped with a circular weir of radius $d/2$, and filled with the same soil as the main cylinder. However, the soil does not extend below the level whose position can be varied over the range $H \pm \Delta H$ ($\Delta H \leq 0.1$ m). A gauge tube and a take-off pipe, inserted in the inner space of the weir, can be sealed hermetically just as can the gauge tube be connected to the filter. In shallow lysimeters the diameter of the auxiliary cylinder is chosen so large as to make the 0.1 m range of level fluctuations sufficient to take care of the total discharge for about two weeks.

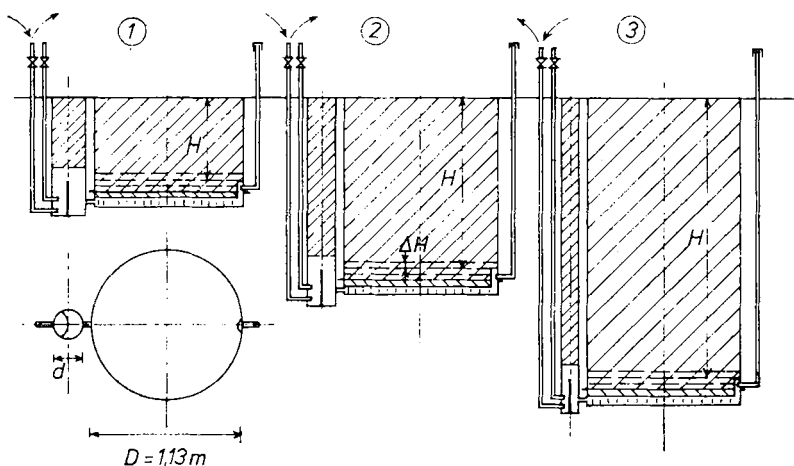


Fig. 1.13. Schematic diagram of a battery of lysimeters.

During that time, the natural development of air pressure in the two cylinders should not be disturbed drastically. A check made after two weeks provides information regarding the necessity of adding to, or removing from, the lysimeter the volume of water V (negative in the former, positive in the latter case). After a time interval Δt in which N additions or removals of water were effected, the average discharge (produced by the effect of climatic factors) at a depth H will be

$$v_0|_{H, \Delta t} = \frac{4 \sum_{i=1}^N V}{\pi(D^2 + d^2) \Delta t} \approx \frac{4 \sum_{i=1}^N V}{\pi D^2 \Delta t}$$

Hence, the required dependence will be, approximately,

$$\frac{v_0(H)}{q_s} = f(H)$$

$$q_s = \frac{\sum_{i=1}^M V_s}{t_2 - t_1}, \quad \Delta t = t_2 - t_1$$

where

V_s — partial total quantities of precipitations. In a time period Δt , M total quantities of precipitations were detected, with $M \geq 1$ in the interval Δt ,
 q_s — average intensity of precipitations in the interval Δt .

Evidently, an important factor is the inclination of the terrain in which a lysimeter battery operates because it is the same as that of the surface of the soil filling the cylinders. If surface run-off is of no interest, the edges of the cylinders can be made to protrude above the terrain surface.

Under conditions prevailing in Central Europe, special attention should be paid to the winter season, when the effect of precipitations plays a major role during the period of snow melting and earth defrosting.

Compensation lysimeters are equipped with electrical circuits for contact signalling, possibly also with sensors to measure the suction pressure, and radiometric probes to record the moisture distribution. But their primary purpose is to enable us to study the effect of fluctuations of the gravitational water surface. If the soil monolith is stratified, the process can be studied even in cases when the diffusion theory fails.

The quantity used in groundwater hydraulics to characterize the storing capacity of soils, is the active porosity μ . This quantity depends on a number of factors; among others, on the distribution of moisture above the free surface of gravitational water. As shown earlier, it may be a function of H .

The value of μ (cf. Section 1.2.2) can be determined in a compensation lysimeter. The determination starts from the state in which the free surface in the auxiliary cylinder is on a level with the weir. If a volume V is suddenly added to it, the surface will first rise Δh above the crest and then settle down at a height $\Delta h'$. The coefficient of saturation deficiency is calculated from the equation

$$\mu_1 = \left(\frac{d}{D}\right)^2 [\Delta h - \Delta h']$$

This value applies in the case of a rising surface of gravitational water. In the determination of μ_2 for a falling surface, the same volume V is removed. The surface will first settle below the level of the crest, and then begin to rise. Data referring to the initial and final position of the surface can be used in the calculation of the coefficient of water yield.

Experiments such as those described above furnish us with fairly objective information about the effect of climatic factors, the effect of the depth of water level below the surface of the terrain, and other factors (especially temperature and pneumatic effects) which contributes to the determination of the mean value of the active porosity μ .

1.7. Force action of groundwater flow on soils

In studies of groundwater motion in saturated or unsaturated soils, the phenomenon is represented by a model in which a model liquid with physical properties (density and compressibility) identical with those of the actual liquid, flows continuously through a given space. The flow velocity of the model liquid is a fictitious quantity because no allowance is made for the soil. However, the resistance forces acting against the motion of the real liquid are taken into account by being referred to unit mass of the real liquid or to unit volume of the space (soil). No shear stresses are considered in the model liquid.

Consider an elementary cylinder of flow and denote by dS the projection of its cross-sectional area onto a plane normal to the direction of flow. The length ds of the cylinder is measured in the direction of flow. The cylinder encloses the model liquid which is subjected to pressures p .

The difference between the total pressures acting on the two bases is $dp dS$. It can be expressed in terms of the pressure head

$$\gamma dS dh \quad (1.7.1)$$

In a real soil, the moisture is w and the resistance force acting on unit mass of real liquid is R_v . Hence the resistance force acting on the entire elementary cylinder is

$$-R_v w dS ds \quad (1.7.2)$$

in uniform motion the two forces must be in equilibrium, i.e.

$$-R_v w dS ds + \gamma dS dh = 0$$

Hence

$$R_v = \frac{\gamma}{w} \frac{dh}{ds} = \frac{\gamma}{w} J_w \quad (1.7.3)$$

Considering the force F per unit volume of space, we have

$$F = w R_v = \gamma J_w \quad (1.7.4)$$

where J_w is the gradient which characterizes, at a given moisture, the effect of the increment of pressure on the boundary surfaces of the element.

Eqn. (1.7.4) applies regardless of the particular law of flow being used. In a saturated zone, $J_w = J$.

In a saturated zone, we have according to eqn. (1.5.30), for the combined law of flow

$$J = \frac{v^2}{k^2} = v^2 \frac{\lambda_c}{2gd} \quad (1.7.5)$$

Using eqn. (1.5.27) we obtain

$$J = \frac{v}{k} \left\{ 1 + k \left[\frac{v}{k_t^2} + (vk_n)^{-1/2} \right] \right\} \quad (1.7.6)$$

Eqn. (1.7.6) forms the basis for deriving relationships applicable to the various regimes of flow. Neglecting, for example, the effect of pre-linearity ($vk_n \rightarrow 0$), we obtain a relation similar to eqn. (1.5.2) for $c_1 = 1/k$, $c_2 = 1/k_t$, viz.

$$J = \frac{v}{k} + \frac{v^2}{k_t^2} \quad (1.7.7)$$

Neglecting moreover the effect of turbulence, and considering it adequate to use the linear law, i.e. on the assumption that $k_t \gg v$, eqn. (1.7.7) becomes

$$J = \frac{v}{k} \quad (1.7.8)$$

In an unsaturated zone, we have according to eqn. (1.6.44)

$$v = k_w J_w = -k_w \frac{\partial h}{\partial s} - D(w) \frac{\partial w}{\partial s} = k_w J - D(w) \frac{\partial w}{\partial s} \quad (1.7.9)$$

The sign of J in eqn. (1.7.9) is chosen in accordance with the direction of motion of gravitational water relative to the selected system of coordinates.

As a rule, J is comparatively small. Substituting eqn. (1.6.45) in eqn. (1.6.41) and this in turn in eqn. (1.7.9) we obtain

$$J_w = - \frac{D(w)}{k_w} \frac{\partial w}{\partial s} = \frac{p_{w0}(1 - \delta) w_0 w_m}{\gamma(w_m - w_0) w^2} \frac{\partial w}{\partial s} \quad (1.7.10)$$

for such cases.

As eqn. (1.7.10) implies, large internal forces are at work in soils during the emptying of pores originally filled with water.

Let us note at this point that in the analysis of forces exerted by water on soil, we have so far given no consideration to deformations of material which are apt to give rise to additional secondary flows.

1.8. Flow of water in media with discrete voids

In rocks and fissured cohesive soils, there exist voids whose dimensions are much larger than those of the pores in the material proper. If interconnected, these voids form a pathway for flowing water. In many cases their seepage activity is so high that in an analysis of water motion we can neglect the seepage through the

material with comparatively small pores and concentrate on the motion of water in the voids. An analysis of the flow through systems of caves in karst formations can serve as an illustrative example.

Localized, un-systematic voids may be regarded as a system of pipes conveying water with a free surface or under pressure. Since the solution of this problem belongs in the area of surface water hydraulics, we shall merely set down a few equations which are used in calculations of systems of circular pipes. In pressure flow the hydraulic gradient J along a certain segment is determined from a formula implied by the Bernoulli equation

$$J = \left[\frac{D}{l} (1 + \Sigma \xi_D) + \lambda_D \right] \frac{1}{D} \frac{v_D^2}{2g} \quad (1.8.1)$$

where

D — pipe diameter,

ξ_D — coefficient of local losses,

l — pipe length,

λ_D — friction factor in pipe (coefficient of friction in pipe),

v_D — mean cross-sectional velocity.

In laminar flow, the coefficient of friction is calculated from the equation

$$\lambda_D = \frac{64}{Re_D}, \quad Re_D < 2320 \quad (1.8.2)$$

where Re_D is the Reynolds number determined relative to the pipe diameter

$$Re_D = \frac{v_D D}{\nu} \quad (1.8.3)$$

where ν is the coefficient of kinematic viscosity.

At $Re > 2320$, the motion is usually assumed to be turbulent. For smooth pipes, Kármán and Prandl derived the relation

$$\frac{1}{\sqrt{\lambda_D}} = 2 \log \frac{Re_D \sqrt{\lambda_D}}{2.51} \quad (1.8.4)$$

According to Blasius, in a related case

$$\lambda_D = \frac{0.3164}{Re_D^{1/4}} \quad (1.8.4a)$$

In pipes with regular roughness characterized by the degree of roughness Δ , Nikuradse obtained the relation

$$\frac{1}{\sqrt{\lambda_D}} = 2 \log \frac{3.7D}{\Delta} \quad (1.8.5)$$

A pipe is assumed to have smooth walls if the thickness of the laminar sub-layer is at least five times the degree of roughness.

Altshul [6] has theoretically proved the correctness of the relation recommended by Colebrook and White. Starting from a synthesis of eqns. (1.8.4) and (1.8.8), these authors obtained the following relation which is universally applicable to regions of turbulent flow and requires no analysis of the thickness of the boundary layer

$$\frac{1}{\sqrt{\lambda_D}} = -2 \log \left(\frac{2.51}{Re_D \sqrt{\lambda_D}} + \frac{A}{3.7D} \right) \quad (1.8.6)$$

A discussion of local losses and relations used to calculate the coefficient ξ_D can be found in textbooks on hydraulics.

Systems of voids with branches characterized by the above relations, are solved on the basis of the piping theory, stemming essentially from Kirchhoff's laws. This subject is treated in the specialized literature.

Voids in natural materials are, of course, rarely circular. Hence the relation set out above must be modified, and the most frequently employed form of modification is the hydraulic radius

$$R = \frac{S}{O} \quad (1.8.7)$$

R — hydraulic radius,

S — cross-sectional area,

O — wetted circumference.

In a circular pipe

$$R = \frac{\pi D^2/4}{\pi D} = \frac{D}{4} \quad (1.8.8)$$

1.8.1. Basic characteristics of water flow in fissures filled with water

A special type of voids are fissures which separate blocks of the basic material. The blocks are in contact in localized joints where the effective stress is concentrated (Griffith strength). These joints are usually neglected in hydraulic considerations, and attention is focused on isolated seepage paths which are likely to determine the seepage activity of the porous medium as a whole.

Assume first that the blocks have a relatively low permeability. Under such conditions the flow occurs exclusively through the fissures. The seepage activity of fissures depends above all on the shape and extent of the network, geometry of the individual fissures, the medium with which they are filled, etc. We shall first be concerned with fissures filled with water flowing under pressure.

The principal source of information on this subject is geology with its scientific description of the formation and composition of rocks, and of the geometrical

characteristics and genesis of fissures. Readers seeking full coverage of the topic are referred to the specialized literature (e.g. Müller [26]).

Of interest from the hydraulic point of view are the properties of a particular continuous fissure, i.e. the opening, longitudinal change of opening, longitudinal curvature and wall roughness. A fissure is considered to be curved if its rise is several times greater than the opening (Fig. 1.14a). Other fissures are regarded as not curved,

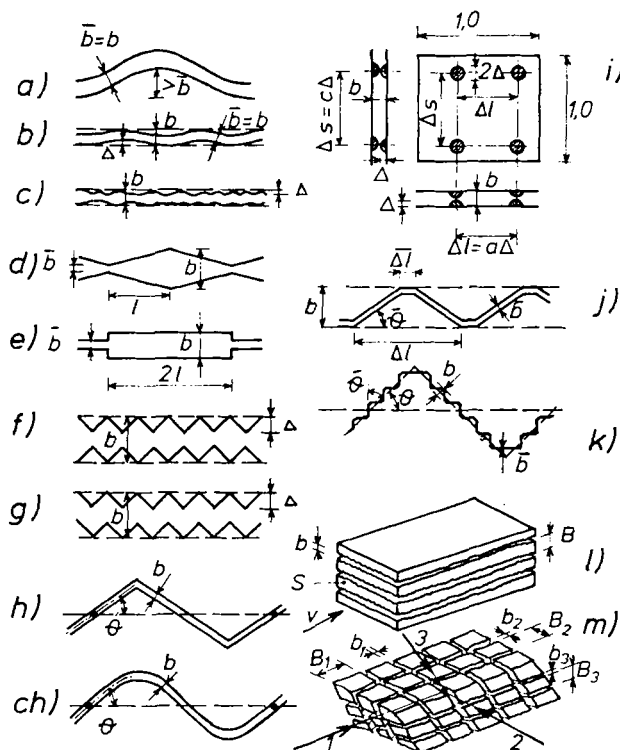


Fig. 1.14. Characteristics of fissures.

and the effect of unevenness is included in the generalized concept of roughness. Roughness is defined by the degree of roughness Δ (Fig. 1.14b). The ratio between the degree of roughness and the opening is the so-called relative roughness.

Roughness of natural fissures varies along the fissure length (Fig. 1.14c). With the exception of extremes which are likely to produce localized losses in pressure head, calculations are made using the statistical mean of the roughness.

Fissures with a linear (longitudinal) change in opening are considered to be wedged if, according to Fig. 1.14d,

$$\frac{b - \bar{b}}{l} \gg \frac{1}{10} \quad (1.8.9)$$

For wedged fissures, local losses are taken into account; otherwise, changes in opening are included in the generalized concept of roughness.

Sudden changes in cross-section (Fig. 1.14e) must also be considered in calculations. If eqn. (1.8.9) does not apply, the flow of water is regarded as motion through a rough fissure.

Hydraulically important are the concepts of symmetrical roughness (Fig. 1.14f) and of non-symmetrical roughness (Fig. 1.14g).

Wedges, sudden change in cross-section and various kinds of roughness can also exist in curved fissures. Certain simplifications must be introduced when solving spatial systems of meandering curved fissures.

Each branch of such systems is assumed to contain a straight fissure, and a so-called curvature factor χ is introduced into the calculations. For angular fissures (Fig. 1.14h) this factor is

$$\chi = \frac{1}{\cos \theta} \quad (1.8.10)$$

In fissures composed of parabolical arcs linking one another (Fig. 1.14ch) the curvature factor becomes

$$\chi = 1 + \frac{1}{6} \operatorname{tg}^2 \theta \quad (1.8.11)$$

The hydraulic radius per unit width of flow in a smooth fissure with opening b , is defined by

$$R = \frac{1 \cdot b}{2 \cdot 1} = \frac{b}{2} \quad (1.8.12)$$

In rough fissures, the hydraulic radius R is defined differently, i.e. as the ratio between the volume of water in the fissure per unit area (looking in the direction normal to the fissure centreline), and the area of both walls in the unit so delineated.

It is assumed that on both walls of the unit area, there are M projections; the bases of these projections have an area S' , a surface S and a volume V . In the interest of the subsequent exposition, consider two fissures, both with symmetrical roughness. The opening of fissure 1 is b_1 , the projections are characterized by the values M_1 , S'_1 , S_1 , V_1 . The opening of fissure 2 is b_2 , and the projections are defined by the values M_2 , S'_2 , S_2 , V_2 . The ratio of the hydraulic radii of the two fissures is

$$\frac{R_1}{R_2} = \frac{\frac{b_1 - M_1 V_1}{2 - M_1(S'_1 - S_1)}}{\frac{b_2 - M_2 V_2}{2 - M_2(S'_2 - S_2)}} = \frac{b_1 - M_1 V_1}{b_2 - M_2 V_2} \cdot \frac{2 - M_2(S'_2 - S_2)}{2 - M_1(S'_1 - S_1)} \quad (1.8.13)$$

If the wetted surfaces of both fissures are such that their size is independent of the number and height of the projections, we can set the second fraction on the right-hand side of eqn. (1.8.13) equal to one. The existence of special surfaces can be demonstrated

by considering a set of hemispheres arrayed in a triangular system (equilateral triangles). Let the distance between the centres of the hemispheres be $a_2\Delta_2$. Then the number of projections per unit area of fissure is $M_2 = 4/a_2^2 \Delta_2^2 \sqrt{3}$. The wetted surface of fissure 2 will be

$$2 - M_2(S'_2 - S_2) = 2 - \frac{4}{a_2^2 \Delta_2^2 \sqrt{3}} (\pi \Delta_2^2 - 2\pi \Delta_2^2) = 2 \left(1 + \frac{4\pi}{a_2^2 \sqrt{3}} \right)$$

If the hemispheres are laterally contiguous, $a_2 = 2$, and the size of the wetted surface is independent of Δ_2 . The same holds of fissure 1 whose roughness may be similarly described. If the hemispheres are not contiguous

$$\frac{R_1}{R_2} = \frac{b_1 - M_1 V_1}{b_2 - M_2 V_2} \frac{1 + 2\pi/(a_2^2 \sqrt{3})}{1 + 2\pi/(a_1^2 \sqrt{3})} \quad (1.8.14)$$

Special surfaces are of importance in engineering. We believe that it was Nikurads's regular roughness which facilitated "laminarization" of pipe flow, with the result that instabilities are observed near the critical Reynolds numbers Re_D . Other surfaces e.g. Shevelev's "technological roughness" possess no properties similar to those described; hence the manifestation of instabilities is far less intensive. Since rock fissures are generally of the second kind, we shall make no gross error by ignoring the instabilities.

Of importance for our subsequent studies is the conception of "roughness" of a smooth fissure, which the fissure 2 is supposed to be. We can assume that it contains projections with $\Delta V_2 = 0$. Their spacing is characterized by the numbers $M_1 = M_2$. Using the notation

$$R_1 = R, \quad R_2 = \frac{b}{2}, \quad b_1 = b_2 = b, \quad M_1 = M_2 = M, \quad V_1 = V$$

we obtain for the hydraulic radius of rough fissure the relation

$$R = \frac{b}{2} \left(1 - \frac{MV}{b} \right) \quad (1.8.15)$$

Consider the special case of a triangular array of hemispheres referred to above. If they are not laterally contiguous, eqn. (1.8.14) gives

$$R = \frac{b}{2} \left(1 - \frac{8\pi}{3a^2 \sqrt{3}} \frac{\Delta}{b} \right) \quad (1.8.16)$$

In the limiting case, $\Delta/b = 0.5$ so that the fissure is braced depth-wise. Such a configuration is of practical significance because adjoining blocks can be linked to one another by means of point braces (bridges).

In this particular case

$$R = \frac{b}{2} \left(1 - \frac{4\pi}{3a^2\sqrt{3}} \right) \quad (1.8.17)$$

For $a > 30$, the effect of bridges plays practically no role and so the action of braces is neglected in most practical problems.

Another extreme is the case when the hemispheres form a special surface with $a = 2$. According to eqn. (1.8.15) we obtain

$$R = \frac{b}{2} \left(1 - \frac{2\pi}{3\sqrt{3}} \frac{\Delta}{b} \right) \quad (1.8.18)$$

and for a simultaneous bridge contact, i.e. $\Delta/b = 0.5$,

$$R = \frac{b}{2} \left(1 - \frac{\pi}{3\sqrt{3}} \right) \quad (1.8.19)$$

In this case, braces are an integral component of fissure roughness.

1.8.2. Turbulent flow in fissures filled with water

Nikuradse who followed the effect of roughness on turbulent flow in pipes, made experiments at relative roughnesses $\Delta/D < 1/30$. Since the relative roughness in fissures is considerably greater, we must take into account the effect of turbulent tortuosity, just as we have done when dealing with turbulent flow in soils (Section 1.5.1). In the case in question, however, the “randomness” of the location of the projections is in a sense orderly, and moreover, the effect of the meandering flow is reduced by solid walls. According to our opinion based on the results of experiments reported in the literature as well as on own experimental work, for a fissure with symmetrical roughness

$$\varepsilon = m^{-1/2} \quad (1.8.20)$$

where

ε — is the coefficient of turbulent tortuosity. The value ε^2 includes the effect of tortuosity as well as the effect of the difference in orientation of the fissure centreline and the average direction of curvilinear flow,

m — fissure porosity affected by the number of projections, M , per unit area, and by the volume V of an average projection,

$$m = \frac{b - MV}{b} = 1 - \frac{MV}{b} \quad (1.8.21)$$

Hence

$$\varepsilon = \left(1 - \frac{MV}{b} \right)^{-1/2} \quad (1.8.22)$$

Assume, for example, that the isolated projections are shaped like hemispheres of radius Δ , and that the lines connecting the centres of their bases form a network of equilateral triangles. The distance between centres is an a -multiple of radius Δ . The number of projections per unit area is

$$M = \frac{4}{a^2 \Delta^2 \sqrt{3}}$$

The volume of a projection is

$$V = \frac{2}{3} \pi \Delta^3$$

Hence

$$\varepsilon = \left(1 - \frac{4}{a^2 \Delta^2 \sqrt{3}} \frac{\frac{2}{3} \pi \Delta^3}{b} \right)^{-1/2} = \left(1 - \frac{8\pi}{3a^2 \sqrt{3}} \frac{\Delta}{b} \right)^{-1/2} \quad (1.8.23)$$

In the case of the hemispheres being laterally contiguous, $a = 2$ and

$$\varepsilon = \left(1 - \frac{2\pi}{3 \sqrt{3}} \frac{\Delta}{b} \right)^{-1/2} \quad (1.8.24)$$

According to eqn. (1.8.15), the coefficient of turbulent tortuosity can be expressed in terms of the hydraulic radius, viz.

$$\varepsilon = \left(\frac{2R}{b} \right)^{-1/2} \quad (1.8.25)$$

The knowledge of the value of ε enables us to judge the relations between the character of flow in a pipe and in a relatively rough fissure. To be able to transfer the mathematical description of pipe-flow to fissure-flow we have to know the relations between the pipe diameter D and the equivalent fissure width b as well as between the friction coefficient λ_D in a pipe and the equivalent coefficient λ_b in a fissure.

Assuming the validity of the relation

$$J = \left[\frac{b}{l} (1 + \sum \xi_t) + \chi \lambda_b \right] \frac{1}{b} \frac{v_*^2}{2g} \quad (1.8.26)$$

where

l — length of fissure,

b — opening of fissure,

J — mean hydraulic gradient,

v_* — mean cross-sectional velocity in fissure,

ξ_t — coefficient of local losses in turbulent flow in fissure,

χ — coefficient of fissure curvature,

λ_b — friction factor in fissure.

Comparing with eqn. (1.8.1) and the related formulae we find that

$$D = 2b, \quad \lambda_D = 2\lambda_b \quad (1.8.27)$$

We have already mentioned the fact that Nikuradse's regular roughness represented by grains of sand on pipe walls, has the character of a special surface. Moreover, a regular distribution of grains of equal diameters in contact with one another gives rise to a certain roughness anisotropy. This is best demonstrated by the example in which roughness is represented by hemispheres so arrayed that the lines connecting their centres form a network of equilateral triangles. All the same, we shall start from Nikuradse's formulation.

The tortuosity influences especially the prolongation of water particles trajectories, affects the value of friction losses. Eqns. (1.8.1) and (1.8.26) assume rectilinear flow in general. This hypothesis is further acceptable if the value of friction coefficient is increased in a corresponding manner. Rearranging eqn. (1.8.5) with respect to (1.8.27) and (1.8.18) we arrive at

$$\frac{1}{\sqrt{\lambda_b}} = -\frac{2\sqrt{2}}{\varepsilon} \log \frac{3\sqrt{3}}{14.8\pi} \left(1 - \frac{2R}{b}\right) \quad (1.8.28)$$

For a smooth fissure, we use the Kármán-Prandtl equation (1.8.4) with the Blasius formula (1.8.4a) substituted for λ_D on the right-hand side. After some manipulations we find that

$$\frac{1}{\sqrt{\lambda_b}} = -2\sqrt{2} \log \frac{2 \cdot 4352}{Re_b^{7/8}} \quad (1.8.29)$$

where Re_b is the Reynolds number defined relative to the fissure opening b

$$Re_b = \frac{v_* b}{\nu} \quad (1.8.30)$$

Following a synthesis carried out in the well-established manner of Colebrook and White, we obtain

$$\frac{1}{\sqrt{\lambda_b}} = -\frac{2\sqrt{2}}{2 \cdot 3} \ln \left\{ \frac{2 \cdot 4352}{Re_b^{7/8}} + \left[\frac{3\sqrt{3}}{14.8\pi} \left(1 - \frac{2R}{b}\right) \right]^{1/\varepsilon} \right\} \quad (1.8.31)$$

where ε is obtained from eqn. (1.8.25) and R from eqn. (1.8.15).

Eqn. (1.8.31) applies equally to smooth and to rough fissures. In (1.8.26) the coefficient of local losses ξ_r is calculated from relations established in piping hydraulics. Thus, for example, according to Saint-Venant, we measure the Bord loss that

arises as a fissure opens suddenly from b_1 to b_2 , by the coefficient

$$\xi_t = c \left[\left(\frac{b_2}{b_1} - 1 \right)^2 + \frac{1}{9} \right] \quad (1.8.32)$$

where c is the Coriolis number.

Of essential significance is the fact that the term in the square brackets on the right-hand side of eqn. (1.8.26) characterizes the total magnitude of the losses. We can assume that we are dealing with the effect of fictitious friction which is larger than the effect of actual friction. We thus rearrange eqn. (1.8.26) in the form

$$J = \lambda_t \frac{1}{b} \frac{v_*^2}{2g} \quad (1.8.33)$$

where λ_t is the coefficient of fictitious friction

$$\lambda_t = \frac{b}{l} (1 + \xi_t) + \chi \lambda_b \quad (1.8.34)$$

With the notation

$$\sqrt{\frac{\lambda_t}{2bg}} = C_{*t} \quad (1.8.35)$$

we obtain the simple equation

$$v_* = \frac{1}{C_{*t}} \sqrt{J} = k_{*t} \sqrt{J} \quad (1.8.36)$$

where

C_{*t} – coefficient of resistance to flow in fissure under turbulent regime of flow,

k_{*t} – coefficient of permeability inside fissure under turbulent regime of flow.

It is important to appreciate the role of the coefficient of local losses as regards the coefficient of fictitious friction, the coefficient of resistance, and possibly, the coefficient of permeability. We illustrate this by considering the following example:

In a fissure of length l and opening b , there exists a contracted segment of length Δl and opening $\bar{b}$. The losses at the inlet to, and at the outlet from the contraction will be neglected.

As the equation of continuity implies, the ratio between the velocity v_{*b} in the fissure, and the velocity $v_{*\bar{b}}$ in the contraction is

$$\frac{v_{*b}}{v_{*\bar{b}}} = \frac{\bar{b}}{b} \quad (1.8.37)$$

Hence the Reynolds number is the same in both parts of the fissure. Assuming the walls to be smooth, we would obtain the same friction factor in both segments. Because of possible differences in relative roughness, we consider the friction factor

to be λ_b in the fissure, and $\bar{\lambda}_b$ in the contraction. Generally speaking, the curvature of the fissure can be expressed by the coefficient χ , and in the contracted segment, by the coefficient $\bar{\chi}$. It can readily be shown that

$$\xi_t = \frac{\Delta l}{b} \chi \lambda_b \left[\frac{\bar{\chi} \bar{\lambda}_b}{\chi \lambda_b} \left(\frac{b}{\bar{b}} \right)^3 - 1 \right] - 1 \quad (1.8.38)$$

$$\lambda_t = \chi \lambda_b \left\{ \frac{\Delta l}{l} \left[\frac{\bar{\chi} \bar{\lambda}_b}{\chi \lambda_b} \left(\frac{b}{\bar{b}} \right)^3 - 1 \right] + 1 \right\} \quad (1.8.39)$$

The magnitude of the coefficient of fictitious friction depends on the cube of the ratio between the openings, and on the ratio between the lengths of the contracted and un-contracted fissure. Provided that $\Delta l/l$ is small, a slight contraction is likely to have a considerable effect on the magnitude of λ_t . For $l \gg \Delta l$, the effect of the contraction is insignificant.

In general, the important factors are the length of the fissure considered, and the local losses occurring in the location in question.

1.8.3. Laminar flow in fissures filled with water

Since the Bernouilli equation also applies to laminar flow, we have

$$J = \left[\frac{b}{l} (1 + \sum \xi) + \chi \lambda_b \right] \frac{1}{b} \frac{v_*^2}{2g} \quad (1.8.40)$$

$$\lambda = \frac{b}{l} (1 + \sum \xi) + \chi \lambda_b \quad (1.8.41)$$

$$J = \lambda \frac{1}{b} \frac{v_*^2}{2g} \quad (1.8.42)$$

where

ξ — coefficient of local losses,

λ — coefficient of fictitious friction which also embraces through the intermediary of the coefficient of local losses ξ the effect of other resistances besides that of friction.

As its structure suggests, the equation is generally non-linear. It may become linear when λ happens to be inversely proportional to Re_b and thus also inversely proportional to v_* :

$$\lambda_b = \frac{N_1}{Re_b}, \quad \sum \xi = \frac{N_2}{Re_b} \quad (1.8.43)$$

where N_1, N_2 are constants.

Assume that a laminar regime exists throughout a very rough fissure. We must then make allowance for the effect of tortuosity of particle paths as well as for the effect of the deviation of the fissure centreline from the mean direction of the actual flow. Both effects are accounted for by the coefficient α^2 . For symmetrical roughness

$$\alpha^2 = \left(1 - \frac{MV}{b}\right)^{-1/4} = \left(\frac{2R}{b}\right)^{-1/4} \quad (1.8.44)$$

By integrating the Navier-Stokes equations directly we find that in the case being considered

$$N_1 = \frac{12\alpha^2 b}{R}$$

Hence

$$\lambda_b = \frac{24\alpha^2}{Re_b} \frac{b}{2R} \quad (1.8.45)$$

or

$$\lambda_b = \frac{24}{Re_b} \left(\frac{b}{2R}\right)^{5/4} \quad (1.8.46)$$

In smooth fissures, $M = 0$, $\alpha^2 = 1$, and

$$\lambda_b = \frac{24}{Re_b} \quad (1.8.47)$$

Fissures whose origin can be traced to the disintegration of large pieces of rocks, are a special case, with projections usually in the form of continuous prisms. If we assume that their longitudinal orientation is normal to the direction of flow, we can — when calculating the effect of tortuosity — regard the projections as composed of systems of closely adjoining but separate elementary ribs. Since $V = 0$ for each rib, eqn. (1.8.44) gives

$$\alpha^2 = 1$$

The friction factor is obtained from eqn. (1.8.45)

$$\lambda_b = \frac{24}{Re_b} \frac{b}{2R}$$

Assume that the projections are shaped as indicated in Fig. 1.14j. Their number per unit area is

$$M = \frac{2}{\Delta l}$$

and according to Section 1.8.1

$$V = (b - \bar{b}) \frac{\Delta l + \bar{\Delta l}}{2}$$

$$S = \frac{\Delta l}{\cos \bar{\theta}} + \bar{\Delta l} \left(1 - \frac{1}{\cos \bar{\theta}} \right)$$

Comparing the meandering fissure 1 of opening $\bar{b}$ with the straight fissure 2 of opening b , we obtain from eqn. (1.8.13) for $R_1 = R$

$$R = \frac{\bar{b}}{2} \frac{\frac{b}{\bar{b}} \left(1 + \frac{\bar{\Delta l}}{\Delta l} \right) - \frac{\bar{\Delta l}}{\Delta l}}{\frac{1}{\cos \bar{\theta}} \left(1 - \frac{\bar{\Delta l}}{\Delta l} \right) + \frac{\bar{\Delta l}}{\Delta l}}$$

Hence

$$\lambda_b = \frac{24}{Re_b} \frac{\frac{1}{\cos \bar{\theta}} \left(1 - \frac{\bar{\Delta l}}{\Delta l} \right) + \frac{\bar{\Delta l}}{\Delta l}}{\frac{\bar{b}}{b} \left(1 + \frac{\bar{\Delta l}}{\Delta l} \right) - \frac{\bar{\Delta l}}{\Delta l}}$$

For $\bar{\Delta l} = 0$, i.e. for a case analogous to that shown in Fig. 1.14h

$$\lambda_b = \frac{24b}{Re_b \bar{b}} \frac{1}{\cos \bar{\theta}} = \frac{24}{Re_b} \frac{b}{\bar{b}} \bar{\chi},$$

and it can readily be shown that the result proves eqn. (1.8.10) to be correct.

In fact, we are able to describe two kinds of fissure curvature. The effect of the overall configuration is contained in the coefficient χ which appears in eqn. (1.8.41). The effect of "internal" curvature is covered by the coefficient $\bar{\chi}$ which is already included in eqn. (1.8.45). The concept of double curvature is illustrated schematically in Fig. 1.14k.

To demonstrate the significance of local losses, consider a fissure of length l and opening b , with coefficients λ_b , χ , α^2 . Within the segment l is a contracted section $\bar{l}$ long, of opening $\bar{b}$, with coefficients $\bar{\lambda}_b$, $\bar{\chi}$, $\bar{\alpha}^2$.

It is easy to see that without the inlet and outlet losses

$$\xi = \frac{\Delta l}{b} \chi \lambda_b \left[\frac{\bar{\chi}}{\chi} \frac{\bar{\lambda}_b}{\lambda_b} \left(\frac{\bar{\alpha}}{\alpha} \right)^2 \left(\frac{b}{\bar{b}} \right)^3 - 1 \right] - 1 \quad (1.8.48)$$

Hence

$$\lambda = \chi \lambda_b \left\{ \frac{\Delta l}{l} \left[\frac{\bar{\chi}}{\chi} \frac{\bar{\lambda}_b}{\lambda_b} \left(\frac{\bar{\alpha}}{\alpha} \right)^2 \left(\frac{b}{\bar{b}} \right)^3 - 1 \right] + 1 \right\} \quad (1.8.49)$$

Since the fissure and its contracted section have the same Reynolds number,

$$\lambda = \frac{24}{Re_b} N \quad (1.8.50)$$

where N is a constant.

It was established that in laminar flow the coefficient of fictitious friction is inversely proportional to Re_b . Considering eqn. (1.8.30) and expressing by means of the constant N the effects of roughness and of local losses, we have

$$J = \frac{1}{b} \frac{24\nu}{v_* b} N \frac{v_*^2}{2g} = \frac{12\nu N}{gb^2} v_* \quad (1.8.51)$$

Using the notation

$$\frac{12\nu N}{gb^2} = C_* \quad (1.8.52)$$

where C_* is the coefficient of resistance to laminar flow in a fissure, we obtain the linear relation.

$$v_* = \frac{1}{C_*} J = k_* J \quad (1.8.53)$$

The value C_* is a constant in the absence of local turbulence, or when the effect of local turbulence (and simultaneously, of the velocity head $v_*^2/2g$) can be neglected.

The letter k_* denotes the coefficient of permeability of the fissure under the laminar regime of flow.

1.8.4. Combined regime of flow in fissures filled with water

The character of the $\lambda_b \Leftrightarrow Re_b$ dependence is a feature which enables us to distinguish between the various laws of flow. The literature refers to many practical measurements which were processed into diagrams of the type $\lambda_b = f(Re_b)$. As these diagrams suggest, the relationships applicable at fairly large values of Re_b and comparatively small relative roughnesses are analogous to those of piping hydraulics. The theory outlined in the foregoing sections can also be used in numerical determinations of the required functional dependence. To illustrate, we present below the results of calculations made for the case where the roughness consists of a system of laterally contiguous hemispheres with the lines connecting their centres arranged in a triangular network.

Fig. 1.15a shows the $\lambda_b \Leftrightarrow Re_b$ dependence for the chosen values of relative roughness. The line connecting the breaks in the solid curves forms the boundary between the laminar and the turbulent regime, if — as mentioned earlier — no con-

sideration is given to possible instabilities. For smooth fissures, the critical Reynolds number $Re_b = 950$. For rough fissures, the boundary shifts towards lower values of Re_b . In laminar flow, roughness is practically immaterial, provided that $\Delta/b < 0.05$.

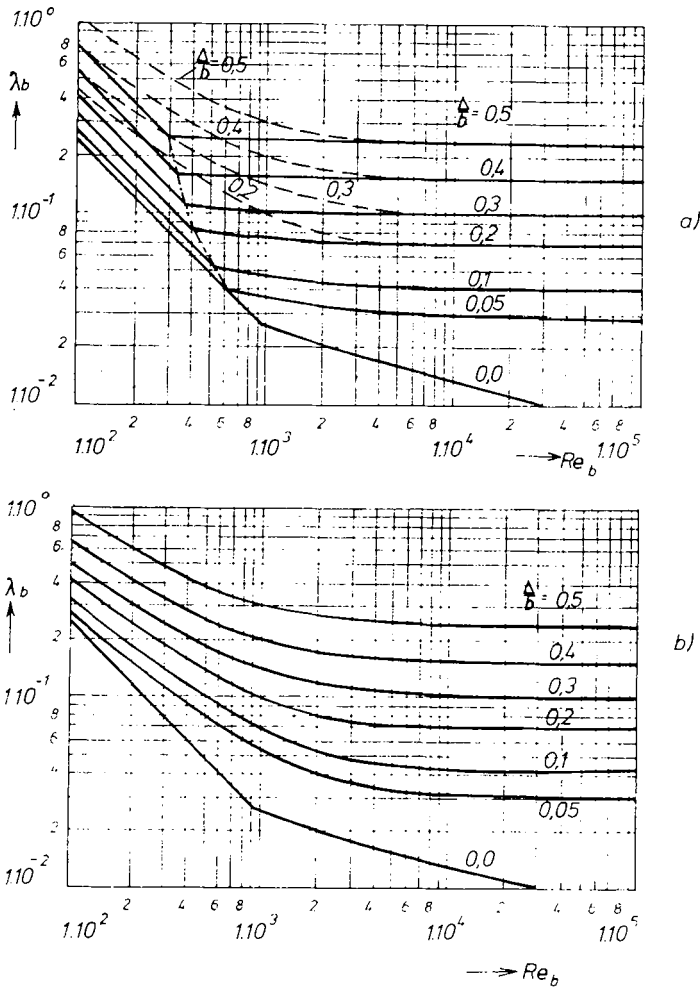


Fig. 1.15. The dependence $\lambda_b := f(Re)_b$ for laminar and turbulent flow.

According to the results of experiments, particularly those reported by Louis [25], smooth fissures give rise to a sharp break of the $\lambda_b \Leftrightarrow Re_b$ curves.

With increasing roughness, the break becomes less sharp, the effect of instabilities gradually disappears, and the character of the flow resembles that of water motion in soils. On the strength of these facts, we made an attempt to set down a description of the combined regime. Proceeding from the principles outlined in

Section 1.5.3, we obtained

$$\lambda_b = \frac{12\alpha^2 b}{Re_b R} + \frac{0.66125}{\ln^2 \left\{ \frac{2.4352}{Re_b^{7/8}} + \left[0.11176 \left(1 - \frac{2R}{b} \right) \right]^{1/2} \right\}} \quad (1.8.54)$$

For the case of hemispherical projections, laterally contiguous and forming a triangular system, the curves $\lambda_b = f(Re_b)$ are shown in Fig. 1.15a (dash lines). Turbulent motion where λ_b is independent of Re_b , is seen to arise at values of the order of 1×10^3 . The linear law of flow may be expected to apply at low Re_b .

A comparison with the results of Louis suggests the possibility of using the combined law of flow at a relative roughness $\Delta/b > 0.15$. However, the results reported in the literature fail to cover the entire range of values of Δ/b . A more detailed check might show that the combined law of flow is applicable to $\Delta/b \geq 0.05$.

Since the term with the exponent $1/\varepsilon$ is predominant for the range of roughness under consideration, we can simplify eqn. (1.8.54) to

$$\lambda_b = \frac{12\alpha^2 b}{Re_b R} + \left\{ \frac{0.813\varepsilon}{\ln \left[0.11176 \left(1 - \frac{2R}{b} \right) \right]} \right\}^2 \quad (1.8.55)$$

The resulting curves calculated using eqn. (1.8.55) for the case of hemispherical projections, laterally contiguous and forming a triangular system, are shown in Fig. 1.15b. They differ only slightly from those obtained using eqn. (1.8.54).

The approximate solution is regarded as relatively acceptable mainly for the reason that roughness, curvature and factors influencing the onset of local losses, are difficult to determine in natural conditions. However, an estimate can be made of the total effect of all these agents.

Consider the hydraulic gradient J and the velocity v_* measured at the inlet to, or at the outlet from, a fissure to be known quantities. For the combined regime of flow, the coefficient of fictitious friction λ' is obtained from the relation

$$\lambda' = \frac{2J}{Fr_b} \quad (1.8.56)$$

where

Fr_b is the Froude number defined relative to the fissure opening b ,

λ' is not a constant, but generally a function of Re_b ,

$$Fr_b = \frac{v_*^2}{gb} \quad (1.8.57)$$

According to the foregoing discussion, we also have

$$\lambda' = 2gb(C'_*)^2 = \frac{2gb}{(k'_*)^2} \quad (1.8.58)$$

$$v_* = k'_* \sqrt{J} \quad (1.8.59)$$

$$k'_* = \sqrt{\frac{2gb}{\lambda'}} \quad (1.8.60)$$

C'_* , k'_* are the coefficients of resistance and permeability, respectively, under the combined regime of flow.

In regions where J/Fr_b is inversely proportional to Re_b , the flow can be claimed to be laminar; in regions where $J/Fr_b = \text{const.}$, it can be expected to be turbulent. In other cases it is uncertain which of the effects predominate.

1.8.5. Continualization of the effect of flow in networks of fissures filled with water

In nature, one often comes across media in which the networks of fissures are shaped fairly regularly. In such cases, the medium is sometimes for convenience regarded as a continuum, and the fictitious velocity v as a continuous function throughout the entire medium.

Take, for example, a system of parallel fissures at a distance B apart (Fig. 1.14l). In a continuum, the flow in the direction of the fissures is described by

$$v_* = v \frac{B}{b} \quad (1.8.61)$$

so that

$$v = \frac{b}{B} \sqrt{\left(\frac{2gb}{\lambda'}\right)} \sqrt{J} = k' \sqrt{J} \quad (1.8.62)$$

where

v — fictitious velocity of flow in the continuum,

k' — coefficient of non-linear (combined) permeability in the continuum,

λ'_B — coefficient of fictitious friction referred to spacing B ,

$$\lambda'_B = \left(\frac{B}{b}\right)^3 \lambda', \quad k' = \sqrt{\frac{2gB}{\lambda'_B}}, \quad C' = \frac{1}{k'} \quad (1.8.63)$$

C' is the coefficient of resistance.

Similarly as in the preceding section, we write

$$\lambda'_B = \frac{2J}{Fr_B} \quad (1.8.64)$$

where Fr_B is the Froude number defined relative to the spacing B

$$Fr_B = \frac{v^2}{gB} \quad (1.8.65)$$

It is easily shown that the Reynolds number defined relative to B is equal to the Reynolds number defined relative to b . If Fr_B is inversely proportional to Re_B , the motion through the continuum is laminar. In that case

$$v = k_* \frac{b}{B} J = kJ \quad (1.8.66)$$

where k is the constant coefficient of permeability in laminar motion.

1.8.6. Flow of water in fissures filled with soil

A fissure filled with soil leaks water just as does a tube filled with soil. The soil filling the fissure is usually of a relatively low permeability, and it is therefore possible that the seepage activity of blocks will also come into play. The motion practically follows the linear law of flow. In a substitute continuum the flow in the direction of the fissure centreline is described by

$$v = \left[k_B + \frac{b}{B} (k_b - k_B) \right] J = kJ, \quad k_b > k_B \quad (1.8.67)$$

where

- k_B — coefficient of permeability of block material,
- k_b — coefficient of permeability of material in fissure,
- b — fissure opening,
- B — spacing of parallel fissures.

1.8.7. Mathematical description of flow in fractured media

A rock massif is divided by fissures into generally oblique blocks. The direction of the centrelines of the fissures is identical with that of the edges of the blocks. Thus there exist three mutually oblique directions (Fig. 1.14m), with the corresponding spacings of fissures B_1, B_2, B_3 and openings b_1, b_2, b_3 . The respective gradients are J_1, J_2, J_3 . Amalgamate the small blocks into larger ones; a fictitious oblique block

produced in this way, will have sides $\psi_1 B_1$, $\psi_2 B_2$, $\psi_3 B_3$. The total quantities of flow Q_i in the directions of the sides (principal directions) of the block are obtained from the equations

$$\begin{aligned} Q_1 &= B_2 \psi_2 B_3 \psi_3 k'_1 \sqrt{J_1} = S_{23} k'_1 \sqrt{J_1} = K'_1 \sqrt{J_1} \\ Q_2 &= B_1 \psi_1 B_3 \psi_3 k'_2 \sqrt{J_2} = S_{13} k'_2 \sqrt{J_2} = K'_2 \sqrt{J_2} \\ Q_3 &= B_1 \psi_1 B_2 \psi_2 k'_3 \sqrt{J_3} = S_{12} k'_3 \sqrt{J_3} = K'_3 \sqrt{J_3} \end{aligned} \quad (1.8.68)$$

where

ψ_1, ψ_2, ψ_3 — number of original blocks making up the fictitious block in the respective direction,

k'_1, k'_2, k'_3 — coefficients of non-linear permeability in the directions of the axes of the oblique system,

S — areas in the direction normal to the chosen principal directions.

Equations (1.8.68) have features in common with relations familiar from hydraulics of pipes. We can, therefore, assume that the flow through the block is represented by the flow through pipes meeting at the point O at the centre of the substitute block.

The entire massif is replaced by a system of blocks, and each block is represented by piping. The system thus forms a piping network. According to Kirchhoff's law, we have that at each node the algebraic sum of the discharge into the node is equal to zero. Consider, for example, one of the nodes O and its neighbouring nodes 1, 2, 3, $\bar{1}$, $\bar{2}$, $\bar{3}$. Then

$$\sum_{i=1}^3 K'_{i0} \sqrt{\frac{h_i - h_0}{\psi_i B_i}} + \sum_{\bar{i}=\bar{1}}^{\bar{3}} K'_{\bar{i}0} \sqrt{\frac{h_{\bar{i}} - h_0}{\psi_{\bar{i}} B_{\bar{i}}}} = 0 \quad (1.8.69)$$

where

$h_i, h_{\bar{i}}$ — piezometric heads at the nodes $i = 1, 2, 3$ and at the nodes $\bar{i} = \bar{1}, \bar{2}, \bar{3}$,
 $K'_{i0}, K'_{\bar{i}0}$ — coefficients applying to the segments $i0$ and $\bar{i}0$.

Analogous equations may be written for each node of the network; the solution of the entire system of equations will yield the distribution of the piezometric heads in each node.

The solution can also be effected by successive approximations. The basic iteration equations applying to each node are

$$(h_0)_n = \frac{\sum_{i=1}^3 \frac{K'_{i,n-1}(h_i)_{n-1}}{\sqrt{[\psi_i B_i (h_i - h_0)_{n-1}]}} + \sum_{\bar{i}=\bar{1}}^{\bar{3}} \frac{K'_{\bar{i},n-1}(h_{\bar{i}})_{n-1}}{\sqrt{[\psi_{\bar{i}} B_{\bar{i}} (h_{\bar{i}} - h_0)_{n-1}]}}}{\sum_{i=1}^3 \frac{K'_{i,n-1}}{\sqrt{[\psi_i B_i (h_i - h_0)_{n-1}]}} + \sum_{\bar{i}=\bar{1}}^{\bar{3}} \frac{K'_{\bar{i},n-1}}{\sqrt{[\psi_{\bar{i}} B_{\bar{i}} (h_{\bar{i}} - h_0)_{n-1}]}}} \quad (1.8.70)$$

Subscripts $n, n - 1$ refer to values in the n -th and $(n - 1)$ th step.

The procedure is as follows: estimate the values of h_0 in all nodes. These estimated piezometric heads are taken for the initial values and used in eqn. (1.8.70)

set up for each node. The equation yields the value $(h_0)_1$ which is generally not the same as the initial value. Consider now the value $(h_0)_1$ as initial and calculate the values $(h_0)_2$ in all nodes. The values obtained in this second step, if not equal to the initial ones, are used in the next approximation. The solution is repeated until $(h_0)_n \approx (h_0)_{n-1}$.

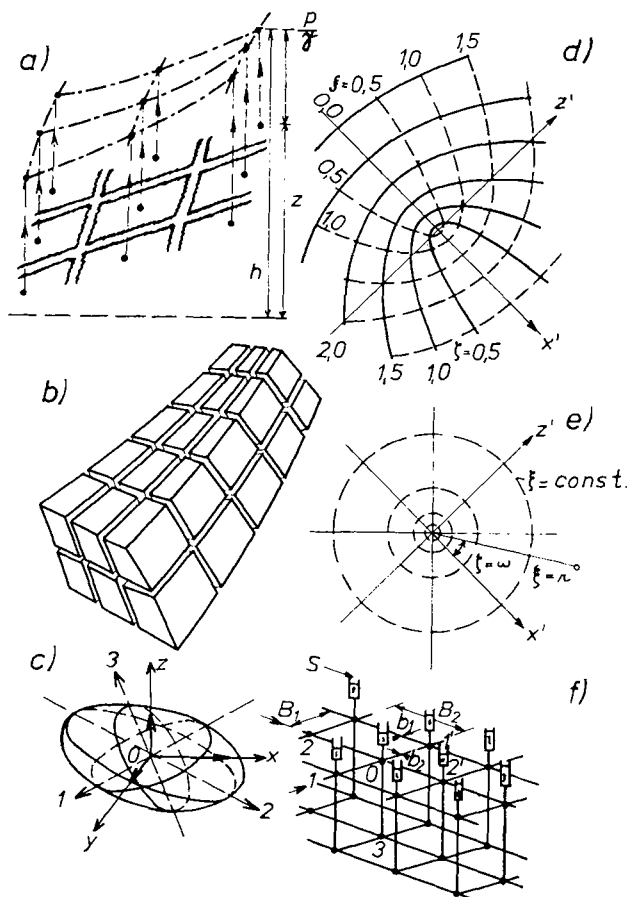


Fig. 1.16. Scheme of calculation of flow in fractured media.

Evidently, each step involves another iteration procedure, for in most cases, K_i, K_f cannot be determined directly. The values of h_0 ascertained by the procedure, enable us to establish the values of h at other points of the blocks. To illustrate, we have prepared the two-dimensional diagram shown in Fig. 1.16a. At the points at which $(h_0)_n$ were calculated, we drew perpendiculars and marked on them the ordinates p/γ obtained from the relation

$$\frac{p}{\gamma} = h - z \quad (1.8.71)$$

The resulting pattern of pressure surfaces provides information on the distribution of pressures at other points, or possibly, on the character of the lines $h = \text{const}$.

The basic property of a medium permeated by a network of fissures, is the total anisotropic permeability. This fact becomes most pronounced in the continualization process. The principal directions of seepage are parallel to the sides of the blocks and are generally oblique. A mathematical description employing Cartesian coordinates is then quite difficult.

The question is how to define the seepage activity given by eqn. (1.8.62) in the oblique system whose axes are taken to be identical with the principal directions of seepage. The velocity v is a vector. The gradient $\sqrt{J}$ is also a vector, and $k' \sqrt{J}$ is the scalar product. Hence k' is a second-degree tensor. In an anisotropically permeable medium the direction of the gradient vector is generally not collinear with the direction of the velocity vector. This property is accentuated still further by the non-linearity of the law of flow (cf. Section 1.5.1).

An idea about the dependence of the value of k' on the chosen direction can be gained by considering the ellipsoid of seepage (Fig. 1.16c). One of its axes lies in one of the principal directions of seepage, preferably in that in which the value of k' is maximal. The semi-major axis of the ellipsoid is then represented by an abscissa corresponding to this maximum value. The corresponding component of k' in another direction is obtained by drawing from the centre of the ellipsoid, O , a ray parallel to the specified direction. The ray intersects the ellipsoid at the point A . The length OA characterizes the corresponding k' .

At any given point, there is generally associated with a value of Re_b a single ellipsoid. A single ellipsoid at a given point is of use in two cases. First, when the flow regime is purely laminar, second, when the flow is turbulent, with the magnitude of the coefficient of fictitious friction independent of Re_b .

In media in which the principal directions of flow vary from one point to another (Fig. 1.16b), the problem is more difficult still.

Eqns. (1.8.68) derived for a discretized medium, are of considerable theoretical significance. In passing from differences to differentials we can — with the help of the Taylor series — evolve partial differential equations describing the flow in a continuum. The resulting relationships are simpler if the linear law of flow applies everywhere and if the principal directions of seepage are parallel to the principal axes of the ellipsoid of seepage. This is essentially a definition of a system of two-dimensional fissures. In the principal directions, the opening of the fissures is constant. We choose the Cartesian coordinate system x', y', z' rotated relative to the system x, y, z (with vertical z axis) and with axes parallel to the principal directions of seepage. The equation of continuity is set up as follows:

$$k_{x'} \frac{\partial^2 h}{\partial (x')^2} + k_{y'} \frac{\partial^2 h}{\partial (y')^2} + k_{z'} \frac{\partial^2 h}{\partial (z')^2} = 0 \quad (1.8.72)$$

This equation can be simplified by considering the coefficients of anisotropy referred,

e.g. to the z' axis:

$$A_{x'z'} = \frac{k_{x'}}{k_{z'}}, \quad A_{y'z'} = \frac{k_{y'}}{k_{z'}} \quad (1.8.73)$$

With the help of the transformations

$$\bar{x} = x' A_{x'z'}^{-1/2}, \quad \bar{y} = y' A_{y'z'}^{-1/2} \quad (1.8.74)$$

we obtain the Laplace equation for a homogeneous medium in the system $\bar{x}, \bar{y}, \bar{z}'$.

The method of isotropization outlined above, represents a solution to the special case when the principal directions of seepage are spatially curved, i.e. when the areas of fissures intersecting orthogonally, are defined in terms of the curvilinear coordinates ξ, η, ζ . The corresponding equation of continuity is of the form

$$\frac{\partial}{\partial \xi} (L_2 L_3 v_\xi) + \frac{\partial}{\partial \eta} (L_1 L_3 v_\eta) + \frac{\partial}{\partial \zeta} (L_1 L_2 v_\zeta) = 0 \quad (1.8.75)$$

where L_1, L_2, L_3 are Lamé's differential parameters:

$$\begin{aligned} L_1 &= \sqrt{\left[\left(\frac{\partial x'}{\partial \xi}\right)^2 + \left(\frac{\partial y'}{\partial \xi}\right)^2 + \left(\frac{\partial z'}{\partial \xi}\right)^2\right]}, \quad v_\xi = -k_\xi \frac{1}{L_1} \frac{\partial h}{\partial \xi} \\ L_2 &= \sqrt{\left[\left(\frac{\partial x'}{\partial \eta}\right)^2 + \left(\frac{\partial y'}{\partial \eta}\right)^2 + \left(\frac{\partial z'}{\partial \eta}\right)^2\right]}, \quad v_\eta = -k_\eta \frac{1}{L_2} \frac{\partial h}{\partial \eta} \\ L_3 &= \sqrt{\left[\left(\frac{\partial x'}{\partial \zeta}\right)^2 + \left(\frac{\partial y'}{\partial \zeta}\right)^2 + \left(\frac{\partial z'}{\partial \zeta}\right)^2\right]}, \quad v_\zeta = -k_\zeta \frac{1}{L_3} \frac{\partial h}{\partial \zeta} \end{aligned} \quad (1.8.76)$$

The differential parameters essentially describe the effect of curvature of the element walls. The equations can be applied to practical problems if one knows the relation of the coordinates ξ, η, ζ to the Cartesian system of coordinates x', y', z' and the values of the coefficients of permeability k_ξ, k_η, k_ζ in the ξ, η, ζ directions:

$$\xi = f_1(x', y', z'), \quad \eta = f_2(x', y', z'), \quad \zeta = f_3(x', y', z') \quad (1.8.77)$$

Substituting eqn. (1.8.76) in eqn. (1.8.75) we obtain the equation

$$\frac{\partial}{\partial \xi} \left(k_\xi \frac{L_2 L_3}{L_1} \frac{\partial h}{\partial \xi} \right) + \frac{\partial}{\partial \eta} \left(k_\eta \frac{L_1 L_3}{L_2} \frac{\partial h}{\partial \eta} \right) + \frac{\partial}{\partial \zeta} \left(k_\zeta \frac{L_1 L_2}{L_3} \frac{\partial h}{\partial \zeta} \right) = 0 \quad (1.8.78)$$

A simpler case arises when one of the principal directions is linear throughout the region. Assume, e.g. that the direction of k_η is the same as the y' direction. Then $L_2 = 1$, $\partial y' / \partial \xi = \partial y' / \partial \zeta = 0$.

For a two-dimensional flow

$$\frac{\partial}{\partial \xi} (L_2 L_3 v_\xi) + \frac{\partial}{\partial \zeta} (L_1 L_2 v_\zeta) = 0 \quad (1.8.79)$$

$$L_1 = \sqrt{\left[\left(\frac{\partial x'}{\partial \xi} \right)^2 + \left(\frac{\partial z'}{\partial \xi} \right)^2 \right]}, \quad L_3 = \sqrt{\left[\left(\frac{\partial x'}{\partial \zeta} \right)^2 + \left(\frac{\partial z'}{\partial \zeta} \right)^2 \right]} \quad (1.8.80)$$

$$\frac{\partial}{\partial \xi} \left(k_\xi \frac{L_3}{L_1} \frac{\partial h}{\partial \xi} \right) + \frac{\partial}{\partial \zeta} \left(k_\zeta \frac{L_1}{L_3} \frac{\partial h}{\partial \zeta} \right) = 0 \quad (1.8.81)$$

Consider, for example, the case when the functions (1.8.77) are actually of the form

$$\begin{aligned} x' &= \xi^2 - \zeta^2, \quad z' = 2\xi\zeta, \quad x' = \xi^2 - \frac{(z')^2}{4\xi^2}, \quad x' = \frac{(z')^2}{4\zeta^2} - \zeta^2 \\ \xi^2 &= \frac{x' \pm \sqrt{[(x')^2 + (z')^2]}}{2}, \quad \zeta^2 = \frac{-x' \pm \sqrt{[(x')^2 + (z')^2]}}{2} \end{aligned} \quad (1.8.82)$$

Eqns. (1.8.82) characterize two systems of curves. One system contains confocal parabolas with the focus at the origin, and the axes of the parabolas identical with the negative x' axis. The other system also consists of confocal parabolas whose axes, however, are identical with the positive x' axis (Fig. 1.16d). According to eqns. (1.8.76) and (1.8.78)

$$L_1 = L_3 = 2\sqrt{(\xi^2 + \zeta^2)} \quad (1.8.83)$$

$$\frac{\partial}{\partial \xi} \left(k_\xi \frac{\partial h}{\partial \xi} \right) + \frac{\partial}{\partial \zeta} \left(k_\zeta \frac{\partial h}{\partial \zeta} \right) = 0 \quad (1.8.84)$$

Consider the coefficient of anisotropy

$$A = \frac{k_\zeta}{k_\xi} \quad (1.8.85)$$

and carry out the transformation

$$\bar{\xi} = \xi \sqrt{A}, \quad \bar{\zeta} = \zeta \quad (1.8.86)$$

Substituting in eqn. (1.8.84) we obtain

$$\frac{\partial}{\partial \bar{\xi}} \left(k_\xi \frac{\partial h}{\partial \bar{\xi}} \right) + \frac{\partial}{\partial \bar{\zeta}} \left(k_\zeta \frac{\partial h}{\partial \bar{\zeta}} \right) = 0 \quad (1.8.87)$$

If $k_\zeta = \text{const. everywhere}$, we obtain the Laplace equation for a homogeneous

isotropic medium

$$\frac{\partial^2 h}{\partial \xi^2} + \frac{\partial^2 h}{\partial \zeta^2} = 0 \quad (1.8.88)$$

After transformation, a point with coordinates x', z' in the original region, will have in the same Cartesian system new coordinates $\bar{x}', \bar{y}'$ defined by

$$\bar{x}' = A\xi^2 - \zeta^2, \quad \bar{z}' = \xi\zeta\sqrt{A} \quad (1.8.89)$$

The above transformation is applicable, for example, to problems of flow in foldings of strata. The condition is that the dividing surfaces between blocks should — approximately at least — be in the form of parabolical cylinders. The condition $k_\zeta = \text{const.}$ should apply in the direction normal to the dividing surfaces.

Let us also note the case when the dividing surfaces are shaped like circular cylinders. In eqn. (1.8.83) set $\xi = r, \zeta = \omega$. In the Cartesian system x', z' introduce the polar coordinates

$$x' = r \cos \omega = \xi \cos \zeta, \quad z' = r \sin \omega = \xi \sin \zeta \quad (1.8.90)$$

Therefore $L_1 = 1, L_3 = \xi$ and

$$\frac{\partial}{\partial \xi} \left(k_\zeta \xi \frac{\partial h}{\partial \xi} \right) + \frac{\partial}{\partial \zeta} \left(k_\zeta \frac{1}{\xi} \frac{\partial h}{\partial \zeta} \right) = 0 \quad (1.8.91)$$

Consider the coefficient of anisotropy

$$A = \frac{k_\xi}{k_\zeta} \quad (1.8.92)$$

and use the transformation functions

$$\bar{\xi} = \xi^{1/\sqrt{A}}, \quad \bar{\zeta} = \zeta \quad (1.8.93)$$

Substituting in eqn. (1.8.91) gives

$$\frac{\partial}{\partial \bar{\xi}} \left(k_\zeta \bar{\xi} \frac{\partial h}{\partial \bar{\xi}} \right) + \frac{1}{\bar{\xi}} \frac{\partial}{\partial \bar{\zeta}} \left(k_\zeta \frac{\partial h}{\partial \bar{\zeta}} \right) = 0 \quad (1.8.94)$$

For $k_\zeta = \text{const.}$, the equation for a homogeneous isotropic medium becomes

$$\frac{\partial}{\partial \bar{\xi}} \left(\bar{\xi} \frac{\partial h}{\partial \bar{\xi}} \right) + \frac{1}{\bar{\xi}} \frac{\partial^2 h}{\partial \bar{\zeta}^2} = 0 \quad (1.8.95)$$

As the form of the transformation functions indicates, the transformation causes the region of flow to deform in the radial direction only (Fig. 1.16e).

The equations derived above assume the open fissure or its filling to be saturated with water. The boundary conditions can be prescribed as a pressure on the boundary

of the region, zero velocity component in the direction of the normal to impervious surfaces, etc. Atmospheric pressure and an additional discharge are assumed on the free surface in the fissure or in a continualized medium.

The process of setting up the equations of the free surface is easier when the entire region is regarded as a continuum. The basic principle of mathematical description was expounded in Section 1.4.4 and subsequent sections.

In a medium divided into blocks and further, into fictitious piping systems, all the nodes are assumed to be underwater. Capacities representing the effect of active porosity and additional discharge in a concentrated form, are incorporated into the branches protruding above the surface (Fig. 1.16f).

1.8.8. Force action of water flowing in fissures

In a strongly anisotropic medium we must always keep in mind the orientation of the system of fissures. The force actions of flow are calculated for real fissures in the untransformed region.

In a single fissure filled with water, the force R_V acting on unit volume of liquid is given by

$$R_V = \gamma J \quad (1.8.96)$$

This equation applies regardless of the law of flow. The shear stress τ transferred to the walls of the fissure is defined by

$$\tau = \frac{b}{2} \gamma J \quad (1.8.97)$$

where

γ — unit weight of liquid,

b — opening of fissure,

B — spacing of parallel fissures.

In a continualized medium the force acting on unit volume of the medium is

$$F = \frac{b}{B} R_V = \gamma \frac{b}{B} J \quad (1.8.98)$$

By definition, the gradient J is an average slope (inclination) along a segment of a definite length. Therefore the forces F , R_V calculated from the equations evolved above, are average values representing the conditions in a certain segment or block in the direction of one of the fissures. The forces acting in the remaining directions are obtained in a similar manner; the resulting action is established by composing these forces according to the principles of mechanics. The direction of the resultants might not be the same as the principal directions of seepage.

CHAPTER 2

ONE-DIMENSIONAL STEADY FLOW OF GROUNDWATER

The hydraulic theory of groundwater motion proposed in Chapter 1 has the incontestable advantage of combining clarity and comprehensiveness with the ability of satisfying the demands imposed on the accuracy of solution by practising engineers. To show that this is indeed the case, we undertake an analysis of water motion in a stratified medium bounded from below by the surface of a relatively impervious subsoil. As shown in Fig. 2.1, we assume that the interfaces between the various layers run parallel to the surface of the relatively impervious subsoil. The coordinate $h_p(x)$ of the bottom edge of the permeable stratum is measured from a definite, chosen, reference plane. The free surface of the groundwater flow lies in the $(n + 1)$ th layer. The thickness of the saturated portion of the permeable stratum is generally $h(x)$. The inclination of the free surface to the reference plane is generally not the same as the inclination of the surface to the relatively impervious subsoil; hence the groundwater motion is non-uniform. The flow velocity is different in each section drawn vertically through the soil profile. The motion would become uniform only in the special case of the inclination of the groundwater free surface being the same as the slope of the surface of the relatively impervious subsoil. Such a case very seldom occurs in nature, however, and for this reason we shall concentrate on the non-uniform motion of groundwater.

In groundwater hydraulics we come across different concepts which are all intended to characterize the quantitative side of groundwater motion. The literature refers to the quantity of seepage and this is understood to mean the quantity of water which flows through a given area per unit time. The same quantitative datum is sometimes embodied in the concept of volume flow. In ascertaining the quantity of water flowing through a strip of unit width and having a depth given by the thickness of the saturated stratum, we usually call this quantity the specific volume flow.

The usage of any particular term is motivated by a variety of considerations. Sometimes we are merely trying to be succinct in our description, but what really matters, in our opinion, is the physical idea. Thus, for example, when solving the one-dimensional non-uniform motion we study the conditions along a vertical line drawn through the whole saturated stratum, and, accordingly, refer the quantitative data to the whole thickness of the relatively permeable saturated medium.

2.1. Non-uniform groundwater motion along the slope of the surface of a relatively impervious subsoil (Fig. 2.1a)

According to eqn. (1.4.84), we find the quantity of water which percolates through a strip of unit width per second using the relation

$$q_x = - \left(\frac{\partial h_p}{\partial x} + \frac{\partial h}{\partial x} \right) [k_{n+1}h + \sum_{i=1}^n T_i(k_i - k_{i+1})] =$$

$$= \frac{\partial}{\partial x} \left\{ -h \left[k_{n+1} \frac{h}{2} + \sum_{i=1}^n T_i(k_i - k_{i+1}) \right] - \frac{\partial h_p}{\partial x} [k_{n+1}h + \sum_{i=1}^n T_i(k_i - k_{i+1})] \right\} \quad (2.1.1)$$

As the formal structure of eqn. (2.1.1) suggests, we are working with ideas mentioned in Section 1.4 in connection with the exposition dealing with the potential G .

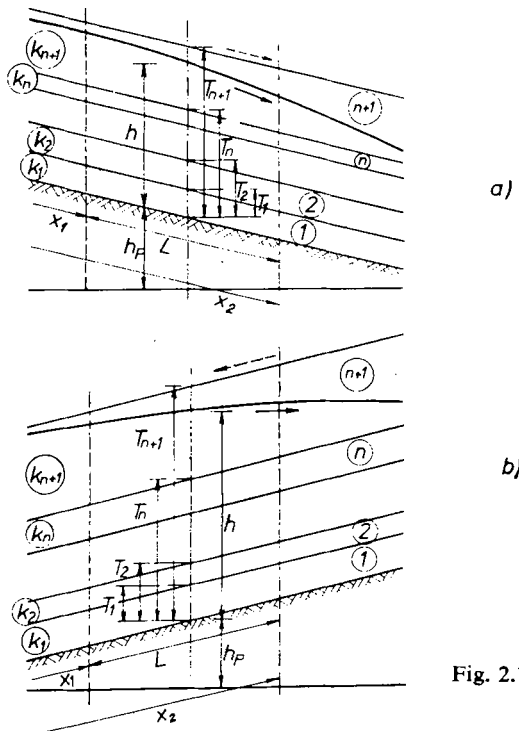


Fig. 2.1. Non-uniform motion.

For uniform motion we obtain

$$q_x = - \frac{\partial h_p}{\partial x} [k_{n+1}h_0 + \sum_{i=1}^n T_i(k_i - k_{i+1})] \quad (2.1.2)$$

where h_0 denotes the thickness of the saturated portion of the stratum.

To steady non-uniform flow applies the equation of continuity which states that the quantity of seepage is the same at any vertical section. Hence the non-uniform flow is quantitatively comparable with a certain fictitious uniform flow which occurs in the saturated stratum of thickness h_0 . Accordingly, by comparing eqn. (2.1.1) with eqn. (2.1.2) we obtain

$$\frac{\partial h}{\partial x} + \frac{\partial h_p}{\partial x} = \frac{\partial h_p}{\partial x} \frac{k_{n+1}h_0 + \sum_{i=1}^n T_i(k_i - k_{i+1})}{k_{n+1}h + \sum_{i=1}^n T_i(k_i - k_{i+1})} \quad (2.1.3)$$

Writing

$$\frac{k_{n+1}h + \sum_{i=1}^n T_i(k_i - k_{i+1})}{k_{n+1}h_0 + \sum_{i=1}^n T_i(k_i - k_{i+1})} = \sigma(x) \quad (2.1.4)$$

and differentiating with respect to x , expression (2.1.4) yields

$$\frac{\partial}{\partial x} [k_{n+1}h + \sum_{i=1}^n T_i(k_i - k_{i+1})] = \frac{\partial}{\partial x} \{ \sigma(x) [k_{n+1}h_0 + \sum_{i=1}^n T_i(k_i - k_{i+1})] \}$$

or, finally, the relation

$$k_{n+1} \frac{\partial h}{\partial x} = [k_{n+1}h_0 + \sum_{i=1}^n T_i(k_i - k_{i+1})] \frac{\partial \sigma}{\partial x}$$

On substituting the latter in eqn. (2.1.3) we have

$$\frac{1}{k_{n+1}} [k_{n+1}h_0 + \sum_{i=1}^n T_i(k_i - k_{i+1})] \frac{\partial \sigma}{\partial x} = \frac{\partial h_p}{\partial x} \left[\frac{1}{\sigma(x)} - 1 \right]$$

With the notation

$$B = \frac{-1}{k_{n+1}} [k_{n+1}h_0 + \sum_{i=1}^n T_i(k_i - k_{i+1})] \quad (2.1.5)$$

we obtain

$$-B \frac{\partial \sigma}{\partial x} = \frac{\partial h_p}{\partial x} \frac{\sigma(x) - 1}{\sigma(x)} \quad (2.1.6)$$

Equation (2.1.6) represents the basic relation for calculating the non-uniform motion in a stratified medium whenever the flow occurs along the slope of the surface of the relatively impervious medium. Let us assume, for the purpose of integration, that the course of both the surface of the relatively impervious medium and the

interface between the layers of the permeable medium is linear. Then the inclination, J_p , of the layer is

$$J_p = - \frac{\partial h_p}{\partial x} = \text{const.} \quad (2.1.7)$$

and, in one-dimensional flow, eqn. (2.1.6) is replaced by the equation

$$\frac{J_p}{B} dx = \frac{\sigma(x)}{\sigma(x) - 1} d\sigma \quad (2.1.8)$$

Eqn. (2.1.8) is integrated between the limits x_1 and x_2 , and between the limits σ_1 and σ_2 , viz.,

$$\frac{J_p}{B} \int_{x_1}^{x_2} dx = \int_{\sigma_1}^{\sigma_2} \frac{\sigma}{\sigma - 1} d\sigma \quad (2.1.9)$$

The result of the integration of eqn. (2.1.9) depends on the magnitude of σ . For $\sigma > 1$ we have the case when, in a given profile, the free surface in non-uniform motion lies higher than it does in the quantitatively comparable uniform motion. The function $h(x)$ then describes the curve of the rising surface. Integrating between the limits $x = x_1$ and $x = x_2$ we obtain

$$\frac{J_p}{B} [x]_{x_1}^{x_2} = [\sigma + \ln(\sigma - 1)]_{\sigma_1}^{\sigma_2} \quad (2.1.10)$$

and for $x_2 - x_1 = L$

$$\frac{J_p L}{B} = \sigma_2 - \sigma_1 + \ln \frac{\sigma_2 - 1}{\sigma_1 - 1} \quad (2.1.11)$$

For $\sigma < 1$, on the other hand, we have the case when, in a given profile, the free surface in non-uniform motion lies lower than that in the quantitatively comparable uniform motion. The function $h(x)$ then describes the curve of the falling surface. The integration of eqn. (2.1.9) between the limits $x = x_1$ and $x = x_2$ yields

$$\frac{J_p}{B} [x]_{x_1}^{x_2} = [\sigma + \ln(1 - \sigma)]_{\sigma_1}^{\sigma_2} \quad (2.1.12)$$

and for $x_1 - x_2 = L$

$$\frac{J_p L}{B} = \sigma_2 - \sigma_1 + \ln \frac{1 - \sigma_2}{1 - \sigma_1} \quad (2.1.13)$$

2.2. Non-uniform groundwater motion against the slope of the surface of a relatively impervious subsoil (Fig. 2.1b)

When applying the basic equations we must bear in mind that uniform motion is possible only in the direction of the inclined surface of the relatively impervious subsoil. For that reason we compare the quantity of water flowing per second through the flow region in non-uniform motion with the fictitious volume flow which, in uniform motion, would move in the opposite direction, i.e.,

$$q_x = \frac{\partial h_p}{\partial x} [k_{n+1} h_0 + \sum_{i=1}^n T_i (k_i - k_{i+1})] \quad (2.2.1)$$

Comparing the above with eqn. (2.1.1) we obtain

$$\frac{\partial h}{\partial x} + \frac{\partial h_p}{\partial x} = - \frac{\partial h_p}{\partial x} \frac{k_{n+1} h_0 + \sum_{i=1}^n T_i (k_i - k_{i+1})}{k_{n+1} h + \sum_{i=1}^n T_i (k_i - k_{i+1})} \quad (2.2.2)$$

On introducing the function $\sigma(x)$ according to eqn. (2.1.4) and using, as before, the letter B according to expression (2.1.5) we obtain the basic equation of non-uniform flow in the form

$$B \frac{\partial \sigma}{\partial x} = \frac{\partial h_p}{\partial x} \frac{1 + \sigma}{\sigma} \quad (2.2.3)$$

On the assumption that the course of the surface of the relatively impervious subsoil, as well as that of the interface between the layers of the permeable medium, is linear, we accept eqn. (2.1.7) and have

$$\frac{J_p}{B} dx = - \frac{\sigma}{1 + \sigma} d\sigma \quad (2.2.4)$$

Integrating between the limits x_1 and x_2 , and σ_1 and σ_2 we obtain

$$\frac{J_p}{B} [x]_{x_1}^{x_2} = [-\sigma + \ln(1 + \sigma)]_{\sigma_1}^{\sigma_2} \quad (2.2.5)$$

For $x_2 - x_1 = L$ this becomes

$$\frac{J_p L}{B} = \sigma_1 - \sigma_2 + \ln \frac{1 + \sigma_2}{1 + \sigma_1} \quad (2.2.6)$$

2.3. Approximate solution of one-dimensional non-uniform motion

When considering the results obtained in the theoretical analysis of non-uniform motion, we see, at first glance, that the best way of obtaining the solution to practical problems is by means of an iteration procedure, or by calculation with the help of tables. The right-hand sides of eqns. (2.1.11), (2.1.13) and (2.2.6) do not differ from the expressions derived by Pavlovsky [31] for motion in homogeneous soil, whose numerical values are set out in tables.

The value which we need first and foremost in practical solutions is h_0 . We determine this by substituting in the basic equations. The required h_0 is established either by integration, or graphically. The next step consists of obtaining the dependence $\sigma = f(h)$ for various values of h chosen at will; the resulting figure shows the value of h corresponding to a certain σ . The corresponding distances L are obtained by solving the basic equations for definite, chosen, values of h (and hence also for known values of σ). The procedure is clearly analogous to that used in the calculation of rising, or falling, free surfaces in river streams. The tedium of the computation naturally increases with the number of layers in the permeable medium.

It is for this reason that we frequently turn to approximate procedures, including those based on empirical knowledge. We are of the opinion that an approximate solution in which the value of h in the second term on the right-hand side of eqn. (2.1.1) has been replaced by some average value, h_s , will be satisfactory for most practical problems. In expressing the volume flow per unit width of stream we make use of the potential G according to Section 1.4.7 and write

$$q_x = \frac{\partial G}{\partial x} \pm \frac{\partial h_p}{\partial x} [k_{n+1}h_s + \sum_{i=1}^n T_i(k_i - k_{i+1})] \quad (2.3.1)$$

where h_s is the average thickness of the saturated portion of the permeable stratum.

The sign of the second term in eqn. (2.3.1) is chosen to suit the direction of flow and the direction of inclination of the surface of the relatively impervious subsoil. If the water flows in the direction of the inclined surface of the relatively impervious subsoil, the sign is the same as that of $\partial G/\partial x$.

It is first assumed in the calculation of h that the surface of the relatively impervious subsoil is horizontal. The values of h thus determined are then taken for the ordinates of the points of the free surface, measured from the actual linear surface of the relatively impervious subsoil. The solution of eqn. (2.3.2) — obtained by substituting eqn. (2.3.1) in eqn. (1.4.77) —

$$\frac{\partial^2 G}{\partial x^2} = 0 \quad (2.3.2)$$

for the specific boundary conditions, will clearly be sufficient for our purpose.

We draw attention to some elementary cases, such as the flow between two parallel observation profiles: at the point $x = 0$, it is $G = G_1$, at the point $x = L$, it is $G = G_2$. Eqn. (2.3.2) will be satisfied by the relation

$$G(x) = G_1 - (G_1 - G_2) \frac{x}{L} \quad (2.3.3)$$

$$\frac{\partial G}{\partial x} = - \frac{G_1 - G_2}{L} \quad (2.3.4)$$

To illustrate the generality of the proposed solution, consider the flow through a layer of thickness $T_1 = T$. The coefficient of permeability is $k_1 = k$. There is a less permeable material above the permeable layer and the flow occurs under pressure (i.e. the free surface virtually coincides with the pressure line and lies within the range of the less permeable upper layer). We clearly have

$$k_{n+1}h \ll \sum_{i=1}^n T_i(k_i - k_{i+1})$$

$$G = -hTk + \frac{kT^2}{2}$$

$$G_1 - G_2 = -kT(h_1 - h_2)$$

Substitution in eqn. (2.3.3) gives

$$\begin{aligned} -hTk + \frac{kT^2}{2} &= -h_1kT + \frac{kT^2}{2} + kT(h_1 - h_2) \frac{x}{L} \\ h &= h_1 - \frac{h_1 - h_2}{L} x \end{aligned} \quad (2.3.5)$$

and substitution in eqn. (2.3.4) yields

$$\frac{\partial G}{\partial x} = kT \frac{h_1 - h_2}{L} \quad (2.3.6)$$

In eqn. (2.3.1) of non-uniform motion we shall use the average value

$$h_s = \frac{h_1 + h_2}{2} \quad (2.3.7)$$

Consider the other extreme case of pressure-free flow through a single layer with $k_1 = k$. According to the definition (see Section 1.4), the potential G is given by

$$G = -k \frac{h^2}{2}$$

On substituting the above in eqn. (2.3.3) we have

$$-k \frac{h^2}{2} = -k \frac{h_1^2}{2} + k \left(\frac{h_1^2}{2} - \frac{h_2^2}{2} \right) \frac{x}{L}$$

which leads to

$$h = \sqrt{\left[h_1^2 - (h_1^2 - h_2^2) \frac{x}{L} \right]} \quad (2.3.8)$$

Substitution in eqn. (2.3.4) gives

$$\frac{\partial G}{\partial x} = \frac{k}{2L} (h_1^2 - h_2^2) \quad (2.3.9)$$

Equation (2.3.9) may also be written as follows:

$$\frac{\partial G}{\partial x} = \frac{k}{L} \frac{(h_1 - h_2)(h_1 + h_2)}{2} = \frac{k}{L} \left(h_2 + \frac{h_1 - h_2}{2} \right) (h_1 - h_2) \quad (2.3.10)$$

Expression (2.3.10) will become identical with expression (2.3.6) on the assumption that

$$T = h_2 + \frac{h_1 - h_2}{2} = \frac{h_1 + h_2}{2}$$

It is, therefore, recommended that the value of h_s according to eqn. (2.3.7) is used in the other extreme case as well.

2.4. Effect of infiltration in non-uniform flow

The indisputable advantage of the approximate method of solving non-uniform flow, outlined above, is its simplicity. The problem is essentially examined on the assumption that the surface of the relatively impervious subsoil is horizontal. The results so obtained are then supplemented so as also to take account of the effect of the inclination of the layers in the permeable strata. In the discussion that follows we shall concentrate on the solution of one-dimensional flow assuming a horizontal surface of the relatively impervious subsoil.

One of the problems worthy of special attention involves the motion in an earth mass between two rivers, assuming seepage from the surface of the terrain. It is assumed that no continuous layer of water is formed on the terrain level; all the water falling vertically, with a seepage velocity v_0 , penetrates unhindered into the ground. Now suppose that the depth of the free surface plays no role in the process and thus (here departing from reality) all the water which has percolated into the ground reaches the free surface (Fig. 2.2a).

Using eqn. (1.4.84) we modify the initial equation (1.4.67) written for the steady state ($\partial h / \partial t = 0$) to

$$\frac{\partial^2 G}{\partial x^2} - v_0 = 0 \quad (2.4.1)$$

The general integral of the above equation is

$$G = \frac{v_0}{2} x^2 + C_1 x + C_2 \quad (2.4.2)$$

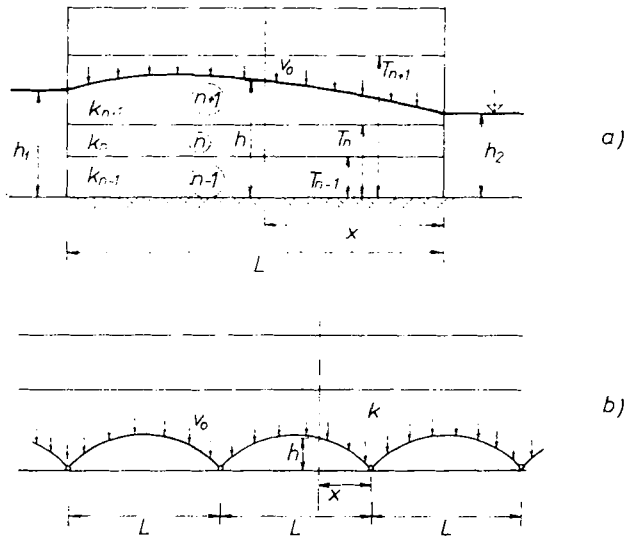


Fig. 2.2. Examples of one-dimensional flow problems.

At the point $x = 0$, $G = G_2$, and hence

$$G_2 = C_2$$

at the point $x = L$,

$$G_1 = \frac{v_0}{2} L^2 + C_1 L + G_2, \quad C_1 = \frac{G_1 - G_2}{L} - \frac{v_0 L}{2}$$

For the given boundary conditions we evidently obtain

$$G = \left[\frac{G_1 - G_2}{L} - \frac{v_0}{2} (L - x) \right] x + G_2 \quad (2.4.3)$$

The quantity of water seeping per unit time through a strip of unit width is

$$q_x = \frac{\partial G}{\partial x} = \frac{G_1 - G_2}{L} - v_0 \left(\frac{L}{2} - x \right) \quad (2.4.4)$$

On the left-hand edge, where $x = L$, it is given by

$$q_1 = \frac{G_1 - G_2}{L} + v_0 \frac{L}{2} \quad (2.4.5)$$

and on the right-hand edge, where $x = 0$, it becomes

$$q_2 = \frac{G_1 - G_2}{L} - v_0 \frac{L}{2} \quad (2.4.6)$$

The depression curve may possibly have an extreme which we determine from the condition that

$$\frac{\partial G}{\partial x} = 0 \quad (2.4.7)$$

This condition is satisfied by the relation

$$x = \frac{L}{2} - \frac{G_1 - G_2}{Lv_0} \quad (2.4.8)$$

If x turns out to be positive, a practical extreme exists. For a positive value of x in eqn. (2.4.8) we obtain the following value of G .

$$G = \frac{1}{2} \left[(G_1 - G_2) \left(1 - \frac{G_1 - G_2}{L^2 v_0} \right) - \frac{v_0 L^2}{4} \right] + G_2 \quad (2.4.9)$$

We wish to reiterate that v_0 denotes the flow velocity of water which enters the free surface from above. If we consider a definite, constant quantity of water seeping from the earth massif region, v_0 will be constant along the whole depression curve only if this quantity reaches the depression curve along its whole length, no matter how far down from the surface of the region its points lie. In this case, the description of the phenomenon with the help of the potential G is of significance, for it offers the possibility of a unified exposition of both the pressure and the pressure-free flows. It is sufficient to read from the graph of the function $G = f(h)$ the values of h corresponding to the ascertained values of G .

As a simple example consider the flow into a system of drains according to Fig. 2.2b, where $G_1 = G_2 = 0$. In this case, according to eqn (2.4.3) we obtain

$$G = - \frac{v_0 x}{2} (L - x) \quad (2.4.10)$$

$$G = -k \frac{h^2}{2} \quad (2.4.11)$$

from which it follows that

$$h = \sqrt{\left[\frac{v_0 x}{k} (L - x) \right]} \quad (2.4.12)$$

Midway between the drains $x = L/2$, so that

$$h_{\max} = \frac{L}{2} \sqrt{\frac{v_0}{k}} \quad (2.4.13)$$

The quantity of seepage in the section given by the abscissa x will be

$$q_x = -v_0 \left(\frac{L}{2} - x \right) \quad (2.4.14)$$

Using eqns. (2.4.5) and (2.4.6) for the left-hand and the right-hand edge, respectively, the quantity of seepage is given as

$$q_1 = v_0 \frac{L}{2}, \quad q_2 = -v_0 \frac{L}{2} \quad (2.4.15)$$

The result is consistent with the physical idea; the flow is, in fact, produced only by the contribution of the infiltration from above.

2.5. Quasi-one-dimensional flow

The theory of the potential G assumes the equipotential lines to be vertical. In many instances, especially whenever continuous seams of less permeable soils are found in a permeable stratum, this assumption is not likely to be fulfilled. To illustrate this, consider the situation depicted in Fig. 2.3a. The flow in the massif between two rivers takes place through a permeable stratum. On the very bottom is a highly permeable layer with a comparatively high coefficient of permeability. This is overlaid with a less permeable layer of thickness T_{II} , the material of which has a coefficient of permeability k_{II} . The third layer from the bottom is relatively permeable, with a thickness $T_{III} = T$, and a coefficient of permeability $k_{III} = k$. There follows a less permeable layer of thickness T_{IV} with a coefficient of permeability k_{IV} . On the very top is a highly permeable layer in which lies a nearly horizontal free surface. The earth massif is closed in a way that allows the water to enter the ground only through the edges of the middle layer.

In this case there exist three permeable horizons in the flow region. The first belongs to the pressure regime in the lowest layer (line h_I). We shall describe the situation in the middle permeable layer with the help of the pressure line which connects onto the levels in the left-hand, and the right-hand, reservoirs. The last horizon relates to the free surface in the uppermost layer (line h_V).

It is immaterial from where the water in the lower and upper horizons comes. We assume that the conditions as outlined above correspond to an instantaneous situation. Through the middle layer the water flows mainly in the horizontal direction.

Through the two adjoining, less permeable layers, the water percolates mainly in the vertical direction. The flow is essentially one-dimensional, but there exist several one-dimensional streams of various senses in the flow region. Such a motion is called quasi-one-dimensional.

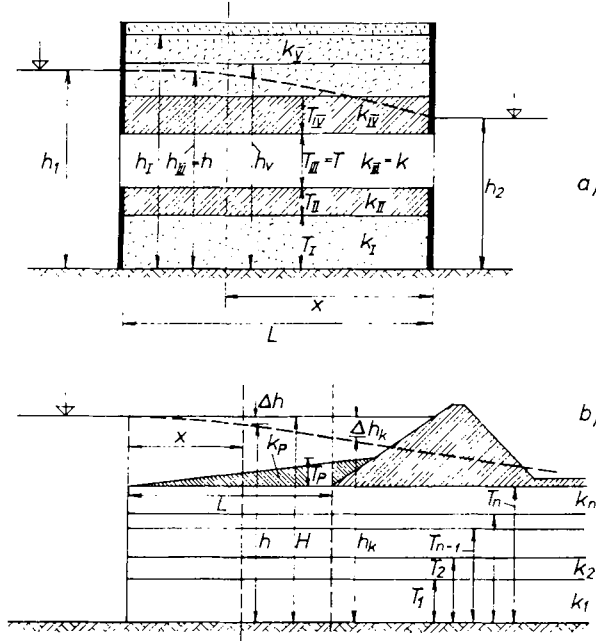


Fig. 2.3. Examples of quasi-one-dimensional flow.

In the sense of eqn. (1.4.61) we write the continuity equation as follows:

$$-kT \frac{\partial^2 h}{\partial x^2} = \frac{h_I - h}{T_{II}} k_{II} + \frac{h_V - h}{T_{IV}} k_{IV} \quad (2.5.1)$$

From this we obtain

$$\begin{aligned} -\frac{\partial^2 h}{\partial x^2} &= \frac{1}{kT} \left(\frac{k_{II} h_I}{T_{II}} + \frac{k_{IV} h_V}{T_{IV}} \right) - \frac{h}{kT} \left(\frac{k_{II}}{T_{II}} + \frac{k_{IV}}{T_{IV}} \right) = \\ &= \frac{1}{kT} \frac{k_{II} h_I T_{IV} + k_{IV} h_V T_{II}}{T_{II} T_{IV}} - \frac{h}{kT} \left(\frac{k_{II} T_{IV} + k_{IV} T_{II}}{T_{II} T_{IV}} \right) = \\ &= \frac{k_{II} T_{IV} + k_{IV} T_{II}}{k T T_{II} T_{IV}} \left(\frac{k_{II} T_{IV} h_I + k_{IV} T_{II} h_V}{k_{II} T_{IV} + k_{IV} T_{II}} - h \right) = \\ &= \frac{1}{kT} \left(\frac{k_{II}}{T_{II}} + \frac{k_{IV}}{T_{IV}} \right) \left(\frac{k_{II} T_{IV} h_I + k_{IV} T_{II} h_V}{k_{II} T_{IV} + k_{IV} T_{II}} - h \right) \end{aligned} \quad (2.5.2)$$

To simplify this we use the notation

$$A^2 = \frac{1}{kT} \left(\frac{k_{II}}{T_{II}} + \frac{k_{IV}}{T_{IV}} \right) \quad (2.5.3)$$

$$B = \frac{k_{II}T_{IV}h_I + k_{IV}T_{II}h_V}{k_{II}T_{IV} + k_{IV}T_{II}} \quad (2.5.4)$$

and introduce the function

$$\Delta h = B - h \quad (2.5.5)$$

so that

$$\frac{\partial^2(\Delta h)}{\partial x^2} = -\frac{\partial^2 h}{\partial x^2} \quad (2.5.6)$$

Substituting in eqn. (2.5.2) we have

$$\frac{\partial^2(\Delta h)}{\partial x^2} = A^2(\Delta h) \quad (2.5.7)$$

The general integral of this equation (see Section 2.6) is

$$\Delta h = C_1 \cosh Ax + C_2 \sinh Ax \quad (2.5.8)$$

at the point $x = 0$, $\Delta h = \Delta h_2$ and, thus

$$\Delta h_2 = C_1$$

at the point $x = L$, $\Delta h = \Delta h_1$ so that

$$\Delta h_1 = \Delta h_2 \cosh AL + C_2 \sinh AL$$

and hence

$$C_2 = -\frac{\Delta h_2 \cosh AL - \Delta h_1}{\sinh AL}$$

For the specified boundary conditions we, therefore, obtain the solution

$$\begin{aligned} \Delta h &= \Delta h_2 \cosh Ax - \frac{\Delta h_2 \cosh AL - \Delta h_1}{\sinh AL} \sinh Ax \\ \Delta h &= \Delta h_2 \left(\cosh Ax - \frac{\sinh Ax}{\tanh AL} \right) + \Delta h_1 \frac{\sinh Ax}{\sinh AL} \end{aligned} \quad (2.5.9)$$

For small values of AL we have approximately,

$$\Delta h = \Delta h_2 \left(1 - \frac{x}{L} \right) + \Delta h_1 \frac{x}{L} = \Delta h_2 + (\Delta h_1 - \Delta h_2) \frac{x}{L} \quad (2.5.10)$$

Using eqn. (2.5.5) this may be expressed as

$$h = h_2 + (h_1 - h_2) \frac{x}{L} \quad (2.5.11)$$

Equation (2.5.11) yields the same result as the analysis of a purely one-dimensional flow in the middle layer. The quasi-one-dimensional flow almost changes into a truly one-dimensional flow on the condition that

$$AL = L \sqrt{\left[\frac{1}{kT} \left(\frac{k_{II}}{T_{II}} + \frac{k_{IV}}{T_{IV}} \right) \right]} < 0.25 \quad (2.5.12)$$

If inequality (2.5.12) applies, the existence of the outer horizons has virtually no effect on the flow in the middle layer. The same conclusion also follows from the calculation of the quantity of water flowing per unit time through a strip of unit width

$$\begin{aligned} q_x &= -kT \frac{\partial h}{\partial x} = kT \frac{\partial(\Delta h)}{\partial x} \\ q_x &= kTA \left[\Delta h_2 \left(\sinh Ax - \frac{\cosh Ax}{\tanh AL} \right) + \Delta h_1 \frac{\cosh Ax}{\sinh AL} \right] \end{aligned} \quad (2.5.13)$$

For $AL < 0.25$ this becomes

$$q_x = kTA \left(-\Delta h_2 \frac{1}{AL} + \Delta h_1 \frac{1}{AL} \right) = \frac{kT}{L} (\Delta h_1 - \Delta h_2), \quad (2.5.14)$$

and in view of eqn. (2.5.5) we obtain

$$q_x = -\frac{kT}{L} (h_1 - h_2) = \text{const.} \quad (2.5.15)$$

The quantity of seepage does not vary along the length of the middle layer. This means that the outer horizons have no effect on the flow.

The example just discussed is actually an extreme, seldom encountered in nature. Yet it has served us well in showing that a thorough analysis of the bearing conditions in a permeable stratum is worth doing. Its results will suggest the theoretical approach to the given problem and, in turn, the choice of a suitable theoretical scheme of the solution.

The principle of the solution of the quasi-one-dimensional flow plays a significant role in studies dealing with the effect of the permeability of upstream blankets. Fig. 2.3b shows a blanket of triangular shape joined to a relatively impervious dam. At the place of contact with the dam (i.e. at the root) the thickness of the blanket is T_p . The coefficient of permeability of the blanket material is k_p . The subsoil of the dam contains permeable strata. The water flows underneath the dam horizontally, directly

from the river, and also takes up the water which has passed through the blanket vertically, from above. The blanket gives rise to a pressure loss which attains the value Δh_k at the root of the blanket. The pressure line connects onto the river level and has an ordinate h_k at the root.

We shall assume that the strata in the dam subsoil have such properties that the water motion may be described in terms of the theory of the potential G . Since from the point of view of the flow underneath the dam we follow the pressure seepage, we shall write the expression of the potential G (1.4.83) for $k_{n+1} = 0$, viz.

$$G = -h \sum_{i=1}^n T_i(k_i - k_{i+1}) + \frac{1}{2} \sum_{i=1}^n T_i^2(k_i - k_{i+1}) \quad (2.5.16)$$

Taking the velocity of water flow through the blanket to be v_z , we can write, for steady flow, according to eqn. (1.4.67)

$$\frac{\partial G}{\partial x} = v_z \quad (2.5.17)$$

Using, simultaneously, eqn. (1.4.84) we obtain

$$\frac{\partial^2 G}{\partial x^2} = v_z \quad (2.5.18)$$

with

$$v_z = \frac{H - h}{T_p \frac{x}{L}} k_p \quad (2.5.19)$$

Differentiating eqn. (2.5.16) twice with respect to x and substituting eqn. (2.5.19) in eqn. (2.5.18) we have

$$-\frac{\partial^2 h}{\partial x^2} \sum_{i=1}^n T_i(k_i - k_{i+1}) = \frac{k_p L}{T_p x} (H - h) \quad (2.5.20)$$

On introducing the substitution

$$\Delta h = H - h \quad (2.5.21)$$

and the notation

$$\sum_{i=1}^n T_i(k_i - k_{i+1}) = kT \quad (2.5.22)$$

we obtain

$$\frac{\partial^2(\Delta h)}{\partial x^2} = \frac{k_p}{kT T_p} \frac{L}{x} \Delta h \quad (2.5.23)$$

For

$$\alpha = \sqrt{\frac{k_p L}{kT T_p}} \quad (2.5.24)$$

the basic differential equation takes the form

$$\frac{\partial^2(\Delta h)}{\partial x^2} - \alpha^2 \frac{\Delta h}{x} = 0 \quad (2.5.25)$$

The general integral of this equation contains a linear combination of the modified Bessel functions of the first and the second kind

$$\Delta h = \sqrt{(x)} Z_1(2\alpha \sqrt{x}) ; \quad (2.5.26)$$

in particular, for the case being considered

$$\Delta h = \sqrt{(x)} [C_1 I_1(2\alpha \sqrt{x}) + C_2 K_1(2\alpha \sqrt{x})] \quad (2.5.27)$$

where $I_1(2\alpha \sqrt{x})$ is the first-order modified Bessel function of the first kind, and $K_1(2\alpha \sqrt{x})$ is the first-order modified Bessel function of the second kind. C_1, C_2 are constants. According to the specification of the problem $\Delta h = 0$ in $x = 0$, so that $C_2 = 0$. At the point $x = L$, we have $\Delta h = \Delta h_k$ and thus

$$\begin{aligned} \Delta h_k &= \sqrt{(L)} C_1 I_1(2\alpha \sqrt{L}) \\ C_1 &= \frac{\Delta h_k}{\sqrt{(L)} I_1(2\alpha \sqrt{L})} \end{aligned}$$

The particular solution for our boundary conditions is of the form

$$\Delta h = \Delta h_k \sqrt{\frac{x}{L}} \frac{I_1(2\alpha \sqrt{x})}{I_1(2\alpha \sqrt{L})} \quad (2.5.28)$$

We shall also require the first and the second derivative of eqn. (2.5.28) with respect to x , namely

$$\frac{\partial(\Delta h)}{\partial x} = \frac{\alpha(\Delta h_k) I_0(2\alpha \sqrt{x})}{\sqrt{(L)} I_1(2\alpha \sqrt{L})} \quad (2.5.29)$$

where I_0 is the zero-order modified Bessel function of the first kind, and

$$\frac{\partial^2(\Delta h)}{\partial x^2} = \frac{\alpha^2(\Delta h_k) I_1(2\alpha \sqrt{x})}{\sqrt{(xL)} I_1(2\alpha \sqrt{L})} \quad (2.5.30)$$

Using eqn. (2.5.29) we calculate the volume flow which moves per unit time below the dam

$$q_x = kT \frac{\partial(\Delta h)}{\partial x} \quad (2.5.31)$$

At the point $x = L$, i.e. at the root of the blanket

$$q_L = \frac{kT\alpha(\Delta h_k) I_0(2\alpha \sqrt{L})}{\sqrt{L} I_1(2\alpha \sqrt{L})} \quad (2.5.32)$$

The increment of volume flow which percolates through the blanket per unit time and unit plan area of the blanket is obtained with the aid of eqn. (2.5.30), viz.

$$q_z = kT \frac{\alpha^2(\Delta h_k) I_1(2\alpha \sqrt{x})}{\sqrt{(xL)} I_1(2\alpha \sqrt{L})} \quad (2.5.33)$$

at the point $x = L$, i.e. at the root of the blanket this is

$$q_z = \frac{kT\alpha^2(\Delta h_k)}{L} = \frac{k_p}{T_p} \Delta h_k \quad (2.5.34)$$

Of interest from the engineering point of view is the gradient, J , of the flow in the location where the blanket contacts the subsoil stratum. We clearly have

$$J = \frac{\Delta h}{T_p \frac{x}{L}} = \frac{\Delta h L}{T_p x}$$

$$J = \frac{\Delta h_k}{T_p} \sqrt{\frac{L}{x}} \frac{I_1(2\alpha \sqrt{x})}{I_1(2\alpha \sqrt{L})} \quad (2.5.35)$$

Also of interest is the result for small values of the argument. For $\alpha \sqrt{L} < 0.1$, we have, approximately

$$I_1(2\alpha \sqrt{x}) = \alpha \sqrt{x}$$

$$I_0(2\alpha \sqrt{x}) = 1$$

Accordingly, in place of eqn. (2.5.28) we obtain the broader relation

$$\Delta h = \Delta h_k \sqrt{\frac{x}{L}} \frac{\alpha \sqrt{x}}{\alpha \sqrt{L}} = \Delta h_k \frac{x}{L} \quad (2.5.36)$$

and, similarly, in place of eqn. (2.5.32) the broader relation

$$q_x = \frac{kT\alpha(\Delta h_k)}{\sqrt{L}} \frac{1}{\alpha \sqrt{L}} = \frac{kT(\Delta h_k)}{L} = \text{const.} \quad (2.5.37)$$

For small values of the argument the seepage through the blanket will hardly manifest itself at all. From this conclusion follows the criterion of relative imperviousness in the form

$$L \sqrt{\frac{k_p}{kTT_p}} < 0.1 \quad (2.5.38)$$

If we choose the material of the blanket and the latter's thickness at the root in a way that will make inequality (2.5.38) hold good in our case, we can regard the effect of the permeability of the blanket as being very slight on the whole.

2.6. Method of fragments for one-dimensional and quasi-one-dimensional flow

The example calculation for the triangular blanket, outlined in the previous section, was intended primarily to serve as an illustration because a structure along such lines could never be realized in practice for technological reasons. Moreover, no attention has so far been paid to the situation underneath and downstream of the dam.

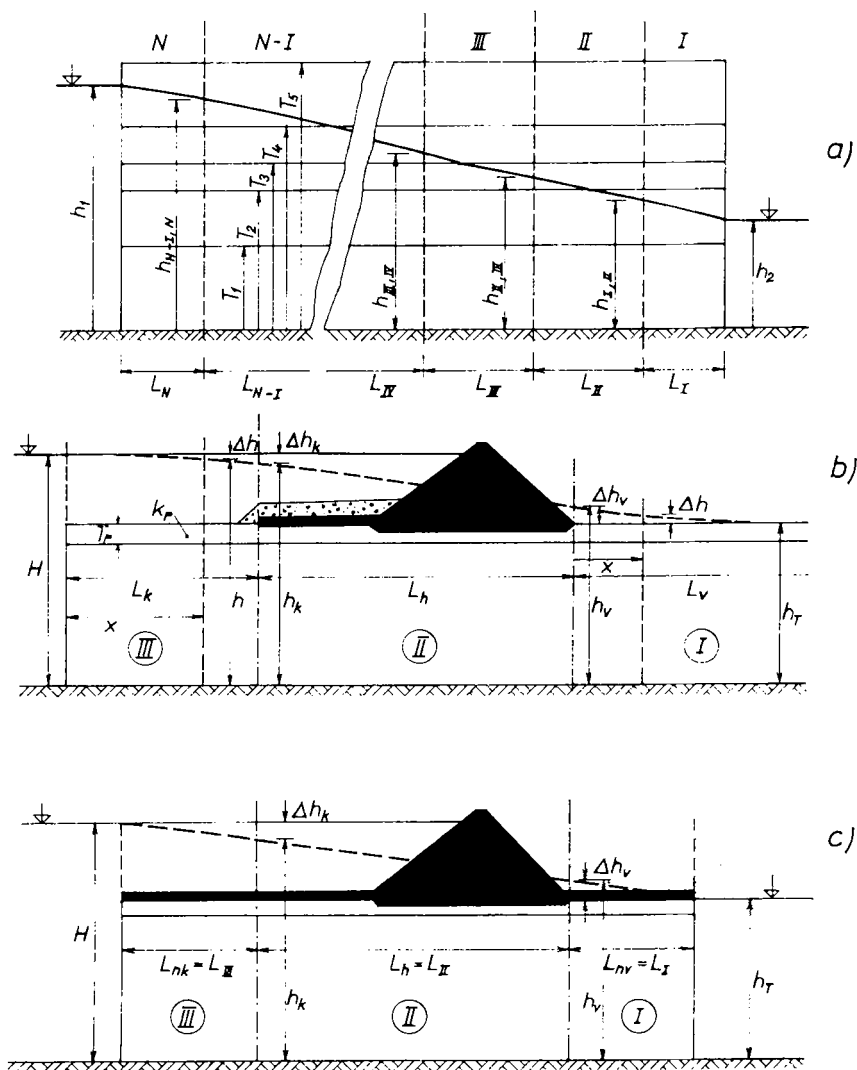


Fig. 2.4. Solution of quasi-one-dimensional flow by the method of fragments.

To describe the conditions there, we shall make use of the fact that a flow field may be divided along the potential lines into parts (fragments), each of which may be examined separately. In one-dimensional flow, the equipotential lines are assumed to be vertical. The vertical dividing sections are chosen in accordance with Fig. 2.4a showing a total of N fragments. Within a fragment, the assumptions of the G potential theory apply; on the edges of each fragment we have certain values of h and certain values of G corresponding to them. Using the results obtained in Section 2.3, we can determine the volume flow (quantity of water flowing through a given area per unit time) in each fragment. Since in steady flow the equation of continuity applies, we may write for a unit width of the field

$$\frac{G_{I,II} - G_2}{L_I} = \frac{G_{II,III} - G_{I,II}}{L_{II}} = \dots = \frac{G_1 - G_{N-I,N}}{L_N} \quad (2.6.11)$$

which in terms of differences of the values of the function G may be written as

$$\frac{\Delta G_I}{L_I} = \frac{\Delta G_{II}}{L_{II}} = \dots = \frac{\Delta G_N}{L_N} \quad (2.6.2)$$

It follows from these equalities that

$$\Delta G_{II} = \Delta G_I \frac{L_{II}}{L_I}, \quad \Delta G_{III} = \Delta G_I \frac{L_{III}}{L_I}, \quad \dots, \quad \Delta G_N = \Delta G_I \frac{L_N}{L_I} \quad (2.6.3)$$

It also holds that

$$G_1 - G_2 = \sum_{j=1}^n \Delta G_j \quad (2.6.4)$$

Substituting eqn. (2.6.3) in eqn. (2.6.4) we obtain

$$G_1 - G_2 = \Delta G_I \left(1 + \frac{L_{II}}{L_I} + \frac{L_{III}}{L_I} + \dots + \frac{L_N}{L_I} \right) = \Delta G_I \frac{\sum_{j=1}^N L_j}{L_I} \quad (2.6.5)$$

Eqns. (2.6.3) and (2.6.5) were written for a basic fragment of length L_I . Similar reasoning applies to any other fragment chosen to be the basic one. Generally we have for the j -th fragment

$$\Delta G_j = \frac{(G_1 - G_2) L_j}{\sum_{j=1}^N L_j} \quad (2.6.6)$$

Hence on the dividing lines between the fragments

$$G_{j-I,j} = G_2 + (G_1 - G_2) \frac{\sum_{j=1}^{j-I} L_j}{\sum_{j=1}^N L_j} \quad (2.6.7)$$

We have thus arrived at the boundary values of G on the dividing lines between the fragments. If we know the values on the edges, we can easily calculate the course of the function G (and hence also of the height h) inside each fragment.

The applicability and possible modifications of the method of fragments will now be demonstrated by means of an example relating to the flow underneath a dam built on a relatively less permeable overburden (Fig. 2.4b). The dam as well as the upstream blanket are assumed to be relatively impervious. Water percolates into the ground both from the river and through the permeable overburden upstream of the blanket. After passing through the dam subsoil it flows into the hinterland. The overburden downstream of the dam is subjected to the uplift pressure so that the water percolates from below to the surface of the terrain. As the percolated water is supposed to run off the surface of the ground, the terrain forms the free surface.

We shall divide the flow domain into three parts in a way that will make each of them into a typical case of flow. According to Fig. 2.4b, the third part is located in the area upstream of the dam and bounded by a vertical line passing through the end of the blanket. The water is assumed to flow horizontally through the dam subsoil and vertically through the overburden (quasi-one-dimensional flow). The overall effect of the resistance of the third part will manifest itself as a loss, Δh_k . According to our ideas, the overburden is of constant thickness T_p ; hence the velocity of water seeping downward through it, is

$$v_z = k_p \frac{H - h}{T_p} = k_p \frac{\Delta h}{T_p} \quad (2.6.8)$$

The flow is described by eqn. (2.5.18) after substitution of eqn. (2.6.8) and the second derivative of the function (2.5.16). Rearranging with the help of substitution of eqn. (2.5.21), yields the basic equation

$$\frac{\partial^2(\Delta h)}{\partial x^2} = \beta^2 \Delta h \quad (2.6.9)$$

$$\beta = \sqrt{\frac{k_p}{k T T_p}}$$

The value of the product kT is obtained from eqn. (2.5.22).

Eqn. (2.6.9) is readily solved on the assumption that at $x = 0$,

$$\Delta h = C_1$$

where C_1 is a constant.

Setting

$$\frac{\partial(\Delta h)}{\partial x} = \beta C_2$$

(where C_2 is also a constant), we obtain directly from eqn. (2.6.9) that

$$\frac{\partial^2(\Delta h)}{\partial x^2} = \beta^2 C_1$$

We continue the differentiation

$$\frac{\partial^3(\Delta h)}{\partial x^3} = \beta^2 \frac{\partial(\Delta h)}{\partial x} = \beta^2 \beta C_2 = \beta^3 C_2$$

$$\frac{\partial^4(\Delta h)}{\partial x^4} = \beta^4 C_1$$

and substitute the results in the McLaurin's series:

$$\Delta h = C_1 + C_2 \beta x + C_1 \frac{\beta^2 x^2}{2!} + C_2 \frac{\beta^3 x^3}{3!} + C_1 \frac{\beta^4 x^4}{4!} + \dots$$

From the series expanded as follows

$$\Delta h = C_1 \left(1 + \frac{\beta^2 x^2}{2!} + \frac{\beta^4 x^4}{4!} + \dots \right) + C_2 \left(\beta x + \frac{\beta^3 x^3}{3!} + \frac{\beta^5 x^5}{5!} + \dots \right)$$

we obtain directly

$$\Delta h = C_1 \cosh \beta x + C_2 \sinh \beta x \quad (2.6.10)$$

For the value $\Delta h = 0$ at the point $x = 0$, and for the value $\Delta h = \Delta h_k$ at the point $x = L_k$ we obtain

$$\Delta h = \Delta h_k \frac{\sinh \beta x}{\sinh \beta L_k} \quad (2.6.11)$$

The quantity of water passing through an arbitrary vertical line with abscissa x per unit time, is

$$q_x = kT \frac{\partial(\Delta h)}{\partial x} = kT\beta(\Delta h_k) \frac{\cosh \beta x}{\sinh \beta L_k} \quad (2.6.12)$$

at the point $x = L$

$$q_{Lk} = \frac{kT\beta}{\operatorname{tgh} \beta L_k} \Delta h_k \quad (2.6.13)$$

at the point $x = 0$

$$q_{0k} = \frac{kT\beta}{\sinh \beta L_k} \Delta h_k \quad (2.6.14)$$

The quantity seeping through the overburden per unit area is

$$q_z = kT \frac{\partial^2(\Delta h)}{\partial x^2} = kT\beta^2 \frac{\sinh \beta x}{\sinh \beta L_k} \Delta h_k$$

and the volume flow through the whole length of the overburden

$$Q = kT\beta^2 \frac{\Delta h_k}{\sinh \beta L_k} \int_0^{L_k} \sinh \beta x \, dx = kT\beta(\Delta h_k) \operatorname{tgh} \frac{\beta L_k}{2} \quad (2.6.15)$$

Clearly, eqn. (2.6.15) can also be obtained by subtracting eqn. (2.6.14) from eqn. (2.6.13). The relationships deduced imply that

$$\beta L_k = L_k \sqrt{\frac{k_p}{kTT_p}} > 3, \quad (2.6.16)$$

the quantity q_{0k} tends relatively to zero so that a substantial amount of seepage passes through the overburden. The infiltration from the river is minimal.

This conclusion is of great practical value, for on the strength of it, we may allow — in very long headwater areas — disturbances in the overburden and plan for example, with the construction of material pits. The condition which they must satisfy is that they should be situated at a distance

$$L_k \geq \frac{3}{\beta} \quad (2.6.17)$$

In part *I*, placed downstream of the dam (Fig. 2.4b) we shall examine the upward seepage of water from the subsoil, through the overburden to the surface of the terrain. As we consider the process to be quasi-one-dimensional, we shall use the same mathematical description as in part *III*. In this particular case the solution of the basic equation (2.6.9) is

$$\Delta h = \frac{\sinh \beta(L_v - x)}{\sinh \beta L_v} \Delta h_v \quad (2.6.18)$$

where

L_v — length of the segment downstream of the dam, in which upward seepage occurs,

Δh_v — elevation of the pressure line above the surface of the terrain on the downstream toe.

The volume flow which percolates through the subsoil in a profile with abscissa x is determined from the equation

$$q_x = kT\beta \frac{\cosh \beta(L_v - x)}{\sinh \beta L_v} \Delta h_v$$

at the point $x = L_v$

$$q_{L_v} = \frac{kT\beta}{\sinh \beta h_v} \Delta h_v$$

at the point $x = 0$

$$q_{0_v} = \frac{kT\beta}{\tanh \beta L_v} \Delta h_v$$

The total flow through the overburden on the downstream side of the dam is

$$Q = kT\beta(\Delta h_v) \tanh \frac{\beta L_v}{2}$$

In steady flow, the region of upward seepage is usually comparatively long, with the $3/\beta$ -long portion downstream of the dam taking a fairly active part in the seepage. The rest of the length L_v shows little activity. Large values of βL_v are a justifiable reason for simplifying our equations. Following the necessary manipulations, the elevation of the pressure line above the surface of the terrain is determined from the equation

$$\Delta h = \Delta h_v \text{Exp}(-\beta x)$$

A quantitative evaluation of the flow is achieved with the help of the relations

$$q_{L_v} = 0$$

$$q_{0_v} = Q = kT\beta(\Delta h_v)$$

Through the overburden, the water flows in the upward direction. The corresponding hydraulic gradient is

$$J = \frac{\Delta h}{T_p} = \Delta h_v \frac{\text{Exp}(-\beta x)}{T_p}$$

The maximum gradient

$$J = \frac{\Delta h_v}{T_p}$$

is on the downstream toe of the dam.

Since the process taking place underneath the dam proper is pressure seepage, the flow in part *II* according to Fig. 2.4b is typically one-dimensional, linking with the quasi-one-dimensional flows in parts *I*, *III*. We shall establish the respective connections by replacing the quasi-one-dimensional flows by one-dimensional flows of equivalent effect. We know, for example, that the quantity seeping through the end of the blanket in part *III*, is q_{L_k} according to eqn. (2.6.13). Assume that the imper-

vious blanket extends over the entire headwater area. We require that the quantity of water flowing under the blanket should be the same as that at the edge of part *III*. From this requirement follows the length of the substitute blanket, L_{nk} :

$$kT \frac{\Delta h_k}{L_{nk}} = \frac{kT\beta}{\tanh \beta L_k} \Delta h_k, \quad L_{nk} = \frac{\tanh \beta L_k}{\beta}$$

The quasi-one-dimensional flow in part *III* is replaced by a one-dimensional flow occurring in the horizontal direction. The length of the streamline in the substitute strip will be L_{nk} .

Similarly, we shall replace the quasi-one-dimensional flow downstream of the dam by an equivalent one-dimensional flow. The length of the substitute strip, L_{nv} , is calculated from the equation

$$kT \frac{\Delta h_v}{L_{nv}} = kT\beta(\Delta h_v), \quad L_v \rightarrow \infty$$

which leads to

$$L_{nv} = \frac{1}{\beta}$$

The scheme of the substitute flow region is shown in Fig. 2.4c. The impervious blanket covers the whole subsoil of the dam. The entrance and emergence profiles on the edges of the region are considered to be vertical. On the assumption that $L_{nk} = L_{III}$, $L_h = L_{II}$, $L_{nv} = L_I$, the scheme thus obtained is the same as that shown in Fig. 2.4a.

Now let us examine the pressure flow in the substitute scheme. According to eqn. (2.5.16) it is generally the case on the dividing line between the j -th and $(j - I)$ th fragment that

$$G_{j-I,j} = -h_{j-I,j} \sum_{i=1}^n T_i(k_i - k_{i+1}) + \frac{1}{2} \sum_{i=1}^n T_i^2(k_i - k_{i+1})$$

In the case being considered, eqn. (2.6.7) yields for the dividing lines between the fragments

$$\begin{aligned} G_{j-I,j} = & -h_T \sum_{i=1}^n T_i(k_i - k_{i+1}) + \frac{1}{2} \sum_{i=1}^n T_i^2(k_i - k_{i+1}) - \\ & - (H - h_T) \sum_{i=1}^n T_i(k_i - k_{i+1}) \frac{\sum_{j=I}^{j-I} L_j}{\sum_{j=I}^N L_j} \end{aligned}$$

The two relations apply simultaneously. On comparing them, we obtain for the

dividing line between parts *I* and *II*, where $h_{j-I,j} = h_v$,

$$h_v = h_T + (H - h_T) \frac{L_I}{L_I + L_{II} + L_{III}};$$

hence

$$\Delta h_v = h_v - h_T$$

On the dividing line between parts *II* and *III*, where $h_{j-I,j} = h_k$, we obtain similarly

$$h_k = h_T + (H - h_T) \frac{L_I + L_{II}}{L_I + L_{II} + L_{III}};$$

hence

$$\Delta h_k = H - h_k$$

The boundary values Δh_k , Δh_v obtained in the above manner, will now be substituted into the equations which describe in detail the situation inside the various fragments.

The approximate method outlined above makes use of schematized patterns of flow. As suggested by a comparison of the results obtained using this method, with those of laboratory tests of un-schematized cases, the two sets of data only differ seriously in cases where the thickness of the subsoil layer is larger by far than the width of the dam and of the upstream blanket.

2.7. One-dimensional flow under the combined law of flow

The basic ideas of the methodological approach to engineering problems can also be used to deal with flows described by non-linear equations of motion. As an illustration consider the seepage through an earth massif where the effect of post-linearity plays a major role in some parts of the region.

The scheme shown in Fig. 2.5 indicates the boundary conditions of the solution and the orientation of the axis of abscissae. The medium is assumed to have comparatively large permeability; consequently, the pre-linear component of the combined law can be disregarded. By eqn. (1.5.27)

$$\lambda_c = \frac{2gd}{vk} \left(1 + \frac{vk}{k_i^2} \right); \quad (2.7.1)$$

hence, according to eqn. (1.5.29)

$$k_c = k_i \sqrt{\frac{vk}{vk + k_i^2}} \quad (2.7.2)$$

where

- λ_c — coefficient of friction in combined regime of flow,
- k_t — coefficient of permeability in developed turbulence,
- k — coefficient of permeability in region of validity of linear law,
- k_c — coefficient of permeability in combined regime of flow.

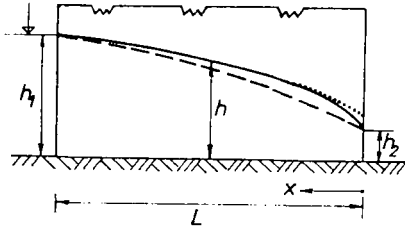


Fig. 2.5. Flow of water in a highly permeable medium.

In keeping with eqn. (1.5.30) we consider the equation of the flow velocity v to be

$$v^2 = k_t^2 \frac{vk}{vk + k_t^2} \frac{dh}{dx} \quad (2.7.3)$$

Assume that the flow occurs predominantly in the horizontal direction so that on any vertical line

$$q = vh \quad (2.7.4)$$

where q is the quantity of seepage per unit time and unit width of the massif [m²/s].

The basic differential equations are obtained on substituting eqn. (2.7.4) in eqn. (2.7.3) and rearranging, viz.

$$\frac{q}{kh} + \frac{q^2}{k_t^2 h^2} = \frac{dh}{dx} \quad (2.7.5)$$

The solution of eqn. (2.7.5) is

$$\frac{q}{k} x = \frac{1}{2} \left(h + \frac{qk}{k_t^2} \right)^2 - \frac{2qk}{k_t^2} \left(h + \frac{qk}{k_t^2} \right) + \left(\frac{qk}{k_t^2} \right)^2 \ln \left(h + \frac{qk}{k_t^2} \right) + c$$

The constant c is calculated from the condition that $h = h_2$ at the point $x = 0$. After some manipulation we obtain

$$\frac{q}{k} x = \frac{1}{2} (h - h_2) \left(h + h_2 - \frac{2qk}{k_t^2} \right) + \left(\frac{qk}{k_t^2} \right)^2 \ln \frac{h + qk/k_t^2}{h_2 + qk/k_t^2} \quad (2.7.6)$$

The value of q is determined by successive approximations, proceeding from eqn. (2.7.6) on substituting $x = L$, $h = h_1$. As the form of the resulting equation suggests, in practical calculations aimed at determining the course of the free surface, it is expedient to choose the values of h and seek the corresponding values of x .

Judging by eqn. (2.7.6), the effect of turbulence will not be too strong provided we can accept that

$$\frac{qk}{k_1^2} \ll h_2, \quad \frac{qk}{k_1^2} \rightarrow 0 \quad (2.7.7)$$

In this particular case we have

$$\frac{q}{k} x = \frac{1}{2}(h^2 - h_2^2) \quad (2.7.8)$$

For $x = L$, $h = h_1$

$$q = \frac{k}{2L}(h_1^2 - h_2^2) \quad (2.7.9)$$

so that

$$h = \sqrt{\left[(h_1^2 - h_2^2) \frac{x}{L} + h_2^2\right]} \quad (2.7.10)$$

It can easily be shown that eqn. (2.7.10) is identical with eqn. (2.4.3) applied to the case being considered ($v_0 = 0$, $G = -kh^2/2$).

In coarse materials, on the other hand, the predominating effect is that of turbulence. In such a case

$$\lambda_c = \frac{2gd}{k_1^2}, \quad k_c = k_t = \text{const.} \quad (2.7.11)$$

According to eqn. (1.5.52), in one-dimensional flow

$$\frac{\partial}{\partial x} \sqrt{\frac{\partial G_t}{\partial x}} = 0 \quad (2.7.12)$$

By solving eqn. (2.7.12) for the prescribed boundary conditions: $G_t = G_{t2}$ at the point $x = 0$, $G_t = G_{t1}$ at the point $x = L$, we obtain

$$G_t = (G_{t1} - G_{t2}) \frac{x}{L} + G_{t2} \quad (2.7.13)$$

Since eqn. (1.5.48) simultaneously applies there, we have

$$h = \sqrt{\left[(h_1^3 - h_2^3) \frac{x}{L} + h_2^3\right]} \quad (2.7.14)$$

and according to eqn. (1.5.45)

$$q = \sqrt{\frac{\partial G_t}{\partial x}} = \sqrt{\frac{(G_{t1} - G_{t2})}{L}} = k_t \sqrt{\frac{(h_1^3 - h_2^3)}{3L}} \quad (2.7.15)$$

An idea about the character of the free surface may be gained from Fig. 2.5 which shows the depression curve corresponding to the case of predominating turbulence as a solid line, and that corresponding to the assumed linear law of flow as a dashed line.

The procedure outlined in the foregoing can be modified without difficulty to apply to non-uniform flow in a medium lying on an inclined surface of a relatively impervious subsoil. An analysis of this problem would, however, yield nothing essentially new.

We observe in conclusion the fact that the assumptions of the theory of one-dimensional flow are not always acceptable. Thus, for example, a surface of seepage can arise in the region where the water leaves the earth massif. In the neighbourhood of this surface, the character of the depression curve differs somewhat from that obtained by theoretical calculations (dotted line in Fig. 2.5). The causes and effects of this phenomenon will be discussed in subsequent chapters.

CHAPTER 3

TWO-DIMENSIONAL STEADY FLOW OF GROUNDWATER

In many cases of groundwater flow the liquid particles move in planes parallel to one another. The character of the flow is the same at all points of a straight line drawn at right angles to those planes.

Such a flow is a two-dimensional steady seepage flow, and the corresponding seepage problem can be solved as a two-dimensional one. Since the liquid particles move in a plane, the velocity vectors also lie in that plane. Therefore, we choose any of the planes in which the motion takes place, and obtain a solution in that plane. In the solution, the length of the flow region in the direction normal to the plane of flow, is taken to be equal to unity. The total flow for the entire flow region is then obtained by multiplying the results of the plane problem by the actual length of the region, L .

In this chapter we intend to show how to solve cases of two-dimensional flow in a vertical plane. The method of solving two-dimensional flows in a horizontal plane will be explained in Chapter 5.

The assumption of two-dimensional flow means a great simplification. On the strength of it we can examine many, otherwise intractable cases, because a mathematical treatment of three-dimensional seepage flows is only feasible in a few, very simple problems. Fortunately, the majority of practical problems are essentially cases of two-dimensional flow; for example, the seepage through earth dams, canals, etc., where one dimension of the structure exceeds by far all the other dimensions, and the flow takes place in a plane normal to that dimension. Sometimes a flow of a three-dimensional character can be converted to a two-dimensional flow with the help of a suitable scheme.

3.1. Fundamental equations of two-dimensional steady seepage flow

In a steady two-dimensional seepage flow through a homogeneous and isotropic medium, all quantities depend on two coordinates only. The fundamental equations of this flow are obtained by modifying the general equations of seepage flow derived in Section 1.4.

The coordinates used in the examination of two-dimensional flows in a vertical plane are as shown in Fig. 3.1, where the plane of the seepage flow is identified with

the plane of the coordinate axes x, y . The fundamental equations of the two-dimensional steady seepage flow in a homogeneous isotropic medium then become

$$v_x = \frac{\partial \varphi}{\partial x} = -k \frac{\partial h}{\partial x}, \quad v_y = \frac{\partial \varphi}{\partial y} = -k \frac{\partial h}{\partial y} \quad (3.1.1)$$

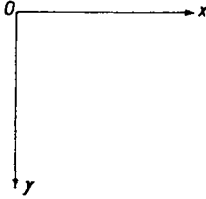
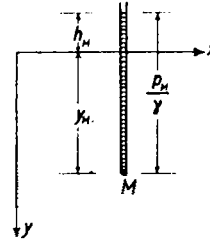


Fig. 3.1.

Fig. 3.2. Hydrostatic pressure and piezometric head at the point M .

where v_x and v_y are the components of seepage velocity in the direction of the coordinate axes, $\varphi(x, y)$ is the potential of seepage flow

$$\varphi = -kh \quad (3.1.2)$$

$h(x, y)$ is the piezometric head at the point (x, y) above the chosen reference plane. For the direction of the coordinate axes being considered

$$h = \frac{p}{\gamma} - y + C \quad (3.1.3)$$

where $p(x, y)$ is the hydrostatic ($\doteq$ total) pressure at the point (x, y) , C is a constant dependent on the choice of the reference plane used in the determination of the piezometric head h . If we put this reference plane on the level of the axis x (Fig. 3.2), $C = 0$.

In our case the continuity equation is

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = 0 \quad (3.1.4)$$

By substituting in eqn. (3.1.4) according to eqn. (3.1.1) or (3.1.2) we obtain Laplace's equation of the potential φ :

$$\Delta \varphi = \frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} = 0 \quad (3.1.5)$$

or of the piezometric head h :

$$\Delta h = \frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2} = 0 \quad (3.1.6)$$

Hence both the potential ϕ and the piezometric head h are harmonic functions of the coordinates of points in the region of seepage. Solving eqn. (3.1.5) or eqn. (3.1.6) under actual boundary conditions gives the magnitude of the potential $\phi = \phi(x, y)$ or the piezometric head $h = h(x, y)$ in the region of seepage (except for an arbitrary additive constant); all the remaining quantities in which we are interested, i.e. v_x , v_y and p can then be determined with the help of eqns. (3.1.1) to (3.1.3).

3.1.1. The potential function and the stream function

Eqn. (3.1.4) may also be written in the form

$$\frac{dv_x}{dx} = -\frac{\partial v_y}{\partial y} \quad (3.1.7)$$

Relation (3.1.7) is a well-known mathematical relation which implies that there exists a certain function of two variables, $f(x, y)$ — which we shall denote by ψ — whose total differential is

$$d\psi = \frac{\partial \psi}{\partial x} dx + \frac{\partial \psi}{\partial y} dy = -v_y dx + v_x dy \quad (3.1.8)$$

Eqn. (3.1.8) yields the relations

$$v_x = \frac{\partial \psi}{\partial y}, \quad v_y = -\frac{\partial \psi}{\partial x} \quad (3.1.9)$$

To establish the physical meaning of the function ψ we shall examine the relation

$$d\psi = -v_y dx + v_x dy = 0 \quad (3.1.10)$$

which we first rearrange in the form

$$\frac{v_y}{v_x} = \frac{dy}{dx} = k_\psi \quad (3.1.10a)$$

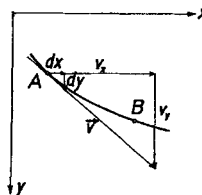


Fig. 3.3. A streamline and its tangent.

The line AB in Fig. 3.3 represents a certain streamline. It is evident that eqn. (3.1.10a) and eqn. (3.1.10) are the differential equations of a streamline. Eqn. (3.1.10a) is simultaneously the equation of the slope k_ψ of a tangent to the streamline. By integrating eqn. (3.1.10) we obtain the equation of the streamline

$$\psi(x, y) = \text{const.} \quad (3.1.11)$$

The function $\psi(x, y)$ is therefore called the stream function. On a streamline, the stream function has a constant value. This value is different for different streamlines.

Comparing eqn. (3.1.1) with eqn. (3.1.9) we obtain the relations

$$\begin{aligned} v_x &= \frac{\partial \varphi}{\partial x} = \frac{\partial \psi}{\partial y} \\ v_y &= \frac{\partial \varphi}{\partial y} = - \frac{\partial \psi}{\partial x} \end{aligned} \quad (3.1.12)$$

which enable us to determine the function $\psi(x, y)$ if we know the function $\varphi(x, y)$, and vice versa.

We differentiate the first of eqns. (3.1.12) with respect to y , the second with respect to x :

$$\begin{aligned} \frac{\partial^2 \varphi}{\partial x \partial y} &= \frac{\partial^2 \psi}{\partial y^2} \\ \frac{\partial^2 \varphi}{\partial y \partial x} &= - \frac{\partial^2 \psi}{\partial x^2} \end{aligned}$$

Subtracting, we have

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = \Delta \psi = 0 \quad (3.1.13)$$

The function $\psi(x, y)$ satisfies Laplace's differential equation and is a harmonic function in the region of seepage. By solving eqn. (3.1.13) for given boundary conditions we determine $\psi = \psi(x, y)$.

Relations (3.1.12) which exist between the potential function $\varphi(x, y)$ and the stream function $\psi(x, y)$ are known as the Cauchy-Riemann (or Euler-d'Alembert) equations. Hence in the region of seepage the potential function $\varphi(x, y)$ and the stream function $\psi(x, y)$ are conjugate harmonic functions.

If we assign a constant value to the potential function

$$\varphi(x, y) = \text{const.} \quad (3.1.14)$$

we obtain the equation of an equipotential line. Differentiation of this equation gives

$$d\varphi = \frac{\partial \varphi}{\partial x} dx + \frac{\partial \varphi}{\partial y} dy = v_x dx + v_y dy = 0 \quad (3.1.15)$$

from which follows the slope of the tangent to the equipotential line

$$k_\varphi = \frac{dy}{dx} = - \frac{v_x}{v_y} \quad (3.1.15a)$$

It is clear from eqns. (3.1.10a) and (3.1.15a) that

$$k_\varphi \cdot k_\psi = -\frac{v_x}{v_y} \cdot \frac{v_y}{v_x} = -1 \quad (3.1.16)$$

This means that the equipotential lines and the streamlines are perpendicular lines and form the so-called flow net. We have thus arrived at the result obtained in Section 1.4.4.

3.1.2. Quantity of flow between two points

In Fig. 3.4 ψ_1 and ψ_2 represent the streamlines passing through the points A and B ; the line AB is an arbitrary line in the region of seepage,

$\mathbf{n}$ is the unit vector normal to the element dS of the line AB and forming an angle α with the $+x$ axis,

$\mathbf{v}$ is the seepage velocity with which the liquid moves through the element ds .

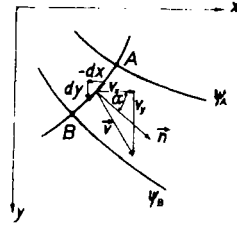


Fig. 3.4. Flow between two streamlines.

Denoting by q_{AB} the quantity of seepage flow between the points A, B (i.e. the quantity of flow in the region whose width in the direction normal to the plane of flow is $L = 1$), we obtain

$$\begin{aligned} q_{AB} &= \int_A^B dq = \int_A^B (\mathbf{v} \cdot \mathbf{n}) ds = \int_A^B [v_x(\mathbf{i} \cdot \mathbf{n}) + v_y(\mathbf{j} \cdot \mathbf{n})] ds = \\ &= \int_A^B (v_x \cos \alpha + v_y \sin \alpha) ds = \int_A^B v_x dy - v_y dx \end{aligned}$$

where $\mathbf{i}$ and $\mathbf{j}$ are the unit vectors in the direction of the axes x and y . Substituting according to eqn. (3.1.9) gives

$$q_{AB} = \int_{\psi_A}^{\psi_B} \left(\frac{\partial \psi}{\partial y} dy + \frac{\partial \psi}{\partial x} dx \right) = \int_{\psi_A}^{\psi_B} d\psi = \psi_B - \psi_A \quad (3.1.17)$$

where ψ_A and ψ_B are the values of the stream function at the points A and B .

The quantity of flow between two points is equal to the difference between the values of the stream function at those points.

3.1.3. Complex potential of seepage flow

Since in the region of seepage the functions $\phi(x, y)$ and $\psi(x, y)$ are conjugate harmonic functions, we can introduce a new function, namely

$$w = \phi + i\psi \quad (3.1.18)$$

called the complex potential of seepage flow; in the region of seepage, this is an analytic function of the complex variable z , where

$$z = x + iy$$

i.e. a function of the complex coordinate of a point in the region of seepage

$$w = \phi(x, y) + i\psi(x, y) = w(z) = w(x + iy) \quad (3.1.19)$$

In operations involving the complex potential w , the region of seepage is often referred to as the (z) region.

The analyticity of the function $w = w(z)$ is disturbed at points at which the Cauchy-Riemann equations fail to be satisfied. In such points, the so-called singular points, the seepage velocity is theoretically either zero or infinitely large: $v = 0$ or $v = \infty$. Note, however, that this only holds theoretically. In practice there exist no points in the region of seepage, in which the contour undergoes a sharp break. Eventually, every edge becomes round. Hence the seepage velocity there can be either very small or very large (depending on the configuration in question) but never extreme. Moreover, at singular points the theory of groundwater flow as developed above ceases to apply. At high velocities resulting from an increase in the effect of inertia forces, the linear Darcy's law, and hence all conclusions based on its assumed validity, no longer apply. The regions in which the seepage velocities are very high, have only a local character and do not affect the overall pattern of flow in any substantial manner.

We have thus converted the solution of the seepage problem to the solution of the problem of finding in the (z) region an analytical function $w = w(z)$ that will satisfy the given boundary conditions, i.e. the known values of the functions ϕ and ψ on the boundaries of the region of seepage.

If we know the complex potential $w = w(z)$, separating it into its real and imaginary parts enables us to determine the potential function $\phi(x, y)$ as well as the stream function $\psi(x, y)$

$$\begin{aligned} \phi &= \operatorname{Re} w(z) = \phi(x, y) \\ \psi &= \operatorname{Im} w(z) = \psi(x, y) \end{aligned} \quad (3.1.20)$$

and all the other quantities of interest, i.e. the velocities v_x and v_y using eqn. (3.1.1), or the so-called "complex velocity" as will be explained in Section 3.1.4, the pressure p using eqns. (3.1.2) and (3.1.3), and the quantity of flow q using eqn. (3.1.17).

On establishing the function inverse to the function $w = w(z)$, i.e.

$$z = z(w) \quad (3.1.21)$$

and separating it into its real and imaginary parts, we obtain the relations

$$\begin{aligned} x &= \operatorname{Re} z(w) = x(\varphi, \psi) \\ y &= \operatorname{Im} z(w) = y(\varphi, \psi) \end{aligned} \quad (3.1.22)$$

Substituting $\varphi = C$ in these equations, we obtain the parametric equations of the equipotential line $\varphi = C$ (the independent variable ψ is the parameter). Similarly, by substituting $\psi = C'$, we obtain the parametric equations of the streamline $\psi = C'$ (the independent variable φ is the parameter).

This is the way in which a flow net is calculated.

3.1.4. Complex velocity

Differentiation of the function $w = w(z)$ with respect to z and the use of eqns. (3.1.1) and (3.1.9) lead to the equations

$$\begin{aligned} \frac{dw}{dz} &= \frac{d(\varphi + i\psi)}{d(x + iy)} = \frac{d\varphi + i d\psi}{dx + i dy} = \\ &= \frac{(\partial\varphi/\partial x) dx + (\partial\varphi/\partial y) dy + i(\partial\psi/\partial x) dx + i(\partial\psi/\partial y) dy}{dx + i dy} = \\ &= \frac{v_x dx + v_y dy - i v_y dx + i v_x dy}{dx + i dy} = \frac{(v_x - i v_y)(dx + i dy)}{dx + i dy} = v_x - i v_y \end{aligned} \quad (3.1.23)$$

The number $dw/dz = v_x - i v_y = \bar{v}$ is a complex number conjugate to the number $v = v_x + i v_y$; it is called the complex velocity and is frequently used in the determination of the seepage velocity. Since the derivative of an analytic function is an analytic function, the complex velocity will also be an analytic function in the region of seepage (except at the singular points).

3.1.5. Boundary conditions for two-dimensional steady seepage flow in vertical plane

A general discussion of the boundary conditions was presented in Section 1.4.5. The various types of segments are shown in Fig. 1.5. In this section we shall briefly review the boundary conditions for the flow being considered and the coordinate system used (cf. Fig. 3.1).

a) Permeable segments (Fig. 1.5, the segments M_1M_5 and M_2M_3) are the lines of equal piezometric heads and hence also the equipotential lines. The direction of the liquid flow is perpendicular to the permeable segment. The pressure $p = p(x, y)$ is distributed according to the hydrostatic law,

$$p = \gamma y'$$

where y' is the depth below the level. By eqns. (3.1.3) and (3.1.2)

$$h = \left(\frac{p}{\gamma} - y \right) + C = (y' - y) + C = \text{const.} \quad (3.1.24a)$$

$$\varphi = \text{const.} \quad (3.1.24)$$

b) Impervious segments (the segment M_1M_2 in Fig. 1.5): Since the velocity component orthogonal to the impervious segment must be zero, the velocity there can only have a component parallel to the impervious boundary of the region of seepage. Therefore, the boundary is a streamline for which

$$\psi = \text{const.} \quad (3.1.25)$$

c) Segments of free seepage to the surface (the segment M_3M_4 in Fig. 1.5):

As the pressure there is equal to the atmospheric pressure

$$p = p_{at},$$

we have, according to eqns. (3.1.3) and (3.1.2),

$$h = \frac{p_{at}}{\gamma} - y + C = \text{const.} - y \quad (3.1.26a)$$

$$\varphi = \text{const.} + ky \quad (3.1.26)$$

d) Free surface (the depression curve): The following two conditions apply on the depression curve:

First, the pressure p must be equal to the atmospheric pressure and therefore

$$h = \frac{p_{at}}{\gamma} - y + C = \text{const.} - y \quad (3.1.26a)$$

$$\varphi = \text{const.} + ky \quad (3.1.26)$$

Second, as the velocity on the depression curve only has a component parallel to the curve, the depression curve is a streamline for which

$$\psi = \text{const.} \quad (3.1.25)$$

This holds true in the absence of infiltration or evaporation.

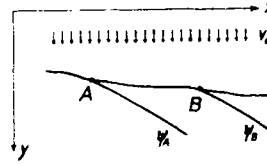
If infiltration or evaporation from the free surface are taking place, i.e. if

$$v_0 = \text{const.} \quad (3.1.27)$$

where v_0 is the volume of liquid which enters the region of seepage through a unit area of the free surface projection onto the horizontal plane per unit time (so that v_0 is the vertical velocity), the condition $\psi = \text{const.}$ no longer applies and another relation will come into play. Between the points A and B (Fig. 3.5) the quantity of liquid which will enter through the free surface is

$$\Delta q = v_0(x_B - x_A)$$

Fig. 3.5. Infiltration between points A , B .



where $x_B > x_A$. The expression $v_0(x_B - x_A)$ describes the flow through the portion of the depression curve lying between the points A and B . By eqn. (3.1.17) this quantity, Δq , is equal to the difference between the values of the stream function at the points A and B . Since

$$v_y = -\frac{\partial \psi}{\partial x} > 0$$

we have

$$\frac{\partial \psi}{\partial x} < 0$$

and so

$$\psi_B < \psi_A$$

Hence

$$\Delta q = \psi_A - \psi_B = v_0(x_B - x_A)$$

$$\psi_A + v_0 x_A = \psi_B + v_0 x_B$$

and the second condition for the depression curve takes the form

$$\psi + v_0 x = \text{const.} \quad (3.1.25a)$$

where

$v_0 > 0$ for infiltration into, and

$v_0 < 0$ for evaporation from, the free surface.

The circumstance that two conditions are available for the characteristics of the seepage flow on the depression curve, makes it possible to determine the unknown character of this curve in the course of the solution.

3.1.6. Conditions on the boundary between soils with different coefficients of permeability

In Fig. 3.6, the line AB forms the boundary between two soils with coefficients of permeability k_1 and k_2 .

By eqn. (3.1.18) in part *I*

$$w_1 = \varphi_1 + i\psi_1$$

and in part *II*

$$w_2 = \varphi_2 + i\psi_2$$

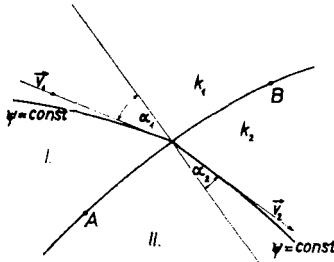


Fig. 3.6. Conditions on the boundary between two layers with different coefficients of permeability.

where — according to eqns. (3.1.2) and (3.1.3) —

$$\varphi_1 = -k_1 \left(\frac{p_1}{\gamma} - y_1 \right) + C_1$$

$$\varphi_2 = -k_2 \left(\frac{p_2}{\gamma} - y_2 \right) + C_2$$

For a point on the boundary, $p_1 = p_2$, $y_1 = y_2$ and

$$\frac{\varphi_1}{k_1} = \frac{\varphi_2}{k_2} + C \quad (3.1.28)$$

Differentiating eqn. (3.1.28) with respect to the direction of the boundary

$$\frac{\partial}{\partial s} \left(\frac{\varphi_1}{k_1} \right) = \frac{\partial}{\partial s} \left(\frac{\varphi_2}{k_2} \right)$$

followed by some rearrangement and use of eqn. (3.1.1) we obtain

$$\frac{v_{1s}}{k_1} = \frac{v_{2s}}{k_2} \quad (3.1.29)$$

Eqn. (3.1.29) describes the relation between the flow velocity components in parts *I* and *II*, parallel to the tangent to the boundary at the considered point of the bound-

ary. The second condition is obtained from the condition that the components of the velocities v_1 and v_2 normal to the boundary, must be the same, viz.

$$v_{1n} = v_{2n} \quad (3.1.30)$$

Eqn. (3.1.30) may also be written in the form

$$\frac{\partial \varphi_1}{\partial n} = \frac{\partial \varphi_2}{\partial n}$$

or, using relations (3.1.12), in the form

$$-\frac{\partial \psi_1}{\partial s} = \frac{\partial \psi_2}{\partial s}$$

After rearrangement and integration we have

$$\psi_1 = \psi_2 \quad (3.1.31)$$

By dividing eqn. (3.1.29) by eqn. (3.1.30) and introducing the relation

$$\frac{v_s}{v_n} = \operatorname{tg} \alpha$$

we obtain

$$\frac{v_{1s}}{k_1 v_{1n}} = \frac{v_{2s}}{k_2 v_{2n}}$$

or

$$\frac{\operatorname{tg} \alpha_1}{\operatorname{tg} \alpha_2} = \frac{k_1}{k_2} \quad (3.1.32)$$

It follows from the relations deduced above that on the boundary between the soils with coefficients of permeability k_1, k_2 the equipotential lines $\varphi = \text{const.}$ are discontinuous. The lines of equal piezometric head, $h = p/\gamma - y = \text{const.}$, and the streamlines $\psi = \text{const.}$ are continuous but possess a break. The angles which their tangents form with the tangent, or the normal, to the dividing curve, are defined by eqns. (3.1.32) (Fig. 3.6).

3.1.7. Reduced complex potential

In the solution of problems involving seepage in a homogeneous isotropic medium, use is made — in place of the quantities φ, ψ, v and q — of the so-called reduced values, i.e. of those quantities divided by the coefficient of permeability k .

The equations of the various quantities deduced in the preceding sections become

$$h = \frac{p}{\gamma} - y + C \quad (3.1.3)$$

$$\varphi_r = \frac{\varphi}{k} = -h = y - \frac{p}{\gamma} + C' \quad (3.1.2')$$

$$v_{rx} = J_x = \frac{v_x}{k} = \frac{1}{k} \frac{\partial \varphi}{\partial x} = \frac{\partial \varphi_r}{\partial x}$$

$$v_{ry} = J_y = \frac{v_y}{k} = \frac{1}{k} \frac{\partial \varphi}{\partial y} = \frac{\partial \varphi_r}{\partial y} \quad (3.1.1')$$

where, by eqn. (1.3.13), J_x and J_y are the hydraulic (piezometric) gradients in the directions of the x and y axes

$$\frac{\partial \varphi_r}{\partial x} = \frac{\partial \psi_r}{\partial y}$$

$$\frac{\partial \varphi_r}{\partial y} = - \frac{\partial \psi_r}{\partial x} \quad (3.1.12')$$

$$\frac{\partial^2 \varphi_r}{\partial x^2} + \frac{\partial^2 \varphi_r}{\partial y^2} = 0, \quad \frac{\partial^2 \psi_r}{\partial x^2} + \frac{\partial^2 \psi_r}{\partial y^2} = 0 \quad (3.1.5')$$

$$(3.1.13')$$

$$q_r = \frac{q}{k} = \int_{\psi_{rA}}^{\psi_{rB}} d\psi_r = \psi_{rB} - \psi_{rA} \quad (3.1.17')$$

$$w_r = \varphi_r + i\psi_r \quad (3.1.18')$$

$$w_r = w_r(z) = w_r(x + iy) \quad (3.1.19')$$

$$\frac{dw_r}{dz} = \frac{1}{k} (v_x - iv_y) = J_x - iJ_y = v_{rx} - iv_{ry} \quad (3.1.23')$$

It is clearly theoretically immaterial whether the actual or the reduced quantities are employed in the calculation. The introduction of the reduced quantities is advantageous in that it simplifies certain types of conformal mapping which we shall use — as will be shown presently — in the solution of seepage problems, especially in the application of the velocity hodograph method.

To obtain the final results, it is necessary at the end of all operations to multiply the reduced quantities by the coefficient of permeability k . This applies particularly to the velocities and quantities of flow.

3.1.8. Determination of the complex potential. Conformal mapping

All the existing methods for solving the Laplace equations (3.1.5), (3.1.6) or (3.1.13) for given boundary conditions, are highly complicated mathematical procedures.

The most popular of the available methods has been that based on the use of functions of a complex variable. By its application the solution of a seepage problem is converted to that of finding the complex potential of the seepage flow according to eqn. (3.1.19) in a way that will make it satisfy the given boundary conditions.

Just as for the Laplace equation, there are several ways of determining the complex potential, $w = w(z)$, each having certain merits and certain drawbacks. Judging by its extensive use in the work of Soviet authors and in the more recent work of American and other authors, conformal mapping seems to be the technique most preferred at present.

Up to now, the following three fundamental methods of applying conformal mapping to the solution of problems of seepage flow have been studied in detail:

the Pavlovsky method,
the Vedernikov-Pavlovsky method,
the velocity hodograph method.

In the application of any of those three methods we meet with the task of determining a certain analytic function of the complex variable ζ

$$\omega = f(\zeta) \quad (3.1.33)$$

under the conditions that we know the shape of the region of the values of the complex variable ζ (in the Gauss plane) as well as the shape of the region of the values of ω corresponding to the various values of the variable ζ , i.e. we know the shape of the boundary of the (ζ) and (ω) regions and have to find the relation (3.1.33) which associates the values of ω with the various values of ζ . Naturally, relation (3.1.33) represents different functions depending on which of the methods is being used.

As is well known from the theory of functions of a complex variable, relation (3.1.33) may be obtained by conformal mapping of the (ζ) region onto the (ω) region, or inversely, of the (ω) region onto the (ζ) region, i.e. we determine either the function $\omega = f(\zeta)$ or the function inverse to it, $\zeta = g(\omega)$, or, under successive conformal mappings, the (ζ) region is mapped onto the (ζ_1) , (ζ_2) , ... up to the (ζ_n) region, and only the final region is mapped onto the (ω) region.

The principles of conformal mapping have received full coverage in publications devoted to this topic, for example [9], [35]. Some of the examples discussed in these publications make it possible to obtain a solution in a number of cases of two-dimensional seepage flow.

3.2. Pavlovsky method

The method is named in honour of N. N. Pavlovsky who was the first to use it in 1922 to solve problems of seepage flow. It is particularly suitable in cases of flow without a free surface,

without segments of free seepage to the surface (i.e. seepage under pressure).

In such cases the boundary of the region of seepage consists of permeable and impervious segments only.

We then have:

for the permeable segments, according to eqn. (3.1.24),

$$\varphi_r = \text{const.} \quad (3.2.1)$$

for the impervious segments, according to eqn. (3.1.25),

$$\psi_r = \text{const.} \quad (3.2.2)$$

Hence the region of the reduced complex potential is a polygon with sides parallel to the coordinate axes $O\varphi_r$, $O\psi_r$.

The region of seepage (z) is specified by the actual boundary conditions of the problem being considered.

The solution consists of determining the reduced complex potential w_r in accordance with eqn. (3.1.19') as a function of the complex coordinate z of the region of seepage, i.e.

$$w_r = w_r(z) \quad (3.2.3)$$

by mapping the region of seepage (z) conformally onto the region of the reduced complex potential (w_r), or possibly, of the function

$$z = z(w_r) \quad (3.2.4)$$

which maps the (w_r) region conformally onto the (z) region.

Sometimes direct mapping of a region onto another region becomes too complicated. If this is the case, it is helpful to map both regions — (z) as well as (w_r) — conformally separately onto an auxiliary half-plane (ζ). The required solution is then obtained in the form of two parametric equations

$$z = f_1(\zeta) \quad \text{and} \quad w_r = f_2(\zeta) \quad (3.2.5)$$

where ζ is the common parameter.

In the sections which follow we shall illustrate the application of the Pavlovsky method in practice by obtaining solutions to some fundamental problems.

3.2.1. Seepage under a weir structure or sheetpile built on permeable subsoil of great (theoretically infinite) thickness

We shall examine the seepage in the (z) region shown schematically in Fig. 3.7. We consider a general case when underneath the weir structure of length l_2 there is built an absolutely impervious cut-off wall to depth s , and to reduce seepage, a blanket of length l_1 is formed upstream of the weir on the bottom of the reservoir. If the blanket is omitted, $l_1 = 0$. The head on the weir structure $H = H_1 - H_2$, the reference plane for the determination of the piezometric heads according to (3.1.3) is placed at the level of the water surface downstream of the weir. The case of a sheetpile is obtained from the general case on setting $l_1 = l_2 = 0$. The coefficient of permeability in the flow domain is k .

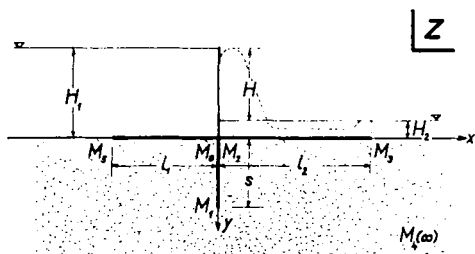


Fig. 3.7. Seepage under a weir structure.

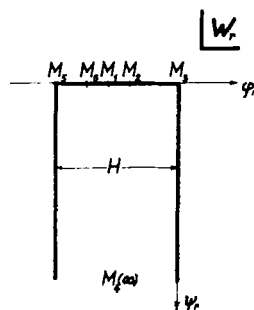


Fig. 3.8. Region of reduced complex potential.

We first determine the shape of the region of the reduced complex potential w_r .

We can choose the value of the stream function ψ_r at will, bearing in mind the validity of eqn. (3.1.17'). For the segment $M_5M_6M_1M_2M_3$ which represents the boundary streamline we choose $\psi_r = 0$. The values of the reduced potential on the segments M_4M_5 and M_3M_4 are determined from eqn. (3.1.2) after we have chosen the constant C so as to obtain $\varphi_r = 0$ for the segment M_3M_4 where $y = 0$ and $p/\gamma = H_2$.

On the boundaries of the region of seepage the following then holds for the (w_r) region:

for the segments

$$M_4M_5: \quad \varphi_r = -H$$

$$M_5M_6M_1M_2M_3: \quad \psi_r = 0$$

$$M_3M_4: \quad \varphi_r = 0$$

and the shape of the region of the reduced complex potential is as shown in Fig. 3.8. This figure also shows the positions of the points corresponding to the points M_k in the region of seepage (z) .

We shall establish the function $w_r = w_r(z)$ according to eqn. (3.2.3) by mapping the (z) region conformally onto the (w_r) region. We introduce the auxiliary half-plane (ζ) , map both the (z) and the (w_r) regions onto it, and obtain the sought relation $w_r = w_r(z)$ by comparison. We effect the representation by successive mappings with the help of simple mapping functions.

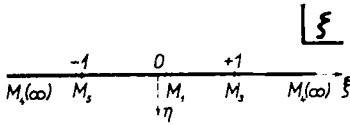


Fig. 3.9.

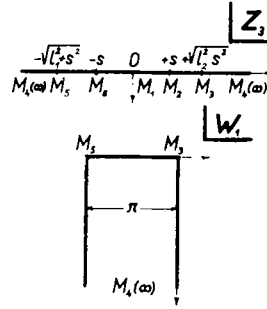


Fig. 3.10.

Figs. 3.9 and 3.10 show the (ζ) region and the regions to which the (z) and the (w_r) regions will be transformed by successive conformal mapping, and indicate the coordinates of the points M_k in the various regions.

The (z) region will be transformed to the (ζ) half-plane by means of the relations

$$z_1 = z^2 \quad (3.2.6a)$$

$$z_2 = z_1 + s^2 = z^2 + s^2 \quad (3.2.6b)$$

$$z_3 = \sqrt{z_2} = \sqrt{(z^2 + s^2)} \quad (3.2.6c)$$

Since in the (z) region, the lengths l_1 and l_2 generally differ from one another, the points M_k on the real axis of the (z_3) half-plane will be distributed arbitrarily. For the purpose of subsequent calculation we require that the following should hold on the auxiliary half-plane (ζ) : for the point M_1 ($\zeta = -1$), for the point M_3 ($\zeta = +1$). This mapping will be effected by the linear function

$$\zeta = \frac{z_3 - a}{b} = \frac{\sqrt{(z^2 + s^2)} - a}{b} \quad (3.2.6d)$$

where

$$\begin{aligned} a &= \frac{1}{2}[\sqrt{(s^2 + l_2^2)} - \sqrt{(s^2 + l_1^2)}], \\ b &= \frac{1}{2}[\sqrt{(s^2 + l_2^2)} + \sqrt{(s^2 + l_1^2)}] \end{aligned} \quad (3.2.6e)$$

For the time being we have mapped the region of seepage (z) onto the auxiliary half-plane (ζ) .

The (w_r) region is transformed to the (ζ) half-plane by the relations

$$w_1 = \frac{\pi w_r}{H} \quad (3.2.7a)$$

$$\zeta = \cos w_1 = \cos \frac{\pi w_r}{H} \quad (3.2.7b)$$

From eqns. (3.2.6d) and (3.2.7b) we obtain

$$\cos \frac{\pi w_r}{H} = \frac{\sqrt{(z^2 + s^2)} - a}{b}$$

and after rearrangement

$$w_r = \frac{H}{\pi} \arccos \frac{\sqrt{(z^2 + s^2)} - a}{b} \quad (3.2.8)$$

We have thus solved our problem, for eqn. (3.2.8) gives us the reduced complex potential at a general point M of the region of seepage with complex coordinate z , and we know how to determine all the other quantities of interest by reference to the relations stated comprehensively in Section 3.1.7.

We shall also derive for our general case the magnitudes of the reduced velocities of seepage flow. For that purpose we shall make use of eqn. (3.1.23'):

$$v_{rx} - iv_{ry} = J_x - iJ_y = \frac{dw_r}{dz} \quad (3.2.9)$$

Differentiation of eqn. (3.2.8) and some manipulation yields

$$v_{rx} - iv_{ry} = -\frac{H}{\pi} \frac{z}{\sqrt{(z^2 + s^2)} \sqrt{[b^2 - z^2 - s^2 - a^2 + 2a \sqrt{(z^2 + s^2)}]}}$$

$$v_x - iv_y = k(v_{rx} - iv_{ry}) \quad (3.2.10)$$

and after separating the equations into its real and imaginary parts (following the substitution $z = x + iy$), the expressions of the components of the seepage velocity in the direction of the coordinate axes.

We can also determine the uplift pressure acting on the weir foundation, i.e. the magnitude of the piezometric heads h on the segments M_5M_6 and M_2M_3 . On those segments $y = 0$; $z = x$, $-l_1 \leq x \leq l_2$; $\varphi_r = -h$; $\psi_r = 0$. Substitution in eqn. (3.2.8) then leads to

$$\frac{h}{H} = -\frac{1}{\pi} \arccos \frac{\pm \sqrt{(x^2 + s^2)} - a}{b} \quad (3.2.11)$$

where the minus sign applies to the portion upstream of the cut-off wall ($-l_1 \leq$

$\leq x \leq 0$), the plus sign to the portion downstream of the cut-off wall ($0 \leq x \leq l_2$), and in view of the considered range of φ_r , the values of $\arccos(\sqrt{(x^2 + y^2) - a}/b)$ lie in the interval $\langle -\pi, 0 \rangle$.

We shall now discuss in detail the various particular cases covered by the general case.

a) Weir structure without a cut-off wall (Fig. 3.11)

In this case we can assume that $l_1 = l_2 = l$; further, $s = 0$, $a = 0$, $b = l$, and from eqn. (3.2.8) we obtain the reduced complex potential

$$w_r = \frac{H}{\pi} \arccos \frac{z}{l} \quad (3.2.12a)$$

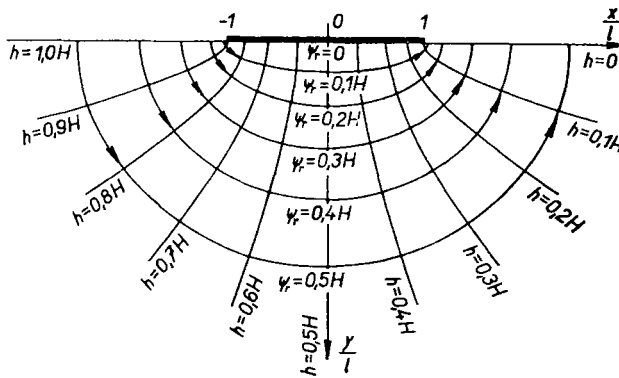


Fig. 3.11. Streamlines in seepage under a weir structure without a cut-off wall.

alternatively, if we wish to establish the pattern of the potential lines and the streamlines, we can use the inverse function

$$z = l \cos \frac{\pi w_r}{H} \quad (3.2.12b)$$

Separating eqn. (3.2.12b) into its real and imaginary parts and substituting for the potential φ_r from eqn. (3.1.2') we obtain the equations

$$\frac{x}{l} = \cos \frac{\pi h}{H} \cosh \frac{\pi \psi_r}{H}, \quad \frac{y}{l} = \sin \frac{\pi h}{H} \sinh \frac{\pi \psi_r}{H} \quad (3.2.12c)$$

where the piezometric head can range between $0 \leq h \leq H$, and the stream function ψ_r can take the values in the range $0 \leq \psi_r \leq \infty$.

Substitution of the chosen values of $h = \text{const.}$ in eqn. (3.2.12c) leads to the parametric equations of the equipotential line (the line of equal piezometric head) in which the variable value of the stream function ψ_r is the parameter. The equipotential lines are halves of the branches of confocal hyperbolas, with foci at the points M_5 and M_3 ($z/l = \pm 1$).

Similarly, substitution of the chosen values of $\psi_r = \text{const.}$ in eqn. (3.3.12c) leads to the parametric equations of the streamline in which the variable piezometric head h is the parameter. The streamlines are confocal semi-ellipses with foci at the points M_5 and M_3 .

The two systems of lines form a flow net. The flow net of the case discussed is shown in Fig. 3.11.

The uplift pressure on the weir foundation is obtained from eqn. (3.2.11). After rearrangement we have

$$\frac{h}{H} = -\frac{1}{\pi} \arccos \frac{x}{l} \quad (3.2.13)$$

where $-l \leq x \leq l$, and the values of $\arccos x/l$ lie in the interval $\langle -\pi, 0 \rangle$.

We are sometimes interested in the velocity with which the water downstream of the weir leaves the flow domain. On the segment M_3M_4 , $y = 0$; $z = x$, $l \leq x < \infty$, and substitution in eqn. (3.2.10) leads in our case to

$$v_{rx} - iv_{ry} = -\frac{H}{\pi} \frac{1}{\sqrt{(l^2 - x^2)}} = \frac{H}{\pi} \frac{i}{\sqrt{(x^2 - l^2)}}$$

from which it follows that

$$v_{rx} = 0; \quad v_{ry} = -\frac{H}{\pi} \frac{1}{\sqrt{(x^2 - l^2)}}, \quad v_y = kv_{ry} \quad (3.2.14)$$

The minus sign of v_{ry} indicates that the water flows upwards.

The quantity of water which will percolate under the weir is readily determined from eqn. (3.1.17'). For the segment M_3M_4 we have $h = 0$. Between the downstream toe of the weir (point M_3) and a general point M lying on the x axis at a distance $z = x$ there will percolate the quantity

$$q_r = \psi_{rM} - \psi_{rM_3}$$

From the first of eqns. (3.2.12c)

$$\psi_{rM_3} = \frac{H}{\pi} \operatorname{argcosh} 1 = \frac{H}{\pi} \cdot 0 = 0$$

and

$$q_r = \psi_{rM} = \frac{H}{\pi} \operatorname{argcosh} \frac{x}{l}, \quad q = kq_r$$

b) Sheetpile (Fig. 3.12)

In this case $l_1 = l_2 = 0$, $a = 0$, $b = s$, and from eqn. (3.2.8) we obtain the reduced complex potential

$$w_r = \frac{H}{\pi} \arccos \frac{\sqrt{(z^2 + s^2)}}{s} \quad (3.2.15a)$$

or the inverse function

$$z = -is \sin \frac{\pi w_r}{H} \quad (3.2.15b)$$

Separating the equation into its real and imaginary parts and substituting for the potential according to eqn. (3.1.2') we obtain the equations

$$\frac{x}{s} = \cos \frac{\pi h}{H} \sinh \frac{\pi \psi_r}{H}, \quad \frac{y}{s} = \sin \frac{\pi h}{H} \cosh \frac{\pi \psi_r}{H} \quad (3.2.15c)$$

where $0 \leq h \leq H$, $0 \leq \psi_r \leq \infty$.

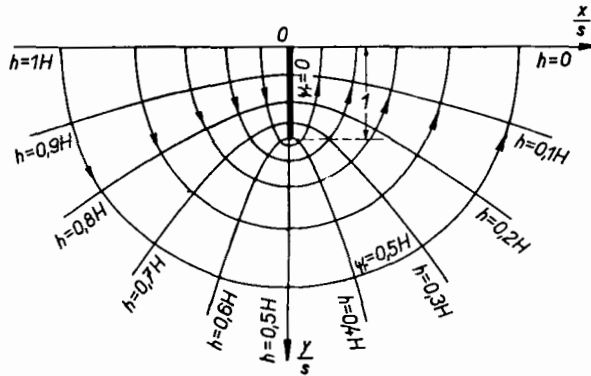


Fig. 3.12. Streamlines in seepage under a sheetpile.

If we substitute constants for h or ψ_r we obtain the parametric equations of the equipotential lines (lines of equal piezometric head) or of the streamlines. The equipotential lines are half-branches of hyperbolas and the streamlines are semi-ellipses; the hyperbolas and the ellipses have common foci at the points $z = \pm is$ (Fig. 3.12).

The velocity of upward seepage downstream of the sheetpile: On the segment M_3M_4 we have $y = 0$; $z = x$, $0 \leq x < \infty$; $h = 0$. From eqn. (3.2.10) we obtain

$$v_{rx} - iv_{ry} = \frac{H}{\pi} \frac{i}{\sqrt{(z^2 + s^2)}} \\ v_{rx} = 0; \quad v_{ry} = -\frac{H}{\pi} \frac{1}{\sqrt{(z^2 + s^2)}}, \quad v_y = kv_{ry} \quad (3.2.16)$$

We now return to the general equation of seepage velocity (3.2.10). If we substitute in it the z -coordinates of the points M_5 , M_1 or M_3 , i.e. the coordinates of the lower edge of the upstream face, of the end of the cut-off wall and of the downstream toe: $z = -l_1$; is ; l_2 , we obtain

$$v_{rx} - iv_{ry} = \infty.$$

The above-mentioned points are the singular points of the region of seepage where the seepage velocity can theoretically become infinite. Actually, however, there exist no "points" in which the contour experiences a sharp break, for even an edge is eventually rounded and the velocity there, though very large, is not infinite.

Regions with very high seepage velocities are, however, quite small and as such, affect the overall pattern of flow in no substantial way — as already noted in Section 3.1.3.

3.2.2. Seepage under a sheetpile with the terrain on its sides at unequal heights — seepage into an excavation

The case to be considered is shown schematically in Fig. 3.13.

The surface of water in the excavation is assumed to be at the level of the bottom, the height of water in the river is H , the depth of the excavation is t .

When seeking the function $w_r = f(z)$ the reference plane for the determination of φ_r or h is placed at the river bed, and the value of ψ_r on the segment $M_4M_1M_2$ is chosen the same as in the previous case.

The following holds for the (w_r) region:

for the segments M_3M_4 : $\varphi_r = -H$

$M_4M_1M_2$: $\psi_r = 0$

M_2M_3 : $\varphi_r = +t$

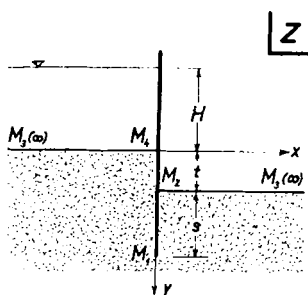


Fig. 3.13. Seepage under a sheetpile on the perimeter of a construction pit.

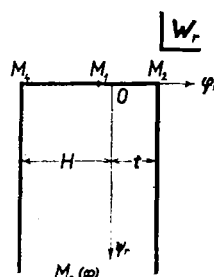


Fig. 3.14.

The shape of the region of the reduced complex potential is as shown in Fig. 3.14.

Just as in the previous case, we introduce the auxiliary half-plane (ζ) and transform the (z) and the (w_r) regions onto it. Since the region of seepage (z) is not symmetrical about the y axis, the distribution of the points M_k about the η axis will not be symmetrical, either.

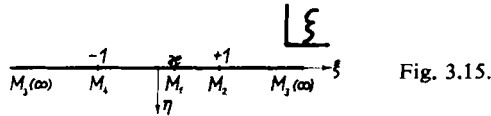
The region of seepage (z) is transformed to the (ζ) half-plane with the help of the Schwarz-Christoffel integral. As we shall show in Section 3.5, point 19, this integral transforms a half-plane into an n -gon. In the transformation we can choose at will the position of three points in the (ζ) region to correspond to three known points in the (z) region. In the table below there are set out the coordinates of the points in the (z) region, the interior angles formed by the boundaries of the region of seepage, and the coordinates of points in the (ζ) region, three of which we have chosen to fit our case (the points M_1 , M_2 and M_4).

M_k	z_k	α_k/π	ζ_k
M_1	$i(t + s)$	2	κ
M_2	it	$\frac{1}{2}$	$+1$
M_3	∞	-1	∞
M_4	0	$\frac{1}{2}$	-1

For κ we have

$$0 < \kappa < 1.$$

The (ζ) half-plane is shown in Fig. 3.15.



For the above choice of the coordinates of the corresponding points, the Schwarz-Christoffel integral takes the form

$$\begin{aligned}
 z &= C \int_{\zeta_0}^{\zeta} \frac{(\zeta - \kappa) d\zeta}{\sqrt{[(\zeta - 1)(\zeta + 1)]}} + C_1 = C \int_1^{\zeta} \frac{\zeta d\zeta}{\sqrt{(\zeta^2 - 1)}} - C\kappa \int_1^{\zeta} \frac{d\zeta}{\sqrt{(\zeta^2 - 1)}} + C'_1 = \\
 &= C \sqrt{(\zeta^2 - 1)} - C\kappa \ln [\zeta + \sqrt{(\zeta^2 - 1)}] + C'_1 \quad (3.2.17a)
 \end{aligned}$$

where C , C'_1 and κ are constants which are determined from the condition of correspondence between the points M_1 , M_2 and M_4 in the (z) and (ζ) regions.

The constants C and C'_1 and κ are obtained as follows:

For M_2 : $\zeta = +1$, $z = it$, and from eqn. (3.2.17a)

$$it = C'_1.$$

For M_4 : $\zeta = -1$, $z = 0$, and from eqn. (3.2.17a)

$$0 = -C\kappa \ln(-1) + it = -C\kappa i\pi + it$$

$$C = \frac{t}{\kappa\pi}$$

Eqn. (3.2.17a) which transforms the (ζ) region onto the (z) region takes the form

$$z = \frac{t}{\kappa\pi} \sqrt{(\zeta^2 - 1)} - \frac{t}{\pi} \ln [\zeta + \sqrt{(\zeta^2 - 1)}] + it \quad (3.2.17b)$$

For M_1 : $\zeta = \kappa$, $z = i(t + s)$, and from eqn. (3.2.17b)

$$i(t + s) = \frac{t}{\kappa\pi} \sqrt{(\kappa^2 - 1)} - \frac{t}{\pi} \ln [\kappa + \sqrt{(\kappa^2 - 1)}] + it$$

or, after rearrangement

$$i \left(\frac{\sqrt{(1 - \kappa^2)}}{\kappa} - \frac{\pi s}{t} \right) = \ln [\kappa + \sqrt{(\kappa^2 - 1)}] = \operatorname{argcosh} \kappa$$

Hence

$$\kappa = \cosh i \left(\frac{\sqrt{(1 - \kappa^2)}}{\kappa} - \frac{\pi s}{t} \right) = \cos \left(\frac{\sqrt{(1 - \kappa^2)}}{\kappa} - \frac{\pi s}{t} \right) \quad (3.2.18a)$$

If we know t and s , we can determine κ from eqn. (3.2.18a). Making the substitution

$$\kappa = \cos \alpha,$$

$$\cos \alpha = \cos \left(\operatorname{tg} \alpha - \frac{\pi s}{t} \right)$$

and performing some manipulations we obtain the equation

$$\operatorname{tg} \alpha - \alpha = \frac{\pi s}{t} \quad (3.2.18b)$$

which is more convenient for the determination of κ than eqn. (3.2.18a).

The transformation of the (w_r) region onto the (ζ) half-plane is effected by the functions

$$w_1 = \frac{\pi(w_r - t)}{H + t}$$

$$\zeta = \cos w_1 = \cos \frac{\pi(w_r - t)}{H + t} \quad (3.2.19)$$

The above equation together with eqn. (3.2.17b) in the form

$$z = \frac{t}{\kappa\pi} \int_1^{\zeta} \frac{(\zeta - \kappa)}{\sqrt{(\zeta^2 - 1)}} d\zeta + it \quad (3.2.17c)$$

define the relation $w_r = w_r(z)$.

The differentiation dw_r/dz is carried out with the help of the inverse functions:

$$\frac{dz}{dw_r} = \frac{1}{v_{rx} - iv_{ry}} = \frac{dz}{d\zeta} \cdot \frac{d\zeta}{dw_r}$$

From eqn. (3.2.17c) we have

$$\frac{dz}{d\zeta} = \frac{t}{\kappa\pi} \frac{\zeta - \kappa}{\sqrt{(\zeta^2 - 1)}}$$

and from eqn. (3.2.19)

$$\frac{d\zeta}{dw_r} = -\frac{\pi}{H+t} \sin \frac{\pi(w_r - t)}{H+t} = -\frac{\pi}{H+t} \sqrt{(1 - \zeta^2)}$$

From the above it follows that

$$\begin{aligned} \frac{dz}{dw_r} &= -\frac{t}{\kappa\pi} \frac{\zeta - \kappa}{\sqrt{(\zeta^2 - 1)}} \frac{\pi}{H+t} \sqrt{(1 - \zeta^2)} = -\frac{it(\zeta - \kappa)}{(H+t)\kappa} \\ v_{rx} - iv_{ry} &= \frac{dw_r}{dz} = i \frac{(H+t)\kappa}{t(\zeta - \kappa)}, \quad v_x - iv_y = k(v_{rx} - iv_{ry}) \end{aligned} \quad (3.2.20)$$

Eqn. (3.2.20) is a parametric equation for the determination of the components of seepage velocity. The variable coordinate ζ of a point in the auxiliary half-plane (ζ) is the parameter.

We shall also determine the velocity of seepage into the excavation. For the bottom of the excavation: $1 \leq \zeta = \xi < \infty$, and from eqn. (3.2.20)

$$v_{rx} = 0; \quad v_{ry} = -\frac{(H+t)\kappa}{t(\xi - \kappa)}, \quad v_y = kv_{ry} \quad (3.2.21)$$

3.2.3. Seepage under a weir structure or sheetpile built on a permeable layer of finite thickness underlain by impervious subsoil

The region of seepage (z) is shown schematically in Fig. 3.16.

The reference plane for the determination of φ_r or h is placed on the water surface downstream of the weir.

The following applies to the various segments of the boundary:

for the segments	M_5M_6 :	$\varphi_r = -H$,
	$M_6M_7M_1M_2M_3$:	$\psi_r = 0$,
	M_3M_4 :	$\varphi_r = 0$,
	M_4M_5 :	$\psi_r = q_r$.

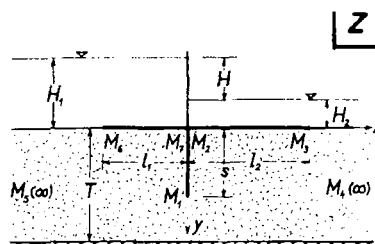


Fig. 3.16. Seepage under a weir structure on a permeable layer of finite thickness.

On the surface of the impervious subsoil, which is the second boundary streamline (the contour of the structure is the first boundary streamline), the value of the stream function $\psi_r = q_r$ so as to satisfy eqn. (3.1.17') — between the two boundary streamlines there must flow the quantity of seepage q_r . This is in fact the flow between two points, M and N , lying on any line in the region of seepage, with the point M lying on the contour of the structure, and the point N on the surface of the impervious subsoil, $q_r = \psi_{rN} - \psi_{rM}$.

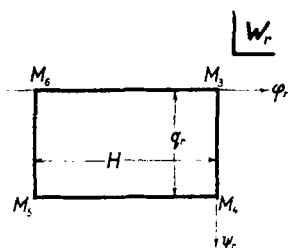


Fig. 3.17.

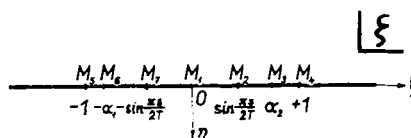


Fig. 3.18.

The shape of the region of the reduced complex potential w_r is shown in Fig. 3.17. It is seen to be rectangular.

The conformal mapping of the (z) region onto the (w_r) region will be effected successively, by means of a series of simple representations.

The (z) region is transformed to the auxiliary (ζ) half-plane (Fig. 3.18) with the help of the function

$$\zeta = \cos \frac{\pi s}{2T} \sqrt{\left(\operatorname{tg}^2 \frac{\pi s}{2T} + \operatorname{tgh}^2 \frac{\pi z}{2T} \right)} \quad (3.2.22a)$$

as can readily be proved by substituting the coordinates of the points on the boundary of the (z) region. On substituting $z = -l_1$ and $z = l_2$ in eqn. (3.2.22a) we have for the coordinates ζ of the points M_6 and M_3

$$\alpha_1 = \cos \frac{\pi s}{2T} \sqrt{\left(\operatorname{tg}^2 \frac{\pi s}{2T} + \operatorname{tgh}^2 \frac{\pi l_1}{2T} \right)}, \quad \alpha_2 = \cos \frac{\pi s}{2T} \sqrt{\left(\operatorname{tg}^2 \frac{\pi s}{2T} + \operatorname{tgh}^2 \frac{\pi l_2}{2T} \right)} \quad (3.2.22b)$$

In the next step we transform the (w_r) region to the auxiliary (ζ) half-plane as well; we do this by transforming it first to the (w_1) region, this in turn to the (w_2) region and only the latter to the (ζ) half-plane (Fig. 3.19).

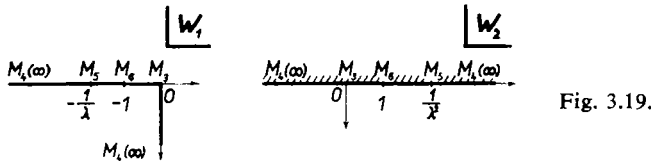


Fig. 3.19.

The transformation of the (w_r) region to the (w_1) region is effected with the help of the elliptic function

$$w_1 = \operatorname{sn} \left(\frac{K w_r}{H}, \lambda \right) \quad (3.2.23a)$$

where K is the complete elliptic integral of the first kind with modulus λ . Another relation that applies to this representation is

$$\frac{q_r}{H} = \frac{K'}{K} \quad (3.2.23b)$$

where K' is the complement to the complete elliptic integral of the first kind. If we know the magnitude of H and q_r , we can determine the modulus λ of the elliptic integral from eqn. (3.2.23b).

The function

$$w_2 = w_1^2 = \operatorname{sn}^2 \left(\frac{K w_r}{H}, \lambda \right) \quad (3.2.23c)$$

transforms the (w_1) region and hence also the (w_r) region to the (w_2) region. To complete the sequence of successive conformal mappings it only remains to represent the (w_2) region on the (ζ) region.

The mapping of the (ζ) half-plane onto the (w_2) half-plane is accomplished with the help of a bilinear function. For this kind of mapping we must know the coordinates of three arbitrary corresponding points in the two half-planes. For our purpose we choose the points M_3 , M_4 and M_6 ; a review of the pertinent data appears in the table which follows.

M_k	ζ_k	w_{2k}
M_3	α_2	0
M_4	1	∞
M_5	-1	$1/\lambda^2$
M_6	$-\alpha_1$	1

The function $w_2 = f(\zeta)$ is

$$w_2 = \frac{(1 + \alpha_1)(\alpha_2 - \zeta)}{(\alpha_1 + \alpha_2)(1 - \zeta)} \quad (3.2.23d)$$

The function for the conformal mapping of the region of the complex potential (w_r) onto the auxiliary half-plane (ζ) is obtained by comparing eqn. (3.2.23d) with eqn. (3.2.23c):

$$\operatorname{sn}^2 \left(\frac{Kw_r}{H}, \lambda \right) = \frac{(1 + \alpha_1)(\alpha_2 - \zeta)}{(\alpha_1 + \alpha_2)(1 - \zeta)} \quad (3.2.23e)$$

Substitution of the coordinates of the point M_5 in the (ζ) and (w_2) regions and some manipulations yield the condition from which we can establish the magnitude of the modulus λ of the complete elliptic integrals of the first kind, K and K' :

$$\lambda = \sqrt{\frac{2(\alpha_1 + \alpha_2)}{(1 + \alpha_1)(1 + \alpha_2)}} \quad (3.2.23f)$$

The problem is solved by establishing, from eqns. (3.2.23e) and (3.2.22a), the required reduced complex potential w_r as a function of the variable complex coordinate z of a point in the region of seepage.

The velocity distribution on the river bed downstream of the weir is found with the help of eqn. (3.1.23'). In the case being considered

$$\frac{dw_r}{dz} = \frac{dw_r}{dw_1} \cdot \frac{dw_1}{dw_2} \cdot \frac{dw_2}{d\zeta} \cdot \frac{d\zeta}{dz}$$

In the determination of the derivative dw_r/dw_1 we shall make use of the function inverse to the elliptic function $\operatorname{sn}(Kw_r/H, \lambda)$. This is the elliptic integral of the first kind

$$w_r = \frac{H}{K} \int_0^{w_1} \frac{dw_1}{\sqrt{[(1 - w_1^2)(1 - \lambda^2 w_1^2)]}} \quad (3.2.24a)$$

or, according to eqn. (3.2.23b)

$$w_r = \frac{q_r}{K'} \int_0^{w_1} \frac{dw_1}{\sqrt{[(1 - w_1^2)(1 - \lambda^2 w_1^2)]}}$$

Then

$$\frac{dw_r}{dw_1} = \frac{H}{K} \frac{1}{\sqrt{[(1 - w_1^2)(1 - \lambda^2 w_1^2)]}} = \frac{q_r}{K'} \frac{1}{\sqrt{[(1 - w_1^2)(1 - \lambda^2 w_1^2)]}} \quad (3.2.24b)$$

Eqn. (3.2.23c) leads to

$$\frac{dw_1}{dw_2} = \frac{1}{2w_1} = \frac{1}{2\sqrt{w_2}} \quad (3.2.24c)$$

and eqn. (3.2.23d) to

$$\frac{dw_2}{d\zeta} = - \frac{(1 + \alpha_1)(1 - \alpha_2)}{(\alpha_1 + \alpha_2)} \cdot \frac{1}{(1 - \zeta)^2} \quad (3.2.24d)$$

Finally, after substituting for w_1 , w_2 and λ according to (3.2.23c), (3.2.23d) and (3.2.23f), we have

$$\begin{aligned} \frac{dw_r}{d\zeta} &= - \frac{q_r}{2K'} \sqrt{\frac{(1 + \alpha_1)(1 + \alpha_2)}{(\alpha_1 + \zeta)(\alpha_2 - \zeta)(1 - \zeta^2)}} = \\ &= - \frac{H}{2K} \sqrt{\frac{(1 + \alpha_1)(1 + \alpha_2)}{(\alpha_1 + \zeta)(\alpha_2 - \zeta)(1 - \zeta^2)}} \end{aligned} \quad (3.2.24e)$$

Differentiation followed by rearrangement and substitution for z yields from eqn. (3.2.22a)

$$\frac{d\zeta}{dz} = \frac{\pi}{2T \cos(\pi s/2T)} \frac{1 - \zeta^2}{\zeta} \sqrt{\left(\zeta^2 - \sin^2 \frac{\pi s}{2T}\right)} \quad (3.2.24f)$$

In accordance with eqns. (3.2.24e) and (3.2.24f) the resulting equations for the velocities are

$$\begin{aligned} v_{rx} - iv_{ry} &= \frac{dw_r}{dz} = \\ &= - \frac{\pi q_r}{4K'T \cos(\pi s/2T)} \frac{1}{\zeta} \sqrt{\frac{(1 + \alpha_1)(1 + \alpha_2)(1 - \zeta^2)[\zeta^2 - \sin^2(\pi s/2T)]}{(\alpha_1 + \zeta)(\alpha_2 - \zeta)}} \\ v_x - iv_y &= k(v_{rx} - iv_{ry}). \end{aligned} \quad (3.2.24g)$$

The distribution of the uplift pressure on the foundation of the weir structure where on the segments M_6M_7 and M_2M_3 we have $z = x$, $w_r = \varphi_r = -h$, $\zeta = \xi$, is established from eqn. (3.2.24a) on substituting for w_1 from eqns. (3.2.23c) and (3.2.23d)

$$\frac{h}{H} = - \frac{1}{K} F \left(\arcsin \sqrt{\frac{(1 + \alpha_1)(\alpha_2 - \xi)}{(\alpha_1 + \alpha_2)(1 - \xi)}}, \lambda \right) \quad (3.2.25a)$$

ξ is found from eqn. (3.2.22a)

$$\xi = \pm \cos \frac{\pi s}{2T} \sqrt{\left(\operatorname{tg}^2 \frac{\pi s}{2T} + \operatorname{tgh}^2 \frac{\pi x}{2T} \right)} \quad (3.2.25b)$$

with the minus sign applying to the section upstream ($-l_1 \leq x \leq 0$), and the plus sign to the section downstream ($0 \leq x \leq l_2$) of the cut-off wall.

The case of seepage discussed in Section 3.2.1 is only a special case of the present one for $T \rightarrow \infty$. It would, however, serve no useful purpose to embark on the solution of this special case from the resulting formulae derived above because the various indeterminate expressions obtained after the substitution of the various quantities would be very difficult to evaluate.

The formulae become simpler in the case of a weir structure without a cut-off wall or in the case of a sheetpile.

a) Weir structure without a cut-off wall

In this case $s = 0$, $l_1 = l_2 = l$.

According to eqn. (3.2.22b)

$$\alpha_1 = \alpha_2 = \operatorname{tgh} \frac{\pi l}{2T} \quad (3.2.26a)$$

and according to eqn. (3.2.23f)

$$\lambda = \sqrt{(1 - e^{-(2\pi l/T)})} \quad (3.2.26b)$$

The uplift pressure according to eqn. (3.2.25a) becomes

$$\frac{h}{H} = -\frac{1}{K} F \left(\arcsin \sqrt{\frac{1 - e^{(\pi/T)(x-l)}}{1 - e^{2\pi l/T}}}, \lambda \right) \quad (3.2.26c)$$

$$-l \leq x \leq l$$

and the velocities of the seepage flow downstream of the weir on the segment M_3M_4 where

$$z = x, \quad l \leq x < \infty; \quad \zeta = \xi = \operatorname{tgh}^2 \frac{\pi x}{2T}; \quad \alpha_2 \leq \xi < 1$$

according to eqn. (3.2.24a), turn out to be

$$v_{rx} - iv_{ry} = i \frac{\pi q_r}{4TK'\xi} \sqrt{\frac{(1 + \alpha_1)(1 + \alpha_2)(1 - \xi^2)\xi^2}{(\alpha_1 + \xi)(\xi - \alpha_2)}} =$$

$$= \frac{i\pi q_r}{4TK'} \frac{e^{\pi l/2T}}{\sqrt{\{\sinh[\pi(x+l)/2T] \sinh[\pi(x-l)/2T]\}}}, \quad v_x - iv_y = k(v_{rx} - iv_{ry}) \quad (3.2.26d)$$

b) Sheetpile with the river bed on its sides at equal levels

In this case

$$l_1 = l_2 = 0, \quad \alpha_1 = \alpha_2 = \sin \frac{\pi s}{2T} \quad (3.2.27a)$$

$$\lambda = \frac{2 \sqrt{\sin(\pi s/2T)}}{1 + \sin(\pi s/2T)} \quad (3.2.27b)$$

and the velocities of the seepage flow downstream of the sheetpile on the segment $M_{2,3}M_4$ where $0 \leq z = x < \infty$, $\zeta = \xi$, $\sin(\pi s/2T) \leq \xi < 1$, with ξ according to (3.2.22a)

$$\xi = \cos \frac{\pi s}{2T} \sqrt{\left(\operatorname{tg}^2 \frac{\pi s}{2T} + \operatorname{tgh}^2 \frac{\pi x}{2T} \right)} \quad (3.2.27c)$$

calculated from (3.2.24g), turn out to be

$$\begin{aligned} v_{rx} - i v_{ry} &= i \frac{\pi q_r}{4K'T} \cos \frac{\pi s}{2T} \frac{1}{\xi} \sqrt{\frac{(1 + \alpha_1)(1 + \alpha_2)(1 - \xi^2)[\xi^2 - \sin^2(\pi s/2T)]}{(\alpha_1 + \xi)(\xi - \alpha_2)}} = \\ &= i \frac{\pi q_r}{4K'T} \left(1 + \sin \frac{\pi s}{2T} \right) \frac{1}{\sqrt{[\sin^2(\pi s/2T) + \sinh^2(\pi x/2T)]}} \\ v_x - i v_y &= k(v_{rx} - i v_{ry}) \end{aligned} \quad (3.2.27d)$$

3.2.4. Seepage under a sheetpile into a sunk excavation in the case where a permeable layer of finite thickness is underlain by impervious subsoil

The method to be outlined in this section can be used to calculate the quantity of seepage whenever the excavation is so large that the flow can be assumed to be two-dimensional, i.e. taking place in vertical planes perpendicular to the sheetpile.

The region of seepage (z) is shown schematically in Fig. 3.20. The meaning of the symbols used in the figure is as follows:

- H – depth of water in the river,
- s – depth of sheetpile embedment below the excavation bottom,
- t – depth of excavation extending below the river bed,
- T – thickness of permeable layer.

The reference plane for the determination of the piezometric head h is chosen at the level of the river bed.

We shall establish the shape of the region of the reduced complex potential. For the segment M_2M_3 where $y = t$ and $p/\gamma = 0$, it is $\varphi_r = -h = +t$; for the segment M_5M_6 , $y = 0$ and $p/\gamma = H$ and hence $\varphi_r = -H$. The values of the reduced stream function for the swept section of the sheetpile and for the surface of the impervious subsoil are chosen the same as in Section 3.2.3. Accordingly, the following will hold for the boundaries of the (w_r) region:

for the segments M_5M_6 : $\varphi_r = -H$, M_2M_3 : $\varphi_r = +t$
 $M_6M_1M_2$: $\psi_r = 0$, $M_3M_4M_5$: $\psi_r = q_r$

The region of the reduced complex potential is a rectangle with sides parallel to the coordinate axes φ_r and ψ_r — see Fig. 3.21.

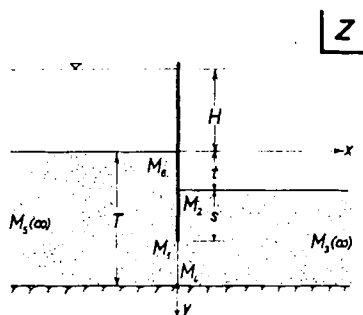


Fig. 3.20. Seepage under a sheetpile on the perimeter of a construction pit sunk into a layer of finite thickness.

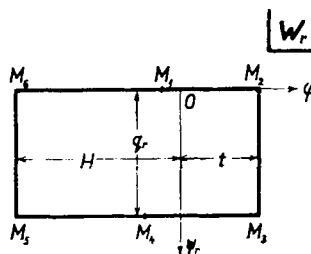


Fig. 3.21.

We shall again map both the (z) region and the (w_r) region conformally onto the auxiliary half-plane (ζ) , and by comparing the relations which will effect this transformation, obtain the reduced complex potential of seepage flow, $w_r = w_r(z)$.

The transformation of the (z) region — a rectangle with sides parallel to the coordinate axes, to the (ζ) half-plane will be effected with the help of the Schwarz-Christoffel integral. When using this integral we can choose the positions of three points in the (ζ) region, corresponding to known positions of identically designated points in the (z) region, or any other three conditions defining the relation between the positions of three points in the (z) region and those of points corresponding to them in the (ζ) region. In our case we shall choose the positions of the points M_3 and M_5 in the (ζ) region (two conditions); the third condition will require that the points M_6 and M_2 should lie in the (ζ) region symmetrically about the η axis. The table below gives the coordinates of the points in the (z) and (ζ) regions and the interior angles formed by the boundaries of the region of seepage (z) .

M_k	z_k	α_k/π	ζ_k
M_1	$+i(t+s)$	2	x
M_2	$+it$	$\frac{1}{2}$	$+a$
M_3	$+\infty$	0	$+1$
M_5	$-\infty$	0	-1
M_6	0	$\frac{1}{2}$	$-a$

It holds for κ — determining the position of the point M_1 in the (ζ) region — that

$$0 \leq \kappa \leq a$$

In this particular case the Schwarz-Christoffel integral is of the form

$$z = C \int_{\zeta_0}^{\zeta} \frac{(\zeta - \kappa) d\zeta}{(\zeta^2 - 1) \sqrt{(\zeta^2 - a^2)}} + C_1 \quad (3.2.28a)$$

where C , C_1 and κ are constants which can be determined from the condition of

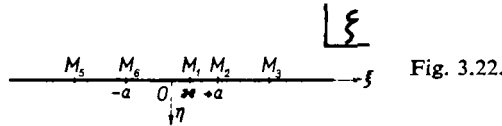


Fig. 3.22.

correspondence of the points in the (z) and (ζ) regions. A general solution to the integral (3.2.28a) is

$$z = A \operatorname{arctg} \sqrt{\left[\frac{(1+a)}{(1-a)} \cdot \frac{(a+\zeta)}{(a-\zeta)} \right]} + B \operatorname{arctg} \sqrt{\left[\frac{(1-a)}{(1+a)} \cdot \frac{(a+\zeta)}{(a-\zeta)} \right]} + C'_1 \quad (3.2.28b)$$

where A and B are constants dependent on C and κ from eqn. (3.2.28a). We shall next determine the constants of the mapping.

Determination of the constants:

For the point M_6 , we have $z = 0$, $\zeta = -a$, and eqn. (3.2.28b) gives

$$0 = A \operatorname{arctg} 0 + B \operatorname{arctg} 0 + C'_1$$

$$C'_1 = 0 \quad (3.2.28c)$$

For the point M_2 , $z = it$, $\zeta = +a$, and with a view to eqn. (3.2.28c), eqn. (3.2.28b) gives

$$it = A \operatorname{arctg} \infty + B \operatorname{arctg} \infty = (A + B) \frac{\pi}{2}$$

$$A + B = \frac{2it}{\pi} \quad (3.2.28d)$$

The third equation for the determination of the three unknown constants is obtained by establishing and comparing the increments of the function z as we pass around the point M_3 , or the point M_5 , i.e. as we pass from the segment M_2M_3 to the segment M_3M_4 , or from the segment M_4M_5 to the segment M_5M_6 . For $\zeta > a$, eqn. (3.2.28b) takes the form

$$\begin{aligned}
 z &= A \operatorname{arctg} i \sqrt{\frac{(1+a)(a+\zeta)}{(1-a)(\zeta-a)}} + B \operatorname{arctg} i \sqrt{\frac{(1-a)(a+\zeta)}{(1+a)(\zeta-a)}} = \\
 &= iA \operatorname{argtgh} \sqrt{\frac{(1+a)(a+\zeta)}{(1-a)(\zeta-a)}} + iB \operatorname{argtgh} \sqrt{\frac{(1-a)(a+\zeta)}{(1+a)(\zeta-a)}}
 \end{aligned}$$

If in the (ζ) region we pass around the point M_3 ($\zeta = +1$) along an infinitely small semi-circle, the increment of the function z will be

$$\begin{aligned}
 \Delta z &= \lim_{\substack{\zeta \rightarrow +1 \\ \Delta \zeta \rightarrow 0}} \frac{dz}{d\zeta} \Delta \zeta: \\
 \Delta z &= \lim_{\substack{\zeta \rightarrow +1 \\ \Delta \zeta \rightarrow 0}} \frac{i}{2} \left[A \sqrt{\left(\frac{1-a^2}{\zeta^2 - a^2} \right)} \frac{\Delta \zeta}{1+\zeta} + B \sqrt{\left(\frac{1-a^2}{\zeta^2 - a^2} \right)} \frac{\Delta \zeta}{1-\zeta} \right] = \\
 &= \frac{i}{2} \left[A \lim_{\substack{\zeta \rightarrow +1 \\ \Delta \zeta \rightarrow 0}} \sqrt{\left(\frac{1-a^2}{\zeta^2 - a^2} \right)} \frac{\Delta \zeta}{1+\zeta} - B \lim_{\substack{\zeta \rightarrow +1 \\ \Delta \ln(1-\zeta) \rightarrow -i\pi}} \sqrt{\left(\frac{1-a^2}{\zeta^2 - a^2} \right)} \Delta \ln(1-\zeta) \right] = \\
 &= -\frac{i}{2} B(-i\pi) = -\frac{\pi}{2} B
 \end{aligned}$$

If in the (z) region we pass from the segment M_2M_3 to the segment M_3M_4 , i.e. pass around the point $M_3(z = \infty)$, the increment of the function z (according to Fig. 3.20) will be

$$\Delta z = i(T - t)$$

Comparing the two expressions of the increment Δz we obtain

$$\Delta z = i(T - t) = -\frac{\pi}{2} B$$

or

$$B = -\frac{2i(T - t)}{\pi} \quad (3.2.28e)$$

Similarly, by passing around the point M_5 we would obtain

$$A = \frac{2iT}{\pi} \quad (3.2.28f)$$

It is clear that eqns. (3.2.28e) and (3.2.28f) really do satisfy eqn. (3.2.28d).

On substituting from eqns. (3.2.28e) and (3.2.28f) in eqn. (3.2.28b) we have the function which transforms the (ζ) region to the region of seepage (z)

$$z = \frac{2iT}{\pi} \operatorname{arctg} \sqrt{\frac{(1+a)(a+\zeta)}{(1-a)(a-\zeta)}} - \frac{2i(T-t)}{\pi} \operatorname{arctg} \sqrt{\frac{(1-a)(a+\zeta)}{(1+a)(a-\zeta)}} \quad (3.2.28g)$$

In the calculations which follow we shall also need the value of the constant $\varkappa$ from eqn. (3.2.28a). We shall determine it by differentiating eqns. (3.2.28a) and (3.2.28b) with respect to ζ and comparing the results. Differentiating eqn. (3.2.28a) we obtain

$$\frac{dz}{d\zeta} = C \frac{(\zeta - \varkappa)}{(\zeta^2 - 1)\sqrt{(\zeta^2 - a^2)}}$$

and differentiating eqn. (3.2.28b)

$$\frac{dz}{d\zeta} = A \frac{\sqrt{(1 - a^2)}}{2(1 + \zeta)\sqrt{(a^2 - \zeta^2)}} + B \frac{\sqrt{(1 - a^2)}}{2(1 - \zeta)\sqrt{(a^2 - \zeta^2)}}$$

Comparing the results and making some rearrangement we have

$$\begin{aligned} C \frac{\varkappa - \zeta}{\sqrt{(\zeta^2 - a^2)}} &= C^* \frac{\varkappa - \zeta}{\sqrt{(a^2 - \zeta^2)}} = A \frac{\sqrt{(1 - a^2)}}{2} \frac{1 - \zeta}{\sqrt{(a^2 - \zeta^2)}} + \\ &+ B \frac{\sqrt{(1 - a^2)}}{2} \frac{1 + \zeta}{\sqrt{(a^2 - \zeta^2)}} \end{aligned}$$

and after factoring out the expression $\sqrt{(a^2 - \zeta^2)}$,

$$C^*\varkappa - C^*\zeta = \frac{\sqrt{(1 - a^2)}}{2} (A + B) - \zeta \frac{\sqrt{(1 - a^2)}}{2} (A - B)$$

Clearly, in the above

$$C^* = \frac{\sqrt{(1 - a^2)}}{2} (A - B)$$

and

$$\varkappa = \frac{A + B}{A - B} = \frac{t}{2T - t} \quad (3.2.29)$$

Since we now know the position of the point M_1 in the (ζ) region, we can find the position of the points M_2 and M_6 which — as we recall — are placed symmetrically about the η axis. For the point M_1 : $z = +i(t + s)$, $\zeta = \varkappa = t/(2T - t)$, and on substituting in eqn. (3.2.28g) and some rearrangement, we obtain a relation from which we can determine a in dependence on the quantities defined by the shape of the region of seepage:

$$\operatorname{arctg} \sqrt{\frac{(1 + a)(a + \varkappa)}{(1 - a)(a - \varkappa)}} - \left(1 - \frac{t}{T}\right) \operatorname{arctg} \sqrt{\frac{(1 - a)(a + \varkappa)}{(1 + a)(a - \varkappa)}} = \frac{\pi}{2} \frac{(t + s)}{T} \quad (3.2.30)$$

We have thus established all the values which we encounter when transforming the (z) region to the (ζ) half-plane.

In our search for the relation which transforms the (w_r) region to the (ζ) half-plane we shall similarly use the Schwarz-Christoffel integral which leads — analogously as in Section 3.2.3 — to the solution of elliptic integrals. The transformation is effected with the help of the relations

$$w_1 = w_r - t \quad (3.2.31a)$$

$$w_2 = \operatorname{sn}\left(\frac{w_1}{C}, \lambda\right) \quad (3.2.31b)$$

$$w_3 = w_2^2 = \operatorname{sn}^2\left(\frac{w_1}{C}, \lambda\right) \quad (3.2.31c)$$

where w_3 is a bilinear function of ζ , viz.

$$w_3 = \frac{\zeta - a}{\zeta - 1} \cdot \frac{a + 1}{2a} \quad (3.2.31d)$$

The coordinates of the three pairs of points corresponding to each other in the (w_3) and (ζ) regions were chosen according to the table below:

M_k	M_2	M_3	M_6
ζ_k	$+a$	$+1$	$-a$
w_{3k}	0	∞	$+1$

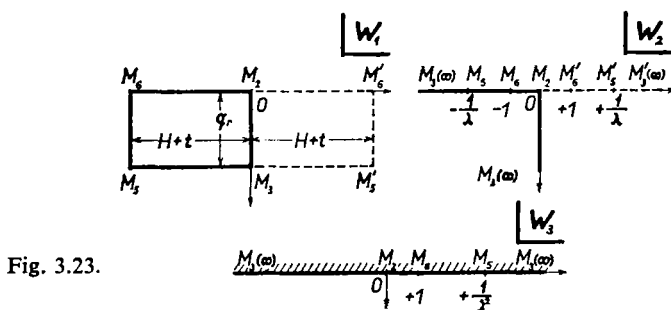


Fig. 3.23.

Relations (3.2.31) finally give us

$$w_2 = \operatorname{sn}\left(\frac{w_1}{C}, \lambda\right) = \sqrt{\left[\frac{(\zeta - a)}{(\zeta - 1)} \cdot \frac{(a + 1)}{2a}\right]} \quad (3.2.31e)$$

The various regions are shown in Fig. 3.23; the (ζ) region is in Fig. 3.22.

In eqn. (3.2.31e) λ is the modulus of the elliptic functions and C is a constant defined by

$$C = \frac{H + t}{K}$$

where K is the complete elliptic integral of the first kind with the modulus λ . The magnitude of the modulus λ is found from eqn. (3.2.31d) by substitution of the values of the point M_5 ; for that point, $w_3 = +1/\lambda^2$, $\zeta = -1$, and after rearrangement we have

$$\lambda = \frac{2\sqrt{a}}{a+1} \quad (3.2.31f)$$

For the purpose of determining the components of seepage velocity according to eqn. (3.1.23') we modify eqn. (3.2.31b) by changing to the inverse function and substituting at the same time for the constant C and for w_1 from eqn. (3.2.31a):

$$w_r - t = \frac{H + t}{K} \int_0^{w_2} \frac{dw_2}{\sqrt{[(1 - w_2^2)(1 - \lambda^2 w_2^2)]}} \quad (3.2.32)$$

We have for the velocity components

$$v_{rx} - iv_{ry} = \frac{dw_r}{dz} = \frac{dw_r}{dw_2} \cdot \frac{dw_2}{d\zeta} \cdot \frac{d\zeta}{dz} \quad (3.2.33a)$$

Differentiation of eqn. (3.2.32) gives

$$\frac{dw_r}{dw_2} = \frac{H + t}{K} \cdot \frac{1}{\sqrt{[(1 - w_2^2)(1 - \lambda^2 w_2^2)]}}$$

and after substitution from eqns. (3.2.31e) and (3.2.31f) we have

$$\frac{dw_r}{dw_2} = \frac{H + t}{K} \cdot \frac{(\zeta - 1)}{(a - 1)} \cdot \frac{\sqrt{[2a(a + 1)]}}{\sqrt{[(a + \zeta)(\zeta + 1)]}}$$

Differentiation of eqn. (3.2.31e) leads to

$$\frac{dw_2}{d\zeta} = \frac{1}{2} \frac{(a - 1)}{(\zeta - 1)^2} \sqrt{\left[\frac{(\zeta - 1)}{(\zeta - a)} \cdot \frac{(a + 1)}{2a} \right]}$$

The derivative of eqn. (3.2.28g) was already taken in the determination of the constant κ . We now rearrange it in the form

$$\frac{dz}{d\zeta} = \frac{\sqrt{(1 - a^2)}}{2\sqrt{(a^2 - \zeta^2)}} \left(\frac{A}{1 + \zeta} + \frac{B}{1 - \zeta} \right) = -\frac{i}{\pi} \cdot \frac{\sqrt{(1 - a^2)}}{\sqrt{(a^2 - \zeta^2)}} \cdot \frac{2T\zeta - t(1 + \zeta)}{1 - \zeta^2}$$

The magnitude of the velocity components is obtained by substitution in eqn. (3.2.33a):

$$v_{rx} - iv_{ry} = \frac{i\pi(H + t)(a + 1)\sqrt{(1 - \zeta^2)}}{2K\sqrt{(1 - a^2)}[2T\zeta - t(\zeta + 1)]}, \quad v_x - iv_y = k(v_{rx} - iv_{ry}) \quad (3.2.33b)$$

For the bottom of the excavation where

$$a \leq \zeta = \xi \leq +1$$

eqn. (3.2.33b) gives

$$v_{rx} = 0; \quad v_{ry} = -\frac{\pi(H + t)}{2K} \cdot \frac{(a + 1)}{\sqrt{(1 - a^2)}} \cdot \frac{\sqrt{(1 - \zeta^2)}}{[2T\zeta - t(\zeta + 1)]}, \quad v_y = kv_{ry}$$

In practice we proceed as follows: for the given values of the quantities describing the shape of the region of seepage, we first determine the value of κ from eqn. (3.2.29) and then the value of a from eqn. (3.2.30). For this a we establish the value of the complete elliptic integral K , and on substituting all the values in eqn. (3.2.33b), obtain the components v_{rx} and v_{ry} at any point of the region of seepage.

The case of seepage solved in Section 3.2.2 is only a special case of the one just discussed when $T \rightarrow \infty$. It is, however, not amenable to the application of the formulae derived above because some of the quantities (for example s) on which seepage depends, vanish when $T = \infty$ is substituted in the various expressions.

3.2.5. Possibilities of additional solutions

In Sections 3.2.1 to 3.2.4 we illustrated with several examples how Pavlovsky's method of conformal mapping may be applied to the solution of groundwater seepage. It need not be stressed that the examples do not include all problems that can conveniently be solved by that method — the limited scope does not permit this, and neither do we intend to make the book a collection of worked problems. All we wished to do in the preceding sections was to show that theoretical solutions to some suitably simplified cases of groundwater flow can be obtained by procedures which are by no means complicated — as even an uninitiated reader will acknowledge on becoming familiar with the principles and fundamental relations of conformal mapping. The professional literature offers solutions to a number of further problems — those interested in their details and particulars are referred to comprehensive publications, e.g. [1], [17], [29], [34] where they will find a number of completely worked examples and additional sources of information as well.

3.3. Vedernikov-Pavlovsky method

The method is named after V. V. Vedernikov and N. N. Pavlovsky who applied it independently at nearly the same time — about 1934 — to the solution of problems of groundwater flow. They started from the method proposed by N. E. Zhukovsky who — more than ten years earlier — had introduced into the solution of problems of seepage flow, a new analytic function called Zhukovsky's function in his honour.

The Vedernikov-Pavlovsky method is especially well suited to analyses of problems in which the region of seepage has:

horizontal permeable segments,
vertical impervious segments,
a depression curve (free surface).

We first explain the principles of the method:

A new analytic function

$$w_{zh} = z - iw_r = u_{zh} + iv_{zh} = x + \psi_r + i(y - \varphi_r) \quad (3.3.1)$$

is introduced, which is a modification of the earlier, well-known Zhukovsky's function. The meaning of the various quantities in eqn. (3.3.1) is the same as in the preceding sections.

According to (3.3.1), the boundary conditions for the region of the reduced complex potential and the region of Zhukovsky's function, i.e. for the (w_r) and (w_{zh}) regions are:

for horizontal permeable segments:

$$\begin{aligned} \varphi_r &= \text{const} ; \quad y = \text{const.} \\ v_{zh} &= \text{const.} \end{aligned} \quad (3.3.2)$$

for vertical impervious segments:

$$\begin{aligned} \psi_r &= \text{const} ; \quad x = \text{const.} \\ u_{zh} &= \text{const.} \end{aligned} \quad (3.3.3)$$

for the depression curve:

$$\begin{aligned} \psi_r &= \text{const.} \\ \varphi_r &= - \left(\frac{p}{\gamma} - y \right) = y + \text{const.} \\ v_{zh} &= \text{const.} \end{aligned} \quad (3.3.4)$$

Hence both the (w_r) and the (w_{zh}) regions are polygons with sides parallel to the coordinate axes.

The solution of the problem consists of finding the function

$$w_{zh} = f(w_r) \quad (3.3.5)$$

Again applying the method of conformal mapping, we map the (w_r) region onto the (w_{zh}) region. Using eqn. (3.3.5) we obtain from eqn. (3.3.1)

$$z = w_{zh} + iw_r = iw_r + f(w_r) \quad (3.3.6)$$

When using successive conformal mappings with the auxiliary half-plane (ζ) we have

$$w_r = f_1(\zeta), \quad w_{zh} = f_2(\zeta) \quad (3.3.7)$$

and after substituting in eqn. (3.3.6)

$$z = if_1(\zeta) + f_2(\zeta) \quad (3.3.8)$$

Using eqn. (3.3.6) or the first of eqns. (3.3.7) and eqn. (3.3.8), we determine the complex potential of seepage flow at any point of the region of seepage (z) and in turn, all the other quantities of interest.

The Vedernikov-Pavlovsky method is frequently used in the so-called semi-reverse version.

In the application of this version, the permeable segments need not be horizontal, nor the impervious segments vertical. We determine the (w_r) region, and specify the portions of the segments of the boundary of the (w_{zh}) region, corresponding to the permeable and impervious segments of the region of seepage in a way that will make the function $w_{zh} = f(w_r)$ easy to construct. Next, we establish the function $z = z(w_r)$ and from the values of w_r on the boundary of the (w_r) region determine — by reverse procedure — the shape of the permeable and impervious segments. If the shape thus ascertained corresponds approximately to the specified shape, we regard the (w_{zh}) region as correct and the problem as solved.

The semi-reverse version is plainly an approximate method.

For the cases where infiltration or evaporation from the free surface is present, Numerov introduced yet another analytic function, viz.

$$w_n = v_{0r}z - iw_r = \mu + iv = v_{0r}x + \psi_r + i(v_{0r}y - \varphi_r) \quad (3.3.9)$$

where $v_{0r} = v_0/k$ is the reduced value of infiltration or evaporation. The boundary conditions for the (w_r) , (w_{zh}) and (w_n) regions then turn out to be

for horizontal permeable segments:

$$\begin{aligned} \varphi_r &= \text{const.}, \quad v_{zh} = \text{const.} \\ v &= \text{const.}, \end{aligned} \quad (3.3.10)$$

for vertical impervious segments:

$$\begin{aligned}\psi_r &= \text{const.}, \quad u_{zh} = \text{const.} \\ \mu &= \text{const.},\end{aligned}\tag{3.3.11}$$

for the depression curve:

$$\begin{aligned}\varphi_r &= -\left(\frac{p}{y} - y\right) + C = \text{const.} + y, \\ \psi_r &= \text{const.} - v_{0r}x, \quad v_{zh} = \text{const.} \\ \mu &= \text{const.}\end{aligned}\tag{3.3.12}$$

Hence the (w_r) , (w_{zh}) as well as the (w_n) regions are polygons having sides parallel to the coordinate axes and can be represented on the auxiliary half-plane (ζ) with the help of the Schwarz-Christoffel integral. From the equations

$$w_n = v_{0r}z - iw_r\tag{3.3.9}$$

and

$$w_{zh} = z - iw_r\tag{3.3.1}$$

we obtain on subtraction

$$z = \frac{w_{zh}(\zeta) - w_n(\zeta)}{1 - v_{0r}}\tag{3.3.13}$$

and further, substituting in eqn. (3.3.1)

$$w_r = \frac{i[w_n(\zeta) - v_{0r} w_{zh}(\zeta)]}{1 - v_{0r}}\tag{3.3.14}$$

This represents the solution to our problem.

The various problems to which the Vedernikov-Pavlovsky method can be applied, will be discussed in the sections which follow.

3.3.1. Seepage from a reservoir with a permeable bottom under a cut-off wall (a sheetpile) on the perimeter

A solution to this problem — under greatly simplified conditions — was first obtained by Zhukovsky. We shall show here a solution due to Vedernikov who included in his treatment the effect of capillarity on the free surface.

We shall assume that the permeable layer is of great, theoretically infinite, extent. Two essentially different cases of seepage (see Fig. 3.24) can potentially occur under the cut-off wall:

seepage with backwater (Fig. 3.24a) when the percolating water downstream of the cut-off wall spreads laterally and approaches asymptotically the original level of groundwater (which is theoretically assumed to be at infinite depth), and

seepage without backwater (Fig. 3.24b) when it is assumed that the permeable layer is underlain at a great (infinite) depth by another layer with a far greater permeability acting as a drain so that the water which has percolated laterally downstream of the cut-off wall, is deflected fast in the downward direction.

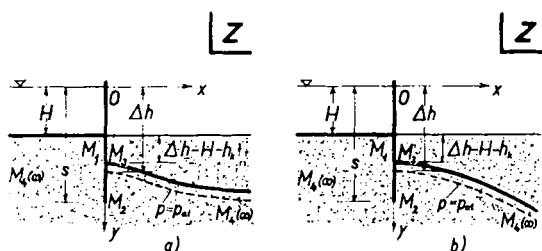


Fig. 3.24. Seepage from a reservoir under a cut-off wall on the perimeter.

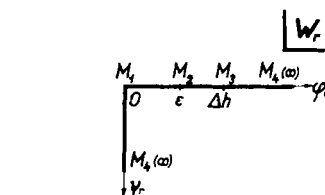


Fig. 3.25. Region of reduced complex potential.

In the analysis done by the Vedernikov-Pavlovsky method we shall determine the shape of the region of the reduced complex potential, w_r , the shape of the region of Zhukovsky's function, w_{zh} , relation (3.3.5) with the help of conformal mapping, and in accordance with eqn. (3.3.6) the dependence

$$z = z(w_r).$$

We shall place the reference plane for the determination of the piezometric heads and the potential at the level of the surface in the reservoir.

In such a case we have for the (w_r) region:

for the bottom of the reservoir — the segment M_4M_1 :

$$\varphi_r = -h = -(p/\gamma - y) = 0;$$

for the swept surface of the cut-off wall — the segment $M_1M_2M_3$:

$$\psi_r = \text{const.}, \text{ we choose: } \psi_r = 0$$

for the free surface — the segment M_3M_4 : since the free surface is a continuation of the streamline flowing around the cut-off wall, $\psi_r = 0$. The shape of the (w_r) region is shown in Fig. 3.25.

For the boundaries of the (w_{zh}) region we have:

for the bottom of the reservoir — the segment M_4M_1 :

$$y = H, \quad \varphi_r = 0$$

$$v_{zh} = H$$

according to (3.3.1)

for the sheetpile — the segment $M_1M_2M_3$:

$$x = 0, \quad \psi_r = 0, \quad u_{zh} = 0;$$

for the depression curve — the segment M_3M_4 : since the effect of capillarity is being taken into account, the pressure will be $p = -p_k$ instead of $p = p_{at} = 0$; hence $\varphi_r = -h = -(p/\gamma - y) = -(-p_k/\gamma - y) = h_k + y$, where $h_k = p_k/\gamma$ is the height of capillary elevation in the soil, and according to eqn. (3.3.1)

$$v_{zh} = y - \varphi_r = y - h_k - y = -h_k$$

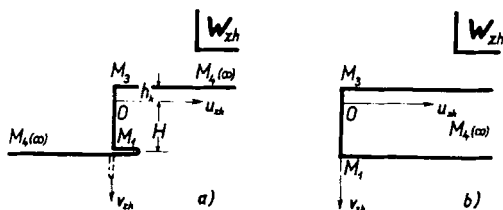


Fig. 3.26. Region of Zhukovsky's function.

For the general case the shape of the region of Zhukovsky's function is as shown in Fig. 3.26a; for the special case of seepage without backwater (Fig. 3.24b) the shape of the (w_{zh}) is shown in Fig. 3.26b. We shall carry out our solution for the general shape (Fig. 3.26a) and obtain the solution for the shape shown in Fig. 3.26b as a special case in the course of the calculations. After substituting the coordinates of the corresponding points in the various regions into the resulting formulae we shall discover whether or not the shape according to Fig. 3.26b was estimated correctly.

It can readily be proved — by substituting the corresponding coordinates — that the conformal mapping of the (w_r) region onto the (w_{zh}) region is effected by the function

$$w_{zh} = iH + iCw_r \sqrt{\left[1 - \left(\frac{w_r}{\Delta h}\right)^2\right]} - i\frac{2}{\pi}(H + h_k) \arcsin \frac{w_r}{\Delta h} \quad (3.3.15)$$

where

C is a constant whose value will be determined in the course of the calculations, Δh is the difference between the heights of the water level in the reservoir and of the free surface (i.e. of the level where $p = p_{at} = 0$) on the downstream side of the sheetpile; this, in fact, is the loss of the piezometric head produced as the flow sweeps past the sheetpile.

According to eqns. (3.3.5) and (3.3.6) we determine the complex potential w , as a function of the complex coordinate z of the region of seepage,

$$z = iw_r + w_z = iH + iw_r + iCw_r \sqrt{\left[1 - \left(\frac{w_r}{\Delta h}\right)^2\right]} - i\frac{2}{\pi}(H + h_k) \arcsin \frac{w_r}{\Delta h} \quad (3.3.16)$$

Eqn. (3.3.16) readily yields the equation of the depression curve after substituting for w_r the values relating to the depression curve, i.e.

$$w_r = \varphi_r = h_k + y:$$

$$x + iy = iH + i(h_k + y) + iC(h_k + y) \sqrt{\left[1 - \left(\frac{h_k + y}{\Delta h}\right)^2\right]} - \\ - \frac{2i}{\pi} (H + h_k) \arcsin \frac{h_k + y}{\Delta h}$$

and after rearrangement

$$x = C(h_k + y) \sqrt{\left[\left(\frac{h_k + y}{\Delta h}\right)^2 - 1\right]} + \frac{2}{\pi} (H + h_k) \operatorname{argcosh} \frac{h_k + y}{\Delta h} \\ \Delta h \leq h_k + y < \infty \quad (3.3.17)$$

The velocity of groundwater flow is established with the help of eqn. (3.1.23'). Differentiating eqn. (3.3.16)

$$\frac{dz}{dw_r} = i \left[1 + C \sqrt{\left[1 - \left(\frac{w_r}{\Delta h}\right)^2\right]} - C \frac{(w_r/\Delta h)^2}{\sqrt{\left[1 - (w_r/\Delta h)^2\right]}} - \right. \\ \left. - \frac{2}{\pi} (H + h_k) \frac{1/\Delta h}{\sqrt{\left[1 - (w_r/\Delta h)^2\right]}} \right] = i \left[1 + \frac{C[1 - 2(w_r/\Delta h)^2] - (2/\pi)(H + h_k)/\Delta h}{\sqrt{\left[1 - (w_r/\Delta h)^2\right]}} \right]$$

we obtain

$$v_{rx} - iv_{ry} = \frac{dw_r}{dz} = i \frac{\sqrt{\left[1 - (w_r/\Delta h)^2\right]}}{\frac{2}{\pi} (H + h_k)/\Delta h - C[1 - 2(w_r/\Delta h)^2] - \sqrt{\left[1 - (w_r/\Delta h)^2\right]}} \\ v_x - iv_y = k(v_{rx} - iv_{ry}). \quad (3.3.18)$$

There are two unknown constants, C and Δh in eqns. (3.3.16) to (3.3.18). We need two conditions to determine them. The procedure to be applied in subsequent calculations depends on which of the cases of seepage — with or without backwater — we intend to solve.

a) Seepage without backwater (Fig. 3.24b)

In this case the depression curve — the segment M_3M_4 — is deflected rapidly in the downward direction and tends ever more to a vertical line as the coordinate y increases.

The slope of the tangent to the depression curve is found from eqn. (3.3.17):

$$\frac{dx}{dy} = \frac{C\{2[(h_k + y)/\Delta h]^2 - 1\} + (2/\pi)(H + h_k)/\Delta h}{\sqrt{\{[(h_k + y)/\Delta h]^2 - 1\}}} \quad (3.3.19)$$

In order to obtain for $y \rightarrow \infty$

$$\lim_{y \rightarrow \infty} \frac{dx}{dy} = 0$$

it must clearly hold that

$$C = 0$$

in eqn. (3.3.19).

Eqns. (3.3.15) to (3.3.18) take the form

$$w_{zh} = iH - i \frac{2}{\pi} (H + h_k) \arcsin \frac{w_r}{\Delta h} \quad (3.3.15a)$$

$$z = iH + iw_r - i \frac{2}{\pi} (H + h_k) \arcsin \frac{w_r}{\Delta h} \quad (3.3.16a)$$

the equation of the depression curve becomes

$$y = \Delta h \cosh \frac{\pi x}{2(H + h_k)} - h_k \quad (3.3.17a)$$

and for the determination of the velocity components we have

$$v_{rx} - iv_{ry} = i \frac{\sqrt{[1 - (w_r/\Delta h)^2]}}{\frac{2}{\pi} (H + h_k)/\Delta h - \sqrt{[1 - (w_r/\Delta h)^2]}}, \quad v_x - iv_y = k(v_{rx} - iv_{ry}) \quad (3.3.18a)$$

There is one more unknown — Δh — in eqns. (3.3.15a) to (3.3.18a) and we determine it as follows:

We start from the fact that the velocity of the flow past the end of the sheetpile is infinitely large

$$(v_{rx} - iv_{ry})_{M_2} = \infty.$$

For this to be so, the denominator in eqn. (3.3.18a) in which $w_r(M_2) = \varepsilon$ was substituted for w_r , must be equal to zero:

$$\frac{2}{\pi} \frac{(H + h_k)}{\Delta h} = \sqrt{\left[1 - \left(\frac{\varepsilon}{\Delta h}\right)^2\right]} \quad (3.3.20)$$

Next, we substitute in eqn. (3.3.16a) the values relating to the point M_2

$$z = is, \quad w_r = \varepsilon :$$

$$s = H + \varepsilon - \frac{2}{\pi} (H + h_k) \arcsin \frac{\varepsilon}{\Delta h} \quad (3.3.21)$$

Eqns. (3.3.20) and (3.3.21) form a system from which we can find both Δh and ε and thus complete the solution of our case.

b) Seepage with backwater (Fig. 3.24a)

It is no longer possible in this case to use the slope of the tangent to the depression curve at the point $y = \infty$ for the determination of the constant C . To find the unknown constants we have to know yet another point of the depression curve. Such a point is either specified in advance by conditions special to the case being considered, or may be chosen, for example, to be a point lying at a sufficiently large distance from the canal, on the level of the original free surface of groundwater. Its coordinates, (x_0, y_0) are substituted in eqn. (3.3.17)

$$x_0 = C(h_k + y_0) \sqrt{\left[\left(\frac{h_k + y_0}{\Delta h}\right)^2 - 1\right]} + \frac{2}{\pi} (H + h_k) \operatorname{argcosh} \frac{h_k + y_0}{\Delta h} \quad (3.3.22)$$

Just as before, we start from the infinitely large velocity of the flow past the end of the sheetpile ($z = is, w_r = \varepsilon$); this means that after this substitution the denominator in eqn. (3.3.18) must be equal to zero. We obtain

$$C = \frac{\frac{2}{\pi} (H + h_k)/\Delta h - \sqrt{[1 - (\varepsilon/\Delta h)^2]}}{1 - 2(\varepsilon/\Delta h)^2} \quad (3.3.23)$$

Substituting the values relating to the point $M_2(z = is, w_r = \varepsilon)$ in eqn. (3.3.16), we have

$$s = H + \varepsilon + C\varepsilon \sqrt{\left[1 - \left(\frac{\varepsilon}{\Delta h}\right)^2\right]} - \frac{2}{\pi} (H + h_k) \arcsin \frac{\varepsilon}{\Delta h} \quad (3.3.24)$$

Eqns. (3.3.22) to (3.3.24) constitute a system of equations that enables us to determine the three unknowns, C , Δh and ε and thus complete the solution.

In practical cases, the calculation may proceed as follows: for a chosen sequence of values of Δh we determine C from eqn. (3.3.22), ε from eqn. (3.3.23), and finally, the depth of sheetpile embedment, s , from eqn. (3.3.24). We then plot the dependence

$$s = f(\Delta h)$$

graphically and use the diagram to determine the values of Δh corresponding to a given s ; the remaining constants are obtained by the reverse procedure.

3.3.2. Seepage through an earth dam with a central cut-off wall and a drain on the downstream side of the foundation

The earth dam is shown schematically in Fig. 3.27. We assume that the permeability of the dam material is the same as that of the subsoil, the cut-off wall is impervious, the drain is a horizontal slot theoretically of zero height at the level of the surface of the terrain downstream of the dam; the water surface downstream of the dam is likewise at the level of the terrain surface.

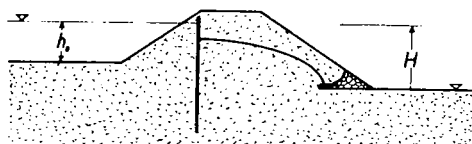


Fig. 3.27. Seepage through an earth dam with a cut-off wall and drain.

To the best of our knowledge, a case of seepage simplified as indicated has so far defied mathematical treatment leading to numerical results. The greatest obstacle in the path to a complete solution, is the existence of the portion of the dam upstream of the cut-off wall (the contour drawn in dashed lines in Fig. 3.28).

We shall show how to solve the case when the portion of the dam upstream of the cut-off wall is missing (Fig. 3.28). The actual case of seepage is intermediate to this case and the case when the bottom of the reservoir upstream of the dam is considered to be at the level of the water surface upstream of the dam, i.e. the depth of water upstream of the dam $h_0 = 0$.

We solve our case using the Vedernikov-Pavlovsky method with the help of Zhukovsky's function

$$w_{zh} = u_{zh} + iv_{zh} = (x + \psi_r) + i(y - \varphi_r)$$

The values on the boundaries of the various regions are:

the river bed upstream of the dam — the segment M_6M_1 :

$$y = h_0, \quad \frac{p}{\gamma} = h_0, \quad \varphi_r = -\left(\frac{p}{\gamma} - y\right) = -(h_0 - h_0) = 0, \quad v_{zh} = h_0$$

the wetted perimeter of the cut-off wall — the segment $M_1M_2M_3$:

$$x = 0, \quad \psi_r = 0, \quad u_{zh} = 0$$

the free surface (the depression curve) — the segment M_3M_4 :

$$\frac{p}{\gamma} = 0, \quad \varphi_r = +y, \quad \psi_r = 0, \quad v_{zh} = 0$$

the surface of the drain and the surface of the terrain through which the water

percolates — the segment $M_4M_5M_6$:

$$y = H, \quad \frac{p}{\gamma} = 0, \quad \varphi_r = +H, \quad v_{zh} = 0$$

The shape of the (w_r) region of the reduced complex potential and the shape of the (w_{zh}) region of Zhukovsky's function are drawn in Fig. 3.29. The figure also shows the (ζ) region, i.e. the auxiliary half-plane which is needed to negotiate the conformal mapping of the (w_r) region onto the (w_{zh}) region.

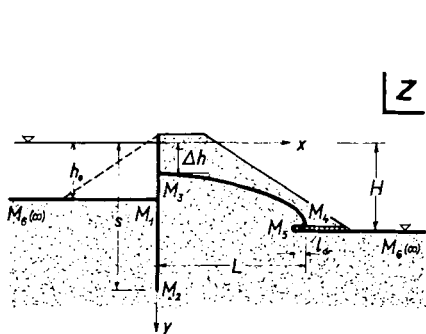


Fig. 3.28.

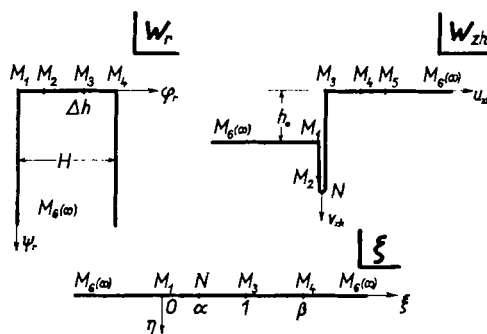


Fig. 3.29.

The conformal mapping of the (ζ) half-plane onto the (w_{zh}) region is effected with the help of the Schwarz-Christoffel integral. The values necessary for its construction are set out in the table below.

M_k	w_{zhk}	α_k/π	ζ_k
M_1	ih_0	$\frac{1}{2}$	0
N	iv_{zhN}	2	α
M_3	0	$\frac{1}{2}$	1
M_6	∞	-1	∞

Hence the function $w_{zh} = f(\zeta)$ is

$$\begin{aligned}
 w_{zh} &= C \int_{\zeta_0}^{\zeta} \zeta^{-1/2} (\zeta - \alpha) (\zeta - 1)^{-1/2} d\zeta = C \int_{\zeta_0}^{\zeta} \frac{\zeta - \alpha}{\sqrt{(\zeta^2 - \zeta)}} d\zeta = \\
 &= C \left[\left| \sqrt{(\zeta^2 - \zeta)} \right|_{\zeta_0}^{\zeta} + \left(\frac{1}{2} - \alpha \right) \int_{\zeta_0}^{\zeta} \frac{d\zeta}{\sqrt{(\zeta^2 - \zeta)}} \right] = \\
 &= C \left[\sqrt{(\zeta^2 - 1)} + \frac{1 - 2\alpha}{2} \operatorname{argcosh} (2\zeta - 1) \right] + C_1
 \end{aligned}$$

For the point M_3 : $\zeta = 1$, $w_{zh} = 0$, and after substitution

$$C_1 = 0$$

For the point M_1 : $\zeta = 0$, $w_{zh} = ih_0$, and

$$C = \frac{ih_0}{\frac{1}{2}(1-2\alpha)i \arccos(-1)} = \frac{ih_0}{\frac{1}{2}(1-2\alpha)i\pi} = \frac{2h_0}{\pi(1-2\alpha)}$$

The resulting form of the function effecting the conformal mapping of the (ζ) region onto the (w_{zh}) region is

$$w_{zh} = \frac{2h_0}{\pi(1-2\alpha)} \left[\sqrt{(\zeta^2 - \zeta)} + \frac{1-2\alpha}{2} \operatorname{argcosh}(2\zeta - 1) \right] \quad (3.3.25)$$

The region of the reduced complex potential will be mapped onto the (ζ) half-plane with the help of the function

$$\zeta = \beta \sin^2 \frac{\pi w_r}{2H}$$

this can readily be proved by substituting the known coordinates of the points on the boundaries of the regions, corresponding in the (ζ) and the (w_r) regions. Further, for the point M_3 : $\zeta = 1$, $w_r = \Delta h$; after the substitution

$$\beta = \frac{1}{\sin^2(\pi \Delta h / 2H)}$$

so that the function $\zeta = f(w_r)$ becomes

$$\zeta = \frac{\sin^2(\pi w_r / 2H)}{\sin^2(\pi \Delta h / 2H)} \quad (3.3.26)$$

where Δh is the loss of piezometric head produced in the flow past the cut-off wall.

Substituting from eqn. (3.3.25) in eqn. (3.3.6) we obtain the relation $z = z(w_r)$ which fully defines our case of flow:

$$z = w_{zh} + iw_r = iw_r + \frac{2h_0}{\pi(1-2\alpha)} \left[\sqrt{(\zeta^2 - \zeta)} + \frac{1-2\alpha}{2} \operatorname{argcosh}(2\zeta - 1) \right] \quad (3.3.27a)$$

ζ is according to (3.3.26).

There is an unknown constant α in eqn. (3.3.27a). It is determined by substituting the coordinates of the point M_4 in the (z) , (w_r) and (ζ) regions, i.e.

$$z = L + iH, \quad w_r = H, \quad \zeta = \beta = \frac{1}{\sin^2(\pi \Delta h / 2H)}$$

in eqn. (3.3.27a):

$$L = \frac{2h_0}{\pi(1-2\alpha)} \frac{\cos(\pi \Delta h/2H)}{\sin^2(\pi \Delta h/2H)} + \frac{h_0}{\pi} \operatorname{argcosh} \frac{1 + \cos^2(\pi \Delta h/2H)}{\sin^2(\pi \Delta h/2H)}$$

$$\alpha = \frac{1}{2} - \frac{\cos(\pi \Delta h/2H)}{\left[\pi(L/h_0) - \operatorname{argcosh} \frac{1 + \cos^2(\pi \Delta h/2H)}{\sin^2(\pi \Delta h/2H)} \right] \sin^2(\pi \Delta h/2H)}$$
(3.3.28)

Substituting for α in eqn. (3.3.27a) we obtain the resulting shape of the function $z = z(w_r)$

$$z = iw_r + \frac{\left[L - (h_0/\pi) \operatorname{argcosh} \frac{1 + \cos^2(\pi \Delta h/2H)}{\sin^2(\pi \Delta h/2H)} \right] \sin(\pi w_r/2H)}{\cos(\pi \Delta h/2H)} \times$$

$$\times \sqrt{\left(\sin^2 \frac{\pi w_r}{2H} - \sin^2 \frac{\pi \Delta h}{2H} \right)} + \frac{h_0}{\pi} \operatorname{argcosh} \left(2 \frac{\sin^2(\pi w_r/2H)}{\sin^2(\pi \Delta h/2H)} - 1 \right)$$
(3.3.27b)

The components of seepage velocity are obtained by differentiating eqn. (3.3.27a) and bearing eqn. (3.3.26) in mind:

$$\frac{dz}{dw_r} = \frac{1}{v_x - iv_y} = i + 2 \frac{h_0}{H} \frac{\sin(\pi w_r/2H) \cos(\pi w_r/2H)}{(1-2\alpha) \sin^2(\pi \Delta h/2H)} \frac{\zeta - \alpha}{\sqrt{(\zeta^2 - \zeta)}} \quad (3.3.29)$$

where eqn. (3.3.26) is substituted for ζ , and eqn. (3.3.28) for α .

The velocity with which the water enters the drain or rises above the terrain is determined from eqn. (3.3.29) on substituting the values relating to the segment $M_4M_5M_6$:

$$z = x + iH, \quad \beta = \frac{1}{\sin^2(\pi \Delta h/2H)} \leq \zeta < \infty, \quad w_r = H + i\psi_r:$$

$$v_{ry} = \frac{1}{1 - \frac{2h_0}{H} \frac{(\zeta - \alpha) \sinh(\pi \psi_r/2H) \cosh(\pi \psi_r/2H)}{(1-2\alpha) \sin^2(\pi \Delta h/2H) \cdot \sqrt{(\zeta^2 - \zeta)}}}, \quad v_y = kv_{ry} \quad (3.3.30a)$$

the position of the point at which the velocity is of this magnitude is found from eqn. (3.3.27a)

$$x = -\psi_r + \frac{2h_0}{\pi(1-2\alpha)} \left[\sqrt{(\zeta^2 - \zeta)} + \frac{1-2\alpha}{2} \operatorname{argcosh}(2\zeta - 1) \right] \quad (3.3.30b)$$

According to (3.3.26) we have in eqns. (3.3.30a) and (3.3.30b)

$$\zeta = \frac{\sin^2 [\pi(H + i\psi_r)/2H]}{\sin^2 (\pi \Delta h/2H)} = \frac{\cosh^2 (\pi\psi_r/2H)}{\sin^2 (\pi \Delta h/2H)} \quad (3.3.30c)$$

We choose the values of ψ_r/H , $0 \leq \psi_r/H < \infty$, determine ζ from eqn. (3.3.30c), and the position of the points and the magnitude of the velocities from eqns. (3.3.30a) and (3.3.30b).

The equation of the depression curve in the dam — the segment M_3M_4 — is obtained from eqn. (3.3.27b) with the substitution of the values of the free surface

$$\begin{aligned} z = x + iy, \quad \psi_r = 0, \quad w_r = y : \\ x = \frac{\left[L - \frac{h_0}{\pi} \operatorname{argcosh} \frac{1 + \cos^2 (\pi \Delta h/2H)}{\sin^2 (\pi \Delta h/2H)} \right] \sin (\pi y/2H)}{\cos (\pi \Delta h/2H)} \times \\ \times \sqrt{\left(\sin^2 \frac{\pi y}{2H} - \sin^2 \frac{\pi \Delta h}{2H} \right) + \frac{h_0}{\pi} \operatorname{argcosh} \left(2 \frac{\sin^2 (\pi y/2H)}{\sin^2 (\pi \Delta h/2H)} - 1 \right)} \\ \Delta h \leq y \leq H \end{aligned} \quad (3.3.31)$$

The length of the drain, l_{dr} , is found from eqn. (3.3.29) as follows: for the point M_5 we set $dz/dw_r = 0$, determine $w_r(M_5)$ — having regard to the fact that $\varphi_r(M_5) = H$ — and after substituting in eqn. (3.3.27b) obtain $L_0 = L - l_{dr}$. As the calculations show, for $L > 2H$, l_{dr} is always less than $0.1H$. It is therefore recommended to use $l_{dr} \doteq 0.1H$ in practical calculations.

So far, the depth of the cut-off wall, s , has not appeared in the calculations. What figures there is the loss Δh of the piezometric head produced by the flow past the cut-off wall; this loss is directly affected by the depth s . We now establish a formula which relates the two quantities.

We start from the fact that in the flow past the cut-off wall the velocity is infinite, i.e. that $dw_r/dz = \infty$, $dz/dw_r = 0$ for the point M_2 . We therefore set $dz/dw_r = 0$ in eqn. (3.3.29), find $w_r(M_2)$ (remembering that $\psi_r(M_2) = 0$), and after substitution in eqn. (3.3.27b), obtain s as a function of L , H , h_0 and Δh .

For the point M_2 , $dz/dw_r = 0$, $0 < \zeta < 1$. Substituting in eqn. (3.3.29) we have

$$\frac{h_0}{H} \frac{2}{1 - 2\alpha} \frac{\sin [\pi w_r(M_2)/2H] \cos [\pi w_r(M_2)/2H]}{\sin^2 (\pi \Delta h/2H)} \frac{(\zeta - \alpha)}{\sqrt{(\zeta - \zeta^2)}} = 1$$

and substituting for ζ from eqn. (3.3.26)

$$\frac{h_0}{H} \frac{2}{1 - 2\alpha} \frac{\cos \frac{\pi w_r(M_2)}{2H} \left(\frac{\sin^2 [\pi w_r(M_2)/2H]}{\sin^2 (\pi \Delta h/2H)} - \alpha \right)}{\sqrt{\left\{ \sin^2 (\pi \Delta h/2H) - \sin^2 [\pi w_r(M_2)/2H] \right\}}} = 1 \quad (3.3.32a)$$

where

$$0 < \frac{w_r(M_2)}{H} < \frac{\Delta h}{H}$$

$w_r(M_2)$ is determined from eqn. (3.3.32) by successive approximations, or better still, graphically. The depth s as a function of L , H , h_0 and Δh is then determined from eqn. (3.3.27b), viz.

$$\begin{aligned} \frac{s}{H} = & \frac{w_r(M_2)}{H} + \frac{h_0}{H} \frac{\frac{L}{h_0} - \frac{1}{\pi} \operatorname{argcosh} \frac{1 + \cos^2(\pi \Delta h/2H)}{\sin^2(\pi \Delta h/2H)}}{\cos(\pi \Delta h/2H)} \sin \frac{\pi w_r(M_2)}{2H} \times \\ & \times \sqrt{\left[\sin^2 \frac{\pi \Delta h}{2H} - \sin^2 \frac{\pi w_r(M_2)}{2H} \right]} + \frac{h_0}{\pi H} \arccos \left[2 \frac{\sin^2 [\pi w_r(M_2)/2H]}{\sin^2(\pi \Delta h/2H)} - 1 \right] \end{aligned} \quad (3.3.32b)$$

It is expedient to draw the diagram of $s/H = f(\Delta h/H)$ for the given values of L/H and h_0/H , and to use it to find the required s for a specified Δh , or to determine Δh for a proposed s . As the calculations reveal, for $L > 2H$ and $s > 1.5H$ the depth h_0 of the water in the reservoir affects the loss Δh and the position of the depression curve in the dam structure in no substantial way.

3.3.3. Seepage from canals with a parabolic (curved) cross-section

As we mentioned at the beginning of Section 3.3, the Vedernikov-Pavlovsky method plays a fairly important role in the so-called semi-reverse application. We shall show an example of this application relating to seepage from leaky canals of curvilinear cross-section.

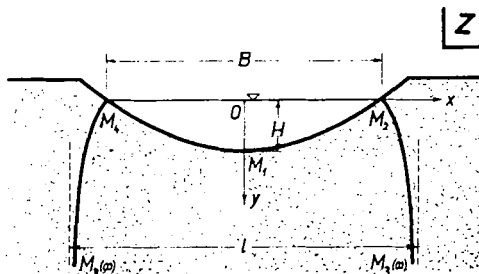


Fig. 3.30. Seepage from a canal of parabolic cross-section without backwater.

In the solution we consider the following two essentially different cases:

a) seepage from a canal without backwater (Fig. 3.30); in this case we assume that the permeable medium in which seepage takes place, is underlain by a layer with a far greater permeability, acting as a drain; the water which has seeped from the canal spreads laterally but is quickly deflected downwards and forms a vertical percolating flow;

b) seepage from a canal with backwater (Fig. 3.36); in this case we assume that the more permeable layer is not present but that the surface of the original groundwater lies at a great — theoretically infinite — depth; the water which has seeped from the canal spreads laterally at a rapid rate and the free surface approaches asymptotically the original free surface. We present solutions for both cases.

a) Seepage from a canal without backwater

This case illustrated schematically in Fig. 3.30 was solved by Pavlovsky and Vedernikov with the help of Zhukovsky's function. Their solution was carried out by the semi-reverse method: for a specified shape of the region of Zhukovsky's function (w_{zh}), they obtained the shape of the canal cross-section by proceeding in reverse.

We first determine the values of the quantities required in the solution, on the boundaries of the various regions:

for the wetted perimeter of the canal — the segment $M_4M_1M_2$:

$$\frac{p}{\gamma} = y, \quad h = 0, \quad \varphi_r = 0$$

for the depression curves — the segment M_2M_3 :

$$\frac{p}{\gamma} = 0, \quad h = -y, \quad \varphi_r = y, \quad \psi_r = \text{const.}; \quad \text{we set } \psi_r = -\frac{q_r}{2}$$

for the segment M_3M_4 :

$$\frac{p}{\gamma} = 0, \quad h = -y, \quad \varphi_r = y, \quad \psi_r = \text{const.}; \quad \text{we set } \psi_r = \frac{q_r}{2}$$

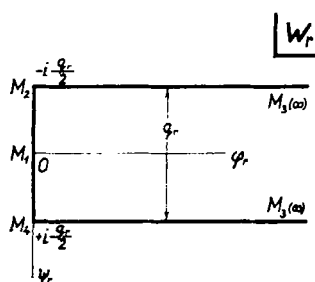


Fig. 3.31.

The values of ψ_r on the depression curves can be chosen as indicated because eqn. (3.1.17') stating that the quantity of flow between the two depression curves is q_r , is satisfied.

The shape of the region of the reduced complex potential is shown in Fig. 3.31.

At the points $y = \infty$, the velocity has vertical direction and the streamlines are parallel to one another. Under these conditions, we have for a point on the free

surface as well as inside the stream

$$h = -y$$

$$v_{rx_{y=\infty}} = 0, \quad v_{ry_{y=\infty}} = -\frac{dh}{dy} = \frac{dy}{dy} = 1$$

and hence

$$l = q_r \quad (3.3.33)$$

Since evidently

$$l > B$$

the value of $(q_r - B)/2$ is positive as well.

For the region of Zhukovsky's function, $w_{zh} = z - iw_r = u_{zh} + iv_{zh} = x + \psi_r + i(y - \varphi_r)$, we have

$$u_{zh}(M_4) = -\frac{B}{2} + \frac{q_r}{2} = \frac{q_r - B}{2}, \quad v_{zh}(M_4) = 0$$

$$u_{zh}(M_1) = 0, \quad v_{zh}(M_1) = iH$$

$$u_{zh}(M_2) = \frac{B}{2} - \frac{q_r}{2} = -\frac{q_r - B}{2}, \quad v_{zh}(M_2) = 0$$

$$u_{zh}(M_3) = 0, \quad v_{zh}(M_3) = 0$$

The authors choose the region of Zhukovsky's function for the permeable segment $M_4M_1M_2$ in the shape of a semi-circle. The (w_{zh}) region is shown in Fig. 3.32.

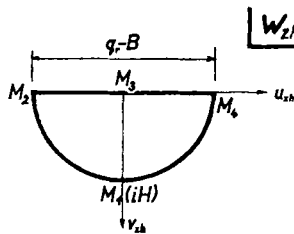


Fig. 3.32.

This choice leads to

$$\frac{q_r - B}{2} = H \quad (3.3.34a)$$

or

$$q_r = B + 2H \quad (3.3.34b)$$

which predetermines the quantity of seepage q_r ; $q = kq_r$

We now carry out the conformal mapping of the (w_r) region onto the (w_{zh}) region:

$$\begin{aligned}
 w_1 &= -w_r \\
 w_2 &= \frac{\pi}{q_r} w_1 = -\frac{\pi}{q_r} w_r \\
 w_3 &= e^{w_2} = e^{-(\pi/q_r)w_r} \\
 w_{zh} &= iHe^{-\pi w_r/q_r}
 \end{aligned} \tag{3.3.35}$$

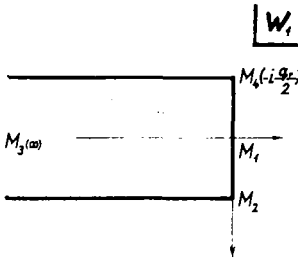


Fig. 3.33.

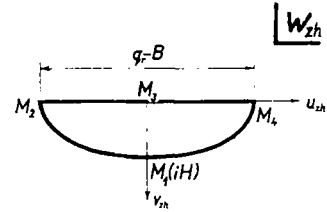
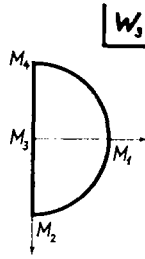


Fig. 3.34.

We have thus determined the function $w_{zh} = f(w_r)$.

By eqn. (3.3.6)

$$\begin{aligned}
 z &= w_{zh} + iw_r = iw_r + iHe^{-\pi w_r/q_r} = \\
 &= i\varphi_r - \psi_r + iHe^{-\pi\varphi_r/q_r} \cos \frac{\pi\psi_r}{q_r} + He^{-\pi\varphi_r/q_r} \sin \frac{\pi\psi_r}{q_r}
 \end{aligned} \tag{3.3.36}$$

and after separating the equation into its real and imaginary parts we obtain the equations of the coordinates of the points in the region of seepage:

$$\begin{aligned}
 x &= -\psi_r + He^{-\pi\varphi_r/q_r} \sin \frac{\pi\psi_r}{q_r} \\
 y &= \varphi_r + He^{-\pi\varphi_r/q_r} \cos \frac{\pi\psi_r}{q_r} \\
 0 &\leq \varphi_r \leq \infty; \quad -\frac{q_r}{2} \leq \psi_r \leq \frac{q_r}{2}
 \end{aligned} \tag{3.3.37}$$

The equation of the canal cross-section:

For the segment $M_4M_1M_2$, $\varphi_r = 0$, and substituting in eqn. (3.3.37) we have

$$\begin{aligned}
 x &= -\psi_r + H \sin \frac{\pi\psi_r}{q_r} \\
 y &= H \cos \frac{\pi\psi_r}{q_r}
 \end{aligned} \tag{3.3.38}$$

For the right-hand branch of the depression curve where $\varphi_r = y$, $\psi_r = -q_r/2$, substitution in eqn. (3.3.37) yields the equation of the depression curve

$$x = \frac{q_r}{2} + He^{-\pi y/q_r} \sin\left(-\frac{\pi}{2}\right) = \frac{q_r}{2} - He^{-\pi y/q_r} \quad (3.3.39)$$

where

$$0 \leq y < \infty, \quad x \geq B/2$$

Our problem is thus solved. The value of the quantity q_r and the shape of the depression curve obtained above apply exactly only to canals with cross-sections described by eqns. (3.3.38). For parabolic cross-sections approaching the theoretical shape defined by eqns. (3.3.38), the solution is only approximate.

Pavlovsky and Vedernikov also examined the way in which the canal cross-section calculated by the reverse procedure, varied with the varying ratio $B/H = \beta$. As their study reveals, the cross-section becomes roughly triangular for decreasing β .

For $\beta = 1.0$ to 1.75 the results apply approximately to triangular canals. The theoretical shape of the canal does not approach a parabola except for $\beta > 5.0$.

A more exact method of solution

In the more exact method of solution which we now present, the semi-circular shape of the region of Zhukovsky's function is replaced by a semi-elliptical one. This makes eqn. (3.3.34a) invalid and q_r an unknown quantity which must be determined in the course of the calculations in such a way as to bring the canal cross-section obtained by the reverse procedure, as closely as possible to the specified parabolic profile. While the values of the function w_{zh} will generally be the same as in the previous case, the (w_{zh}) region will be shaped like a semi-ellipse (Fig. 3.34).

The semi-axes of the ellipse are

$$a' = \frac{q_r - B}{2}, \quad b' = H; \quad \text{the eccentricity } e = \sqrt{\left[\left(\frac{q_r - B}{2}\right)^2 - H^2\right]}$$

The semi-axes of the reduced ellipse ($e = 1$) are:

$$\begin{aligned} a &= \frac{\frac{1}{2}(q_r - B)}{\sqrt{\left\{\left[\frac{1}{2}(q_r - B)\right]^2 - H^2\right\}}} = \frac{1}{2} \left(R + \frac{1}{R} \right) \\ b &= \frac{H}{\sqrt{\left\{\left[\frac{1}{2}(q_r - B)\right]^2 - H^2\right\}}} = \frac{1}{2} \left(R - \frac{1}{R} \right) \\ R &= \sqrt{\frac{(q_r - B)/2 + H}{(q_r - B)/2 - H}} \quad (>1) \end{aligned}$$

where R is the radius of a semi-circle in the (w_2) region which will be mapped by the

function

$$w_1 = \frac{1}{2} \left(w_2 + \frac{1}{w_2} \right)$$

onto the reduced semi-ellipse.

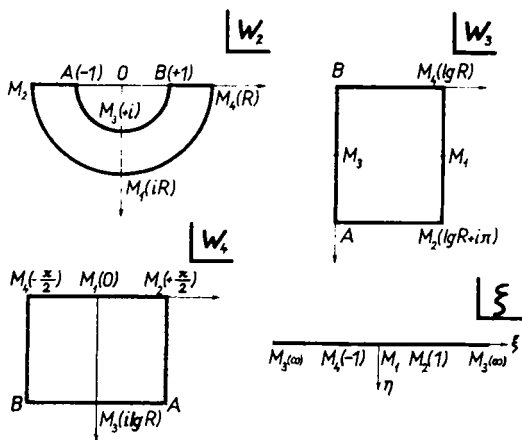


Fig. 3.35.

Fig. 3.35 shows the regions onto which the various functions map the (w_{zh}) region, together with the corresponding points. These functions are:

$$w_{zh} = w_1 \sqrt{\left[\left(\frac{q_r - B}{2} \right)^2 - H^2 \right]}$$

$$w_1 = \frac{1}{2} \left(w_2 + \frac{1}{w_2} \right)$$

$$w_2 = e^{w_3}$$

$$w_{zh} = \sqrt{\left[\left(\frac{q_r - B}{2} \right)^2 - H^2 \right]} \cdot \frac{1}{2} (e^{w_3} + e^{-w_3}) = \sqrt{\left[\left(\frac{q_r - B}{2} \right)^2 - H^2 \right]} \cosh w_3$$

$$w_3 = iw_4 + i \frac{\pi}{2} + \ln R$$

$$\begin{aligned} w_{zh} &= \sqrt{\left[\left(\frac{q_r - B}{2} \right)^2 - H^2 \right]} \cdot \cosh \left(iw_4 + i \frac{\pi}{2} + \ln R \right) = \\ &= \sqrt{\left[\left(\frac{q_r - B}{2} \right)^2 - H^2 \right]} \cdot \cos \left(w_4 - i \ln R + \frac{\pi}{2} \right) = \\ &= - \sqrt{\left[\left(\frac{q_r - B}{2} \right)^2 - H^2 \right]} \cdot \sin (w_r - i \ln R) \end{aligned}$$

We introduce the auxiliary half-plane (ζ). The Schwarz-Christoffel integral which is an elliptic integral of the first kind in our case, transforms the (ζ) half-plane to the (w_4) region

$$w_4 = C \int_0^\zeta \frac{d\zeta}{\sqrt{[(1 - \zeta^2)(1 - \lambda^2 \zeta^2)]}}$$

where

$$C = \frac{\pi}{2K}$$

and λ is determined from the ratio

$$\frac{K'}{K} = \frac{2}{\pi} \ln \sqrt{\frac{\frac{1}{2}(q_r - B) + H}{\frac{1}{2}(q_r - B) - H}} = \frac{2}{\pi} \operatorname{argtgh} \frac{2H}{q_r - B}$$

In the above,

λ is the modulus of the elliptic integrals,

K is the complete elliptic integral of the first kind with the modulus λ ,

K' is the complete elliptic integral of the first kind with the complementary modulus $\lambda' = \sqrt{1 - \lambda^2}$.

The function $w_{zh} = f(\zeta)$ takes the form

$$w_{zh} = - \sqrt{\left[\left(\frac{q_r - B}{2}\right)^2 - H^2\right]} \cdot \sin \left[\frac{\pi}{2K} \int_0^\zeta \frac{d\zeta}{\sqrt{[(1 - \zeta^2)(1 - \lambda^2 \zeta^2)]}} - i \ln \sqrt{\frac{\frac{1}{2}(q_r - B) + H}{\frac{1}{2}(q_r - B) - H}} \right] \quad (3.3.40a)$$

The (w_r) region of the reduced complex potential is transformed to the (ζ) half-plane by the function

$$\zeta = \sin \left(i \frac{\pi w_r}{q_r} \right) = i \sinh \frac{\pi w_r}{q_r} \quad (3.3.40b)$$

We rearrange the integral in eqn. (3.3.40a) as follows:

$$I = \int_0^\zeta \frac{d\zeta}{\sqrt{[(1 - \zeta^2)(1 - \lambda^2 \zeta^2)]}} = \int_0^{\varphi'} \frac{d\varphi'}{\sqrt{[1 - \lambda^2 \sin^2 \varphi']}} = F(\alpha; \varphi') \quad (3.3.40c)$$

where

$$\zeta = \sin \left(i \frac{\pi w_r}{q_r} \right) = \sin \varphi'$$

$$\varphi' = i \frac{\pi w_r}{q_r}$$

$$\alpha = \arcsin \lambda$$

Substituting in eqn. (3.3.40a) from eqn. (3.3.40c) we obtain

$$w_{zh} = - \sqrt{\left[\left(\frac{q_r - B}{2}\right)^2 - H^2\right]} \cdot \sin \left[\frac{\pi}{2K} F(\alpha; \varphi') - i \ln \sqrt{\frac{\frac{1}{2}(q_r - B) + H}{\frac{1}{2}(q_r - B) - H}} \right] \quad (3.3.40d)$$

The final form of eqn. (3.3.6) turns out to be

$$z = - \sqrt{\left[\left(\frac{q_r - B}{2}\right)^2 - H^2\right]} \cdot \sin \left[\frac{\pi}{2K} F(\alpha; \varphi') - i \ln \sqrt{\frac{\frac{1}{2}(q_r - B) + H}{\frac{1}{2}(q_r - B) - H}} \right] + i w_r \quad (3.3.41)$$

Equation of the cross-section

For the canal cross-section we have $\varphi_r = 0$, $-q_r/2 \leq \psi_r \leq q_r/2$, $w_r = i\psi_r$. Under these conditions $F(\alpha; \varphi')$ is always a real number. Eqn. (3.3.40d) then gives

$$\begin{aligned} w_{zh} &= - \sqrt{\left[\left(\frac{q_r - B}{2}\right)^2 - H^2\right]} \left[\sin \left(\frac{\pi}{2K} F(\alpha; \varphi') \right) \cos i \ln \sqrt{\frac{\frac{1}{2}(q_r - B) + H}{\frac{1}{2}(q_r - B) - H}} - \right. \\ &\quad \left. - \cos \left(\frac{\pi}{2K} F(\alpha; \varphi') \right) \sin i \ln \sqrt{\frac{\frac{1}{2}(q_r - B) + H}{\frac{1}{2}(q_r - B) - H}} \right] = - \sqrt{\left[\left(\frac{q_r - B}{2}\right)^2 - H^2\right]} \times \\ &\quad \times \left[\sin \left(\frac{\pi}{2K} F(\alpha; \varphi') \right) \frac{1}{2} \left(\sqrt{\frac{\frac{1}{2}(q_r - B) + H}{\frac{1}{2}(q_r - B) - H}} + \sqrt{\frac{\frac{1}{2}(q_r - B) - H}{\frac{1}{2}(q_r - B) + H}} \right) - \right. \\ &\quad \left. - i \cos \left(\frac{\pi}{2K} F(\alpha; \varphi') \right) \frac{1}{2} \left(\sqrt{\frac{\frac{1}{2}(q_r - B) + H}{\frac{1}{2}(q_r - B) - H}} - \sqrt{\frac{\frac{1}{2}(q_r - B) - H}{\frac{1}{2}(q_r - B) + H}} \right) \right] = \\ &= - \frac{q_r - B}{2} \sin \left(\frac{\pi}{2} \cdot \frac{F(\alpha; \varphi')}{K} \right) + i H \cos \left(\frac{\pi}{2} \cdot \frac{F(\alpha; \varphi')}{K} \right) \quad (3.3.42) \end{aligned}$$

From eqn. (3.3.41) we obtain the equation of the complex coordinate of the points of the cross-section:

$$z = -\psi_r - \frac{q_r - B}{2} \sin \left(\frac{\pi}{2} \cdot \frac{F(\alpha; \varphi')}{K} \right) + i H \cos \left(\frac{\pi}{2} \cdot \frac{F(\alpha; \varphi')}{K} \right) \quad (3.3.43)$$

Separating eqn. (3.3.43) into the real and imaginary parts leads to the following parametric equations of the canal cross-section

$$\begin{aligned} x &= -\psi_r - \frac{q_r - B}{2} \sin \left(\frac{\pi}{2} \cdot \frac{F(\alpha; \varphi')}{K} \right) \\ y &= H \cos \left(\frac{\pi}{2} \cdot \frac{F(\alpha; \varphi')}{K} \right) \end{aligned} \quad (3.3.44)$$

in which the variable ψ_r is the parameter.

For the limiting case of $q_r = 2H + B$ (i.e. the (w_{zh}) region is a semi-circle) these general equations give

$$\frac{K'}{K} = \frac{2}{\pi} \ln \infty = \infty \quad \text{hence:} \quad \lambda = 0$$

$$\int_0^\zeta \frac{d\zeta}{\sqrt{(1-\zeta^2)}} = \int_0^{\varphi'} d\varphi' = \varphi' = \arcsin \zeta$$

$$K = \int_0^{\pi/2} d\varphi' = \frac{\pi}{2}$$

and the equations of the cross-section become

$$x = -\psi_r + H \sin \frac{\pi\psi_r}{q_r}$$

$$y = H \cos \frac{\pi\psi_r}{q_r}$$

i.e. equations which were deduced directly in the previous case.

The unknown value q_r appearing in the general equations of the cross-section, eqns. (3.3.44), is determined from the condition that any other point of the canal cross-section, e.g. the point for which $\psi_r = -q_r/4$, must satisfy the equation of the parabola.

The quantity of seepage, q_r , is obtained by successive approximations. The first approximate value of q_r is determined from eqn. (3.3.34b) and a point of the cross-section is calculated for $\psi_r = q_r/4$ from eqn. (3.3.34). If it fails to conform to the specified shape, the next q_r is corrected and the procedure is repeated until the calculated point lies on, or in the immediate vicinity of, the contour of the cross-section.

Equation of the depression curve:

For the right-hand branch of the depression curve we have

$$\psi_r = -\frac{q_r}{2}; \quad \varphi_r = y$$

$$w_r = y - i \frac{q_r}{2}$$

$$\zeta = \sin \left[i \frac{\pi}{q_r} \left(y - i \frac{q_r}{2} \right) \right] = \sin \left(\frac{\pi}{2} + i \frac{\pi y}{q_r} \right) = \cosh \frac{\pi y}{q_r}$$

The value of ζ grows from 1 to ∞ ($1 \leq \zeta < \infty$). The interval of ζ is divided into

$$\left\langle 1; \frac{1}{\lambda} \right\rangle \quad \text{and} \quad \left\langle \frac{1}{\lambda}; \infty \right\rangle$$

For the first interval

$$\begin{aligned} & \frac{\pi}{2K} \int_0^{\zeta} \frac{d\zeta}{\sqrt{[(1 - \zeta^2)(1 - \lambda^2 \zeta'^2)]}} = \\ &= \frac{\pi}{2K} \left(K + i \int_0^{\sqrt{(1 - \zeta'^2)/\lambda'^2}} \frac{dy'}{\sqrt{[(1 - y'^2)(1 - \lambda'^2 y'^2)]}} \right) = \\ &= \frac{\pi}{2K} [K + iF(\alpha'; \varphi^*)], \\ & \zeta' = \frac{1}{\zeta} \end{aligned} \tag{3.3.45a}$$

For the second interval

$$\begin{aligned} & \frac{\pi}{2K} \int_0^{\zeta} \frac{d\zeta}{\sqrt{[(1 - \zeta^2)(1 - \lambda^2 \zeta'^2)]}} = \frac{\pi}{2K} \left(iK' + \int_0^{\zeta'/\lambda} \frac{dy'}{\sqrt{[(1 - y'^2)(1 - \lambda'^2 y'^2)]}} \right) = \\ &= \frac{\pi}{2K} [iK' + F(\alpha; \varphi^*)] \end{aligned} \tag{3.3.45b}$$

The equation of the depression curve is obtained on substituting the above values in eqn. (3.3.41).

The equation of the depression curve in the first interval, $\langle 1; 1/\lambda \rangle$ is:

$$\begin{aligned} z &= iw_r - \sqrt{\left[\left(\frac{q_r - B}{2}\right)^2 - H^2\right]} \cdot \sin \left[\frac{\pi}{2} + i \frac{\pi}{2K} F(\alpha'; \varphi^*) - i \ln \sqrt{\frac{\frac{1}{2}(q_r - B) + H}{\frac{1}{2}(q_r - B) - H}} \right] = \\ &= iw_r - \sqrt{\left[\left(\frac{q_r - B}{2}\right)^2 - H^2\right]} \cdot \cosh \left[\frac{\pi}{2K} F(\alpha'; \varphi^*) - \ln \sqrt{\frac{\frac{1}{2}(q_r - B) + H}{\frac{1}{2}(q_r - B) - H}} \right] = \\ &= iw_r - \frac{1}{2} \left[\frac{q_r - B}{2} (e^{(\pi/2K)F(\alpha'; \varphi^*)} + e^{(-\pi/2K)F(\alpha'; \varphi^*)}) - \right. \\ &\quad \left. - H(e^{(\pi/2K)F(\alpha'; \varphi^*)} - e^{(-\pi/2K)F(\alpha'; \varphi^*)}) \right] = \\ &= iw_r - \frac{q_r - B}{2} \cosh \left(\frac{\pi}{2K} F(\alpha'; \varphi^*) \right) + H \sinh \left(\frac{\pi}{2K} F(\alpha'; \varphi^*) \right) \end{aligned}$$

and after rearrangement

$$x = \frac{q_r}{2} - \frac{q_r - B}{2} \cosh \left(\frac{\pi}{2K} F(\alpha'; \varphi^*) \right) + H \sinh \left(\frac{\pi}{2K} F(\alpha'; \varphi^*) \right) \quad (3.3.46a)$$

For the special case of $(q_r - B)/2 = H$ we have

$$\lambda = 0, \quad \lambda' = 1, \quad \frac{1}{\lambda} = \infty$$

$$F(\alpha'; \varphi^*) = \int_0^{\sqrt{[(1-\zeta'^2)/\lambda'^2]}} \frac{dy'}{\sqrt{[(1-y'^2)(1-\lambda'^2 y'^2)]}} = \int_0^{\sqrt{[(\zeta^2-1)/\zeta^2]}} \frac{dy'}{1-y'^2} =$$

$$= [\operatorname{argtgh} y']_0^{\sqrt{[(\zeta^2-1)/\zeta^2]}} = \frac{\pi y}{q_r},$$

$$\left(\sqrt{\frac{\zeta^2 - 1}{\zeta^2}} = \sqrt{\frac{\sinh^2(\pi y/q_r)}{\cosh^2(\pi y/q_r)}} = \operatorname{tg} h \frac{\pi y}{q_r} \right)$$

$$K = \frac{\pi}{2};$$

$$x = \frac{q_r}{2} - \frac{q_r - B}{2} \cosh \frac{\pi y}{q_r} + \frac{q_r - B}{2} \sinh \frac{\pi y}{q_r} = \frac{q_r}{2} - \frac{q_r - B}{2} e^{-\pi y/q_r} \quad (3.3.39)$$

The above is the equation derived in the previous case.

The equation of the depression curve in the second interval, $\langle 1/\lambda; \infty \rangle$ is:

$$z = iw_r - \sqrt{\left[\left(\frac{q_r - B}{2} \right)^2 - H^2 \right]} \cdot \sin \left[i \frac{\pi K'}{2K} + \frac{\pi}{2K} F(\alpha; \varphi^*) - \right.$$

$$\left. - i \ln \sqrt{\frac{\frac{1}{2}(q_r - B) + H}{\frac{1}{2}(q_r - B) - H}} \right] = iw_r - \sqrt{\left[\left(\frac{q_r - B}{2} \right)^2 - H^2 \right]} \cdot \sin \left[\frac{\pi}{2K} F(\alpha; \varphi^*) \right]$$

and after rearrangement

$$x = \frac{q_r}{2} - \sqrt{\left[\left(\frac{q_r - B}{2} \right)^2 - H^2 \right]} \cdot \sin \left[\frac{\pi}{2K} F(\alpha; \varphi^*) \right] \quad (3.3.46b)$$

For the special case of $q_r = B + 2H$:

$$x = \frac{q_r}{2},$$

which is correct because for $\lambda = 0$ the interval $\langle 1/\lambda; \infty \rangle$ reduces to a single infinitely distant point.

b) Seepage from a canal with backwater

We again use Zhukovsky's function w_{zh} together with the semi-reverse method.

Fig. 3.36 shows the region of seepage (z) and the region of the reduced complex potential (w_r); the latter has the same shape as in the previous case.

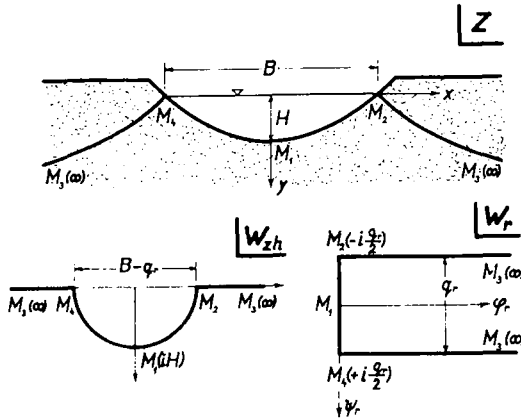


Fig. 3.36. Seepage from a canal of parabolic cross-section with backwater.

The region of the (w_{zh}) function will take a different form. For (w_{zh}) we have: for the segment $M_4M_1M_2$:

$$u_{zh}(M_1) = 0, \quad v_{zh}(M_1) = iH$$

$$u_{zh}(M_2) = \frac{B - q_r}{2}, \quad v_{zh}(M_2) = 0$$

$$u_{zh}(M_4) = -\frac{B - q_r}{2}, \quad v_{zh}(M_4) = 0$$

for the depression curve: $u_{zh}(M_3) = \infty, \quad v_{zh}(M_3) = 0$.

In his solution of this case, Aravin specified the shape of the region of the (w_{zh}) function for the segment $M_4M_1M_2$ as a semi-circle.

Under this condition

$$\begin{aligned} \frac{B - q_r}{2} &= H \\ q_r &= B - 2H \end{aligned} \quad (3.3.47)$$

Determination of the function $w_{zh} = f(w_r)$:

$$w_1 = \frac{\pi}{q_r} w_r, \quad w_2 = e^{\pi w_r / q_r}, \quad w_3 = i e^{\pi w_r / q_r}, \quad w_{zh} = \frac{B - q_r}{2} i e^{\pi w_r / q_r}$$

and eqn. (3.3.6) yields

$$z = i \frac{B - q_r}{2} e^{\pi w_r / q_r} + i w_r = i w_r + i \frac{B - q_r}{2} e^{\pi \varphi_r / q_r} \cdot e^{i(\pi \psi_r / q_r)} \quad (3.3.48)$$

Separating eqn. (3.3.48) into its real and imaginary parts we obtain the equations of the coordinates of the points in the region of seepage:

$$\begin{aligned} x &= -\psi_r - \frac{B - q_r}{2} e^{\pi \varphi_r / q_r} \sin \frac{\pi \psi_r}{q_r} \\ y &= \varphi_r + \frac{B - q_r}{2} e^{\pi \varphi_r / q_r} \cos \frac{\pi \psi_r}{q_r} \end{aligned} \quad (3.3.49)$$

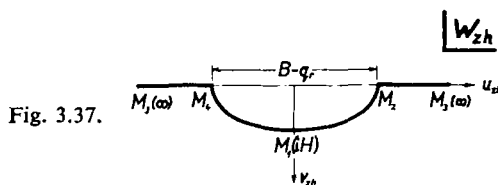
For the canal cross-section $\varphi_r = 0$; $-q_r/2 \leq \psi_r \leq q_r/2$, and the equations of the cross-section become

$$\begin{aligned} x &= -\psi_r - \frac{B - q_r}{2} \sin \frac{\pi \psi_r}{q_r} \\ y &= \frac{B - q_r}{2} \cos \frac{\pi \psi_r}{q_r} \end{aligned} \quad (3.3.50)$$

For the right-hand branch of the depression curve we have $\varphi_r = y$, $\psi_r = -q_r/2$; the equation of the depression curve becomes

$$x = \frac{q_r}{2} + \frac{B - q_r}{2} e^{\pi y / q_r} \quad (3.3.51)$$

The solution defined by eqns. (3.3.47), (3.3.50) and (3.3.51) is an exact solution for canals whose cross-sectional shape is described by eqns. (3.3.50), and an approximate solution for those whose cross-section only approaches that shape.



If we solve the case considered in a more general way, we can derive equations which will bring the cross-section closer to the specified shape. We assume that a semi-ellipse corresponds to the segment $M_4M_1M_2$. On that assumption eqn. (3.3.47) becomes invalid, the value of q_r is not known in advance and must be determined in the course of the solution. The shape of the (w_{zh}) region is shown in Fig. 3.37. The calculation procedure is the same as before.

The eccentricity of the ellipse:

$$e = \sqrt{\left[\left(\frac{B - q_r}{2}\right)^2 - H^2\right]}$$

the axes of the reduced ellipse:

$$a = \frac{\frac{1}{2}(B - q_r)}{\sqrt{\left\{\left[\frac{1}{2}(B - q_r)\right]^2 - H^2\right\}}} = \frac{1}{2} \left(R + \frac{1}{R}\right)$$

$$b = \frac{H}{\sqrt{\left\{\left[\frac{1}{2}(B - q_r)\right]^2 - H^2\right\}}} = \frac{1}{2} \left(R - \frac{1}{R}\right)$$

$$R = \frac{\sqrt{\frac{1}{2}(B - q_r) + H}}{\sqrt{\frac{1}{2}(B - q_r) - H}}$$

R is the radius of the circle in the (ζ) region which is transformed by the function $w_1 = \frac{1}{2}(\zeta + 1/\zeta)$ into the reduced semi-ellipse in the (w_1) region

$$w' = \frac{\pi}{q_r} w_r, \quad w'' = e^{\pi w_r/q_r}, \quad w''' = ie^{\pi w_r/q_r}$$

$$\zeta = \frac{\sqrt{\frac{1}{2}(B - q_r) + H}}{\sqrt{\frac{1}{2}(B - q_r) - H}} \cdot ie^{\pi w_r/q_r}$$

$$w_1 = \frac{w_{zh}}{\sqrt{\left\{\left[\frac{1}{2}(B - q_r)\right]^2 - H^2\right\}}} = \frac{1}{2} \left(\zeta + \frac{1}{\zeta}\right)$$

$$\begin{aligned} w_{zh} &= \frac{1}{2} \sqrt{\left[\left(\frac{B - q_r}{2}\right)^2 - H^2\right]} \left(\sqrt{\frac{\frac{1}{2}(B - q_r) + H}{\frac{1}{2}(B - q_r) - H}} \cdot ie^{\pi w_r/q_r} - \right. \\ &\quad \left. - i \sqrt{\frac{\frac{1}{2}(B - q_r) - H}{\frac{1}{2}(B - q_r) + H}} e^{-\pi w_r/q_r} \right) = \\ &= i \left[\frac{B - q_r}{2} \cdot \frac{1}{2} (e^{\pi w_r/q_r} - e^{-\pi w_r/q_r}) + H \cdot \frac{1}{2} (e^{\pi w_r/q_r} + e^{-\pi w_r/q_r}) \right] = \\ &= i \frac{B - q_r}{2} \sinh \frac{\pi w_r}{q_r} + i H \cosh \frac{\pi w_r}{q_r} \end{aligned} \quad (3.3.52)$$

and the function $z = f(w_r)$ is

$$\begin{aligned} z &= iw_r + i \frac{B - q_r}{2} \left(\sinh \frac{\pi \varphi_r}{q_r} \cos \frac{\pi \psi_r}{q_r} + i \cosh \frac{\pi \varphi_r}{q_r} \sin \frac{\pi \psi_r}{q_r} \right) + \\ &\quad + iH \left(\cosh \frac{\pi \varphi_r}{q_r} \cos \frac{\pi \psi_r}{q_r} + i \sinh \frac{\pi \varphi_r}{q_r} \sin \frac{\pi \psi_r}{q_r} \right) \end{aligned} \quad (3.3.53)$$

Separating eqn. (3.3.53) into the real and imaginary parts we obtain the coordinates of the points in the region of seepage:

$$\begin{aligned} x &= -\psi_r - \frac{B - q_r}{2} \cosh \frac{\pi \varphi_r}{q_r} \sin \frac{\pi \psi_r}{q_r} - H \sinh \frac{\pi \varphi_r}{q_r} \sin \frac{\pi \psi_r}{q_r} \\ y &= \varphi_r + \frac{B - q_r}{2} \sinh \frac{\pi \varphi_r}{q_r} \cos \frac{\pi \psi_r}{q_r} + H \cosh \frac{\pi \varphi_r}{q_r} \cos \frac{\pi \psi_r}{q_r} \quad (3.3.54) \\ 0 &\leq \varphi_r \leq \infty; \quad -\frac{q_r}{2} \leq \psi_r \leq \frac{q_r}{2} \end{aligned}$$

After the substitution $\varphi_r = 0$ in eqn. (3.3.54) the equation of the canal cross-section becomes

$$\begin{aligned} x &= -\psi_r - \frac{B - q_r}{2} \sin \frac{\pi \psi_r}{q_r} \\ y &= H \cos \frac{\pi \psi_r}{q_r} \quad (3.3.55) \end{aligned}$$

The equation of the right-hand branch of the depression curve is obtained from eqn. (3.3.54) for $\varphi_r = y$ and $\psi_r = -q_r/2$:

$$\begin{aligned} x &= \frac{q_r}{2} + \frac{B - q_r}{2} \cosh \frac{\pi y}{q_r} + H \sinh \frac{\pi y}{q_r} \\ x &\geq \frac{B}{2}, \quad 0 \leq y \leq \infty \quad (3.3.56) \end{aligned}$$

The unknown quantity q_r appearing in eqns. (3.3.55) and (3.3.56) can be determined from the condition that one more point, described by eqns. (3.3.55), e.g. the point for which $\psi_r = -q_r/4$ should lie on the wetted perimeter of the cross-section. The required calculation can be carried out directly: for a chosen ψ_r we find y from eqn. (3.3.55), then x from the shape of the cross-section, and finally q_r from the first of eqns. (3.3.55).

3.3.4. Other possible applications of the method

Just as in the discussion of the Pavlovsky method we have restricted the examples of the application of the Vedernikov-Pavlovsky method to the simplest cases which illustrate well the essential features of the method. The Vedernikov-Pavlovsky method is very well suited to the solution of a number of seepage problems with a free surface, symmetrical as well as non-symmetrical cases of seepage from

canals or under cut-off walls of reservoirs with variously arranged withdrawal of percolated water, cases of flow into drainage canals or cuts, etc. For further worked examples and copious references the reader is again directed to [10], [296].

3.4. Velocity hodograph method

The velocity hodograph method has found a wide field of application in theoretical solutions of seepage problems and has been used by many authors in slightly different forms. It is somewhat difficult to decide who was the first to use it.

At the present time there are available four elaborated fundamental versions of the method. Before we turn to their principles we first explain the concept of the velocity hodograph.

3.4.1. Velocity hodograph, its form and construction

We showed in Section 3.1.4 that the reduced complex velocity of seepage

$$\frac{dw_r}{dz} = v_{rx} - iv_{ry} \quad (3.4.1)$$

is an analytic function of the complex coordinate z of the points in the region of seepage.

We now introduce a new variable, $v_r = (v_{rx} + iv_{ry})$ conjugate to the complex velocity. The region in which the variable $(v_{rx} + iv_{ry})$ varies, is the region of the velocity hodograph.

The following conditions apply to the values of the variable $(v_{rx} + iv_{ry})$ at points lying on the boundary of the region of seepage (z):

Along a permeable segment the velocity is normal to that segment. If the segment is a straight line making an angle α with the $+x$ axis, the velocity components are given by

$$\operatorname{arctg} \frac{v_{ry}}{v_{rx}} = \pm \frac{\pi}{2} + \alpha \quad (3.4.2a)$$

or by

$$v_{ry} = -v_{rx} \cotg \alpha \quad (3.4.2b)$$

Therefore, in the velocity hodograph region there corresponds to a straight-line permeable segment a straight line normal to the direction of the permeable segment in the (z) region and passing through the origin of the coordinates v_{rx}, v_{ry} .

Along an impervious segment the velocity is in the direction of that segment. If the segment is a straight line making an angle α with the $+x$ axis, the ve-

locity components are given by

$$\operatorname{arctg} \frac{v_{ry}}{v_{rx}} = \begin{cases} \alpha \\ \pi + \alpha \end{cases} \quad (3.4.3a)$$

or by

$$v_{ry} = v_{rx} \operatorname{tg} \alpha \quad (3.4.3b)$$

Hence in the velocity hodograph region there corresponds to a straight-line impervious segment a straight line parallel to the direction of the impervious segment in the (z) region and passing through the origin of the coordinates v_{rx} , v_{ry} .

For a segment of free seepage to the surface (surface of seepage) which makes an angle α with the $+x$ axis, we have, according to eqn. (3.1.26),

$$\varphi_r = y + \text{const.}$$

The velocity component parallel to the segment is

$$v_{rs} = \frac{\partial \varphi_r}{\partial s} = \frac{\partial y}{\partial s} = \sin \alpha$$

or, if v_{rs} is expressed in terms of v_{rx} , v_{ry} ,

$$v_{rs} = v_{rx} \cos \alpha + v_{ry} \sin \alpha = \sin \alpha$$

and after rearrangement

$$v_{ry} = -v_{rx} \cotg \alpha + 1 \quad (3.4.4)$$

As the above equation implies, in the hodograph there corresponds to a segment of free seepage to the surface a straight line normal to its direction and passing through the point $v_r = +i$.

By eqns. (3.1.26) and (3.1.25a) we have for the depression curve

$$\varphi_r = y + \text{const.}$$

$$\psi_r + v_{r0}x = \text{const.}$$

where v_{r0} denotes the reduced velocity of seepage. The velocity component in the direction of a tangent to the depression curve, v_{rs} , and the velocity component in the direction of a normal, v_{rn} , are

$$v_{rs} = \frac{\partial \varphi_r}{\partial s} = \frac{\partial y}{\partial s} = \cos(s, y)$$

$$v_{rn} = \frac{\partial \varphi_r}{\partial n} = - \frac{\partial \psi_r}{\partial s} = v_{r0} \frac{\partial x}{\partial s} = v_{r0} \cos(s, x)$$

If v_{rs} and v_{rn} are expressed in terms of the components v_{rx} and v_{ry} they become

$$v_{rs} = v_{rx} \cos(s, x) + v_{ry} \cos(s, y) = \cos(s, y)$$

$$\begin{aligned} v_{rn} &= v_{rx} \cos(n, x) + v_{ry} \cos(n, y) = \\ &= -v_{rx} \cos(s, y) + v_{ry} \cos(s, x) = v_{r0} \cos(s, x) \end{aligned}$$

and after rearrangement

$$v_{rx} \cos(s, x) + (v_{ry} - 1) \cos(s, y) = 0 \quad (3.4.5a)$$

$$(v_{ry} - v_{r0}) \cos(s, x) - v_{rx} \cos(s, y) = 0 \quad (3.4.5b)$$

Eliminating the expressions $\cos(s, x)$, $\cos(s, y)$ from the system (3.4.5) we obtain the expression relating the components of seepage velocity on the depression curve,

$$v_{rx}^2 + (v_{ry} - 1)(v_{ry} - v_{r0}) = 0$$

or, more conveniently still,

$$v_{rx}^2 + \left(v_{ry} - \frac{1 + v_{r0}}{2} \right)^2 = \left(\frac{1 - v_{r0}}{2} \right)^2 \quad (3.4.6)$$

Therefore, according to eqn. (3.4.6) there corresponds to the depression curve in the velocity hodograph, a circle (or a part thereof) centred at the point $v_r = +i(1 + v_{r0})/2$ with radius $r = (1 - v_{r0})/2$. In the absence of infiltration, $v_{r0} = 0$ in the last equation, the radius of the circle in the hodograph region is $r = 1/2$ and the circle is centred at the point $v_r = +i/2$.

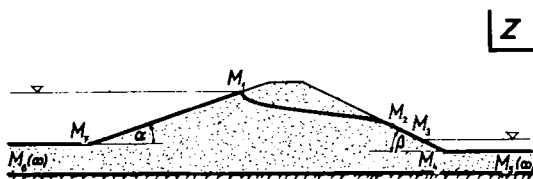


Fig. 3.38. Various types of boundaries of the region of seepage.

If we know the conditions applicable to the various segments of the boundary of the region of seepage (z), we can construct a velocity hodograph and thus obtain a region in which the values of the variable $v_r = (v_{rx} + iv_{ry})$ will vary, i.e. the region of the velocity hodograph, (v_r).

By way of illustration, consider the construction of the velocity hodograph for the region of seepage shown in Fig. 3.38, in the absence of infiltration. The velocity hodograph is shown in Fig. 3.39.

The segment M_5M_6 — the surface of the impervious subsoil — is horizontal; hence $v_{ry} = 0$. Since the water percolates from the upstream reservoir through the

dam and the subsoil to the downstream reservoir, $v_{rx} \geq 0$. In the hodograph, there corresponds to this segment a part of the horizontal axis, $0v_{rx}$, $0 \leq v_{rx}$. At the infinitely distant points of the region of seepage, M_5 and M_6 , the velocity will be zero; hence, in the velocity hodograph, the corresponding points M_5 and M_6 will lie at the origin of the coordinates v_{rx} , v_{ry} . In passing from the point M_5 to the point M_6 , the velocity v_{rx} will increase from zero to a finite value and then decrease to zero. Hence the corresponding segment of the $0v_{rx}$ axis will be traversed twice.

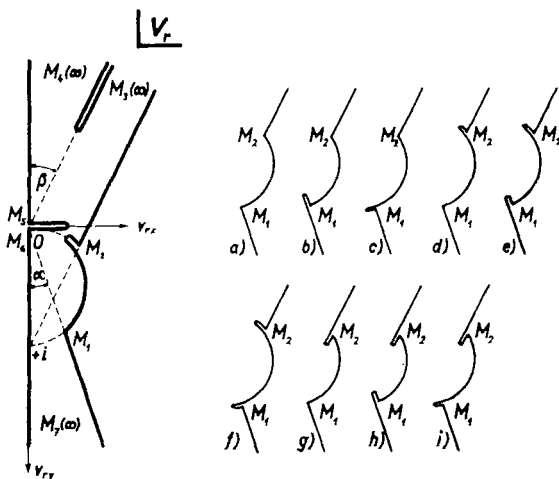


Fig. 3.39. Velocity hodograph.

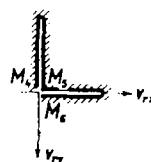


Fig. 3.40.

The segment M_4M_5 — the bottom of the downstream reservoir — is horizontal, the velocity is normal to it, $v_{rx} = 0$. The water percolates upwards, $v_{ry} \leq 0$. In the velocity hodograph there corresponds to this segment a part of the vertical axis $0v_{ry}$, $v_{ry} \leq 0$. At the point M_5 , the velocity is zero. At the point M_4 , the velocity must be either zero or infinitely large for the following reason: to the segment M_3M_4 (the wetted section of the dam side) corresponds a straight line normal to the direction of the segment M_3M_4 in the (z) region and passing through the origin of the coordinates v_{rx} , v_{ry} . The point M_4 lies on the segment M_3M_4 as well as on the segment M_4M_5 . Hence, in the hodograph it must also lie on the vertical line corresponding to the segment M_4M_5 and on the straight line corresponding to the segment M_3M_4 . As is well known from the theory of complex variables, the two lines intersect at the origin of the coordinates ($v_{rx} = v_{ry} = 0$) and at an infinitely distant point of the Gauss plane in which the (v_r) region lies. If the velocity at the point M_4 were zero, a part of the axis $0v_{ry}$, $v_{ry} \leq 0$ traversed twice would correspond to the segment M_4M_5 in the hodograph (Fig. 3.40). The points which lie close to the segment M_4M_5 in the (z) region, also lie close to the image of the segment M_4M_5 in the hodograph. As Fig. 3.40 shows, the horizontal velocity component at the points in the immediate vicinity of the point M_4 would be $v_{rx} < 0$; this can hardly be so because of the overall pattern

of the flow. Hence the velocity at the point M_4 must be $v_r(M_4) = \infty$; in the hodograph, the whole negative half of the coordinate axis Ov_{ry} will correspond to the segment M_4M_5 .

If the segment M_3M_4 (the wetted section of the dam side) makes an angle α with the x axis, the straight line corresponding to it in the hodograph will make an angle α with the axis Ov_{ry} . At the point M_4 , the velocity $v_r(M_4) = \infty$. As will be shown later, the velocity at the point M_3 must also be infinitely large, $v_r(M_3) = \infty$. Hence the part of the straight line in the (v_r) region corresponding to the segment M_3M_4 will be traversed twice. In passing from the point M_4 to the point M_3 , the velocity v_r will decrease from the value $v_r = \infty$ to a finite value and then increase back to $v_r = \infty$. In view of the direction of seepage on this segment, $v_{rx} > 0$, $v_{ry} < 0$.

In the (z) region the segment M_2M_3 (the segment of free seepage to the surface) forms an angle α with the x axis. The segment corresponding to it in the hodograph will be a segment of the straight line making an angle α with the Ov_{ry} axis and passing through the point $v_r = +i$. Hence this straight line is parallel to that corresponding to the segment M_3M_4 . The point M_3 which is common to both straight lines must lie at infinity, $v_r(M_3) = \infty$. The point M_2 lies simultaneously on the depression curve; therefore, in the hodograph it will lie at the point of intersection of the straight line corresponding to the segment M_2M_3 , and the circle corresponding to the depression curve. There exist two such points of intersection: one, i.e. the point $v_r = +i$ does not come into consideration, for otherwise the velocity at the point M_2 would have to be $v_{rx} = 0$, $v_{ry} = +1$, and this is physically impossible in view of the character of the flow. The other point of intersection is $v_r = (\sin \alpha \cos \alpha + i \sin^2 \alpha)$.

The segment M_6M_7 (the bottom of the upstream reservoir) is horizontal, $v_{rx} = 0$, $v_{ry} \geq 0$. The segment corresponding to it in the hodograph is a part of the axis Ov_{ry} , $v_{ry} \geq 0$. The velocity at the point M_6 is $v_r(M_6) = 0$. The velocity at the point M_7 must be zero or infinitely large for the reasons stated in connection with the point M_4 . Just as for the latter we conclude that the velocity at the point M_7 , $v_r(M_7) = \infty$; in the passage from M_6 to M_7 , v_{ry} grows monotonically from $v_{ry} = 0$ to $v_{ry} = \infty$, and to the segment M_6M_7 corresponds the whole semi-axis Ov_{ry} , $v_{ry} \geq 0$.

In the (z) region, the segment M_7M_1 (the wetted upstream side of the dam) makes an angle β with the x axis, $v_{rx} \geq 0$, $v_{ry} \geq 0$. In the hodograph region, there corresponds to it a segment of the straight line which passes through the origin and makes an angle β with the Ov_{ry} axis. The velocity at the point M_7 , $v_r(M_7) = \infty$. The point M_1 lies on the segment M_7M_1 and simultaneously on the depression curve M_1M_2 . Therefore, in the hodograph it will lie at the point of intersection of the straight line corresponding to the segment M_7M_1 , and the circle corresponding to the depression curve. There again exist two points of intersection. The point of intersection $v_r = 0$ does not come into consideration because — as the course of the circle representing the depression curve indicates — the velocity on the depression curve in the immediate vicinity of the point M_1 would have only the horizontal component v_{rx} . This, however, is physically impossible: the depression curve is a stream-

line and therefore perpendicular to the permeable segment M_7M_1 at the point M_1 . The other point of intersection is $v_r = (\sin \beta \cos \beta + i \cos^2 \beta)$.

The segment M_1M_2 (the depression curve) is represented in the hodograph by a section of a circle with radius $r = 1/2$, centred at the point $v_r = +i/2$, lying between the points M_1 and M_2 whose coordinates were given earlier.

In the simplest possible case, i.e. when the velocity on the segment M_7M_1 falls off monotonically, the depression curve M_1M_2 has no point of inflection and the velocity on the segment M_2M_3 increases monotonically, the form of the segments M_7M_1 , M_1M_2 and M_2M_3 is as shown in Fig. 3.39a. Of the more complicated cases which are apt to arise, let us mention that when the velocity on the segment M_7M_1 — instead of falling off monotonically — first decreases from the point M_7 to a certain minimum and then increases to the point M_1 (Figs. 3.39b, e, h), or similarly, when the velocity on the segment M_2M_3 first falls off to a certain minimum and then grows to the point M_3 (Fig. 3.39g, h, i); in another instance, the depression curve can have one or two points of inflection (Figs. 3.39c, d, e, i, f). The various cases depend on the shape of the region of seepage, the ratio between the various dimensions, the ratio between the heights of the surface in the upstream and downstream reservoir, etc. If the ratio between the various dimensions were varied continually, the various shapes of the velocity hodograph would change continually from one to another.

We now have available the shape of the velocity hodograph. In the explanation which follows we shall also need the region of the complex velocity ($v_{rx} - iv_{ry}$). We shall obtain this by inverting the region of the velocity hodograph (v_r) about the real axis Ov_{rx} (Fig. 3.41).

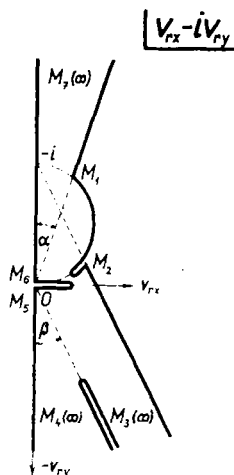


Fig. 3.41. Region of complex velocity.

We recall at this point that — if we know the shape of the boundary of a region in the Gauss plane — the region of the values of the variable being examined can lie either inside or outside that boundary. We know that the complex velocity ($v_{rx} - iv_{ry}$) is an analytic function of the complex coordinate of a point in the region of seepage.

According to the theory of the functions of a complex variable, the $(v_{rx} - iv_{ry})$ region is situated with respect to the boundary of the region in a way that makes the distribution of the individual points on the boundary of the region with respect to the region the same for the $(v_{rx} - iv_{ry})$ region as for the (z) region. This means that if we proceed on the boundary from the point M_1 to the points M_2, M_3 , etc., the region of the variable under examination lies always on the same side (the right-hand side in the case being considered). The sense of the distribution of the points on the boundary of the hodograph region with respect to the (v_r) region, is the opposite.

3.4.2. Four variants of the application of the velocity hodograph method

a) The first variant is particularly well suited for problems in which the boundary of the region of seepage consists only of

permeable segments,
 impervious segments,
 the depression curve, and
 infiltration into, or evaporation from, the free surface of groundwater is absent.

The method can be used in a direct as well as in an indirect manner.

In the direct application, the permeable and the impervious segments of the boundary of the region of seepage are assumed to be straight broken lines. We already know how to determine the velocity hodographs for such cases.

In the indirect application neither the permeable nor the impervious segments of the boundary of the region of seepage (z) need be linear. In such cases, the shape of the velocity hodograph is selected, and the shape of the permeable and impervious segments is obtained as it depends on the shape of the velocity hodograph adopted for those segments.

On these assumptions, the region of the reduced complex potential will be a polygon with straight sides parallel to the coordinate axes $O\varphi_r, O\psi_r$.

We introduce the analytic function

$$\omega = \frac{1}{v_{rx} - iv_{ry}} = \frac{dz}{dw_r} \quad (3.4.7)$$

Recalling the assumptions made in Section 3.4.1, the boundary of the $(v_{rx} + iv_{ry})$ hodograph region, and hence also the boundary of the $(v_{rx} - iv_{ry})$ region, will be composed of straight lines and a circle, all passing through the origin of the coordinates. Hence the (ω) region will be a polygon because in a mapping effected by the reciprocal function (3.4.7), both the straight lines and the circle passing through the origin will transform to straight lines. We need to find — with the help of con-

formal mapping — the function

$$\omega = f(w_r) \quad (3.4.8)$$

which will transform the (w_r) region onto the (ω) region. According to eqn. (3.4.7)

$$z = \int \omega \, dw_r = \int f(w_r) \, dw_r \quad (3.4.9)$$

If we use the auxiliary (ζ) half-plane

$$w_r = f_1(\zeta), \quad \omega = f_2(\zeta), \quad \zeta = f_3(w_r)$$

and

$$\omega = f(\omega_r) = f_2[f_3(w_r)]$$

If an expression $\zeta = f_3(w_r)$ is not feasible, then

$$z = \int f_2(\zeta) f_1(\zeta) \, d\zeta, \quad \text{where } w_r = f_1(\zeta) \quad (3.4.9a)$$

The values to be substituted in eqn. (3.4.9) or eqn. (3.4.9a) can theoretically be determined in every case. If we do not know how to map the (w_r) region onto the (ω) region directly, we can always use the Schwarz-Christoffel integral to map both the (w_r) and the (ω) regions onto their common auxiliary half-plane (ζ) .

In the semi-reverse method of application, the shape of the velocity hodograph on the permeable and impervious segments is specified in a way that will facilitate the construction of the function $\omega = f(w_r)$. If the shape of the permeable and impervious segments, established by backward calculation, are at least approximately correct, the solution is acceptable for practical purposes.

b) The second variant extends the field of application of the velocity hodograph method. It is especially expedient in cases in which the region of Zhukovsky's function is known, i.e. whenever the boundary of the region of seepage (see Section 3.3) contains

horizontal permeable segments,
vertical impervious segments,
segments of free seepage, and
the depression curve.

On these conditions the region of Zhukovsky's function, i.e. the region of the function

$$w_{zh} = z - iw_r, \quad (3.4.10)$$

is a polygon with sides parallel to the coordinate axes Ou_{zh} and Ov_{zh} .

Infiltration or evaporation can generally be present.

The direct application of the second variant assumes the segments of free seepage to be linear. In this case the boundaries of the hodograph region (v_r) consist of straight lines and a circle, all passing through the point $v_r = +i$.

We introduce the new analytic function

$$\omega = \frac{dz}{dw_{zh}} = \frac{i}{v_{rx} - iv_{ry} + i} \quad (3.4.11)$$

and effect the mapping according to the last term of eqn. (3.4.11) which will transform the velocity hodograph region (v_r) to the polygon of the (ω) region. We are required to find the function

$$\omega = f(w_{zh}) \quad (3.4.12)$$

which transforms the region of Zhukovsky's function (w_{zh}) to the (ω) region. Then on the basis of (3.4.11)

$$z = \int \omega dw_{zh} = \int f(w_{zh}) dw_{zh} \quad (3.4.13)$$

and from eqn. (3.4.10)

$$w_r = iw_{zh} - iz \quad (3.4.14)$$

If we use the auxiliary half-plane (ζ), we can write

$$w_{zh} = f_1(\zeta), \quad \omega = f_2(\zeta), \quad \zeta = f_3(w_{zh})$$

$$\omega = f_2[f_3(w_{zh})]$$

If an expression of $f_3(w_{zh})$ is not feasible, then

$$z = \int f_2(\zeta) f_1'(\zeta) d\zeta \quad (3.4.13a)$$

$$w_r = if_1(\zeta) - i \int f_2(\zeta) f_1'(\zeta) d\zeta \quad (3.4.14a)$$

In the indirect method we choose the shape of the segments of the velocity hodograph boundary, corresponding to the segments of free seepage, and then look for their shape in the (z) region. The indirect method again greatly extends the field of application of velocity hodographs.

It is likewise possible to introduce another analytic function, viz.

$$\omega' = \frac{i}{v_{rx} - iv_{ry}} + 1 = 1 + i \frac{dz}{dw_r} = f(w_{zh}) \quad (3.4.15)$$

We then have from eqns. (3.4.15) and (3.4.14)

$$dz = i(1 - \omega') dw_r = (\omega' - 1)(dw_{zh} - dz)$$

and after some manipulation

$$z = \int \left(1 - \frac{1}{\omega'}\right) dw_{zh} \quad (3.4.16)$$

$$w_r = iw_{zh} - iz \quad (3.4.14)$$

c) The third variant is applicable to cases where the boundary of the region of seepage is composed of

permeable segments,
impervious segments, and
segments of free seepage to the surface,

assuming that all the segments are linear. In cases of this sort both the (z) and the $(v_{rx} - iv_{ry})$ regions are rectilinear polygons. We need to find the function

$$(v_{rx} - iv_{ry}) = f(z) \quad (3.4.17)$$

which maps the (z) region onto the $(v_{rx} - iv_{ry})$ region. If we use eqn. (3.1.23) we have

$$(v_{rx} - iv_{ry}) = f(z) = \frac{dw_r}{dz}$$

$$w_r = \int f(z) dz \quad (3.4.18a)$$

If we use the auxiliary half-plane (ζ) we obtain

$$z = f_1(\zeta), \quad (v_{rx} - iv_{ry}) = f_2(\zeta)$$

$$w_r = \int f_2(\zeta) f_1'(\zeta) d\zeta \quad (3.4.18b)$$

d) The fourth variant is applicable to cases where the boundary of the region of seepage is composed of

permeable segments,
impervious segments and
segments of free seepage,

which are all broken straight lines, and of
the depression curve,

with all the straight lines and the circles which form the boundary of the velocity hodograph having at least one point in common at M_0 where $v_r(M_0) = (v_{rx0} + iv_{ry0})$.

The function

$$\omega = \frac{1}{(v_{rx} - iv_{ry}) - (v_{rx0} - iv_{ry0})} \quad (3.4.19)$$

will then map the region of the complex velocity onto a rectilinear polygon. We introduce a new analytic function, viz.

$$\Omega = \Phi + i\Psi = w_r - (v_{rx0} - iv_{ry0})z \quad (3.4.20)$$

where

$$\Phi = \varphi_r - (v_{rx0}x + v_{ry0}y) \quad (3.4.20a)$$

$$\Psi = \psi_r + (v_{ry0}x - v_{rx0}y) \quad (3.4.20b)$$

The boundary conditions for the function Ω :

For a permeable segment making an angle α with the x axis

$$x \sin \alpha = y \cos \alpha + a, \quad (3.4.21a)$$

where a is some real constant; by eqn. (3.4.2b) we have further for this segment

$$v_x \cos \alpha + v_y \sin \alpha = 0 \quad (3.4.21b)$$

By substituting the values for the point M_0 in eqn. (3.4.21b) and multiplying eqn. (3.4.21a) by (3.4.21b) we obtain

$$v_{rx0}x + v_{ry0}y = -\frac{av_{ry0}}{\cos \alpha} \quad (3.4.22)$$

Using eqn. (3.1.24), eqn. (3.4.20a) gives

$$\Phi = \varphi_r + \frac{av_{ry0}}{\cos \alpha} = \text{const.} \quad (3.4.23)$$

For $\alpha = \pi/2$: $v_{ry0} = 0$, $x = a$, and

$$\Phi = \varphi_r - \frac{av_{rx0}}{\cos \alpha} = \text{const.} \quad (3.4.23a)$$

For an impervious segment making an angle α with the x axis we have

$$x \sin \alpha = y \cos \alpha + a \quad (3.4.24a)$$

and further, by eqn. (3.4.3b)

$$v_{rx} \sin \alpha - v_{ry} \cos \alpha = 0 \quad (3.4.24b)$$

After manipulations similar to those used for the permeable segment, the above equations lead to the relation

$$v_{ry0}x - v_{rx0}y = \frac{av_{rx0}}{\cos \alpha} \quad (3.4.25)$$

Using (3.1.25) we obtain from eqn. (3.4.20b)

$$\Psi = \psi_r + \frac{av_{rx0}}{\cos \alpha} = \text{const.} \quad (3.4.26)$$

For $\alpha = \pi/2$: $v_{rx0} = 0$, $x = a$ and

$$\Psi = \psi_r + av_{ry0} = \text{const.} \quad (3.4.26a)$$

For a segment of free seepage to the surface, which makes an angle α with the x axis, we again have

$$x \sin \alpha = y \cos \alpha + a \quad (3.4.27a)$$

further, there apply the relations (3.1.26) and (3.4.4)

$$\varphi_r = y + \text{const.} \quad (3.1.26)$$

$$v_{rx} \cos \alpha = (1 - v_{ry}) \sin \alpha \quad (3.4.27b)$$

Substituting the values for the point M_0 in eqn. (3.4.27b), we obtain from eqn. (3.4.20a) and the above equations

$$\Phi = y + \text{const} - y - \frac{a(1 - v_{ry0})}{\cos \alpha} = \frac{a(v_{ry0} - 1)}{\cos \alpha} + \text{const.} = \text{const.} \quad (3.4.28)$$

For $\alpha = \pi/2$: $v_{ry0} = 1$, $x = a$, and

$$\Phi = y + \text{const.} - (av_{rx0} + y) = -av_{rx0} + \text{const.} = \text{const.} \quad (3.4.28a)$$

The equations applying to the depression curve are (3.1.26), (3.1.25a) and (3.4.6)

$$\varphi_r = y + \text{const.}$$

$$\psi_r = -v_{r0}x + \text{const.}$$

$$v_{rx}^2 + (v_{ry} - 1)(v_{ry} - v_{r0}) = 0$$

Eqns. (3.4.20a) and (2.4.20b) take the form

$$\Phi = y + \text{const.} - (v_{rx0}x + v_{ry0}y) = -v_{r0}x + (1 - v_{ry0})y + \text{const.}, \quad (3.4.29a)$$

$$\Psi = -v_{r0}x + \text{const.} + (v_{ry0}x - v_{rx0}y) = (v_{ry0} - v_{r0})x - v_{rx0}y + \text{const.} \quad (3.4.29b)$$

Eqn. (3.4.29a) is multiplied by v_{rx0} and eqn. (3.4.29b) by $(1 - v_{ry0})$; on adding them

and recalling that according to eqn. (3.4.6)

$$v_{rx0}^2 + (v_{ry0} - 1)(v_{ry0} - v_{r0}) = 0$$

we obtain

$$\Phi v_{rx0} + \Psi(1 - v_{ry0}) = 0 + \text{const.} = \text{const.} \quad (3.4.29)$$

In the (Ω) region, the figure corresponding to the depression curve is a straight line in general position.

The region of the function Ω is thus a rectilinear polygon. In calculations carried out with the help of conformal mapping, use is made of the auxiliary half-plane (ζ) . Then

$$\omega = f_1(\zeta), \quad \Omega = f_2(\zeta)$$

and eqns. (3.4.19) and (3.4.20) are rearranged to

$$\omega = f_1(\zeta) = \frac{1}{(v_{rx} - iv_{ry}) - (v_{rx0} - iv_{ry0})} \quad (3.4.30a)$$

$$\Omega = f_2(\zeta) = w_r - (v_{rx0} - iv_{ry0})z \quad (3.4.30b)$$

w_r is determined from eqn. (3.4.30b) and dw_r/dz is obtained by differentiation:

$$w_r = (v_{rx0} - iv_{ry0})z + f_2(\zeta) \quad (3.4.31)$$

$$\frac{dw_r}{dz} = v_{rx0} - iv_{ry0} + \frac{d}{dz} [f_2(\zeta)] \quad (3.4.31a)$$

dw_r/dz is then calculated from eqn. (3.4.30a)

$$\frac{dw_r}{dz} = v_{rx} - iv_{ry} = v_{rx0} - iv_{ry0} + \frac{1}{f_1(\zeta)} \quad (3.4.31b)$$

and the result is compared with eqn. (3.4.31a):

$$\frac{1}{f_1(\zeta)} = \frac{d}{dz} [f_2(\zeta)] = f_2'(\zeta) \frac{d\zeta}{dz}$$

Following some manipulations and integration this leads to

$$z = \int f_1(\zeta) f_2'(\zeta) d\zeta \quad (3.4.32)$$

and eqn. (3.4.31) to

$$w_r = (v_{rx0} - iv_{ry0}) \int f_1(\zeta) f_2'(\zeta) d\zeta + f_2(\zeta) \quad (3.4.33)$$

e) Application of the method

In the sections which follow the four variants of the velocity hodograph method will be applied to practical cases. We shall pay special attention to, and describe in detail, the various steps of the first variant in its direct as well as indirect form because this is the fundamental variant from which the others are derived. As already noted several times, the velocity hodograph method has found a very wide field of application; in a way, it can be considered to be more versatile than the Pavlovsky or the Vedernikov-Pavlovsky method. Theoretically, it is capable of dealing with problems (though sometimes less straightforwardly) which are within the province of both methods; the converse is not true, however.

The cases we intend to solve are only simple examples of problems to which the method can be applied. Other worked examples can be found in the literature, especially in various comprehensive publications, such as [10], [181], [296], [325], [392].

3.4.3. Seepage from leaky canals

We shall solve this example by the application of the direct form of the first variant of the velocity hodograph method.

We have already examined the seepage from canals at some length. Using mainly approximate methods, we have obtained solutions to the cases of seepage without backwater (Sections 3.3.1 and 3.3.3) as well as to those with backwater. The present case of seepage includes various alternatives: one of the limiting cases considers the horizontal surface of groundwater to be at the level of the surface of the water in the canal, the other, seepage without backwater. All the remaining cases of seepage lie between these two limiting states and can be solved uniquely, provided that one knows a point of the depression curve. For example, a point at a sufficient distance from the canal, at the level of the original free surface of groundwater can be taken for such a point. The intermediate cases will describe various degrees of backwater resulting, e.g. from different heights of the original surface of groundwater.

The solution will be carried out for a canal with a trapezoidal cross-section, neglecting the effect of capillarity. The procedure can be used in practice to determine seepage from a canal assuming that a point of the depression curve is known, for example, the place where the percolating water flows into a surface stream or a drain, and that the permeable layer is fairly thick.

A schematic diagram of the seepage flow is shown in Fig. 3.42. Since the case is symmetrical, it is enough to consider only a half of the region of seepage (z). B_0 denotes the width of the canal on the bottom, B_1 the width of the canal at the level of the water surface, H the depth of the canal, and α the angle of the sides.

The depression curve emerges from the point $M_3(x = B_1/2, y = 0)$ lying on the side of the canal at the level of the water surface; it is perpendicular to the side

of the canal which is a segment of equal piezometric heads (the equipotential line). The inclination of the depression curve $J = 0$ at the point M_4 ($x = \infty$, $y = \infty$); in the case of seepage without backwater, the percolating water forms a vertical stream and the slope of the depression curve rapidly grows larger. In this case, the depression curve at the point M_4 ($y = \infty$) is vertical.

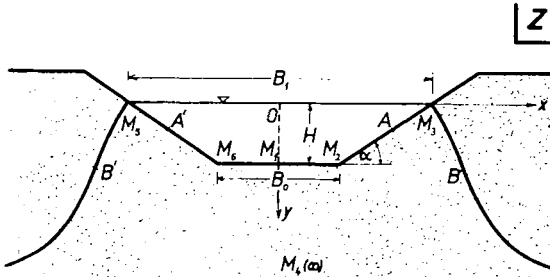


Fig. 3.42. Seepage from a leaky trapezoidal canal.

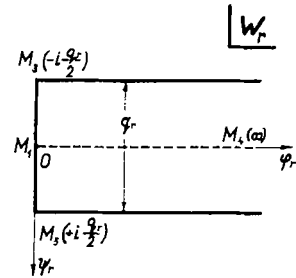


Fig. 3.43. Region of reduced complex potential.

To solve the problem, we must establish the region of the velocity hodograph, (v_r) and the region of the reduced complex potential, (w_r). Next, we determine the shape of the ($v_{rx} - iv_{ry}$) region, and using eqn. (3.4.7) the shape of the (ω) region. Finally, using eqns. (3.4.9) or (3.4.9a) we obtain the relation $z = z(w_r)$. If we wish, we can determine the values of the seepage velocities, using eqn. (3.4.7).

In the case being considered, we shall use the auxiliary half-plane (ζ). Then

$$w_r = f_1(\zeta) \quad (3.4.34)$$

$$\omega = f_2(\zeta) \quad (3.4.35)$$

and the resulting relation becomes

$$z = \int_{\zeta_0}^{\zeta} f_2(\zeta) f_1'(\zeta) d\zeta + C \quad (3.4.36)$$

a) Region of the reduced complex potential (w_r)

The following conditions apply to this region:

for the wetted perimeter of the canal section $\varphi_r = \text{const.}$

for the depression curve $\psi_r = \text{const.}$

If the reference plane for the determination of the value of φ_r is placed at the level of the water surface in the canal, we have:

for the wetted perimeter, according to eqns. (3.1.2) and (3.1.3), $\varphi_r = 0$

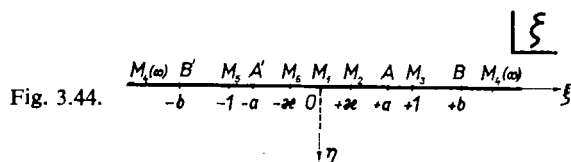
and, recalling the meaning of the streamlines and the fact that the total quantity q_r flows between the left-hand and the right-hand branch of the depression curve,

for the right-hand branch of the depression curve $\psi_r = -q_r/2$

for the left-hand branch of the depression curve $\psi_r = +q_r/2$

Hence the shape of the (w_r) region is as shown in Fig. 3.43.

If we choose the auxiliary half-plane (ζ) in accordance with Fig. 3.44, we can



show that the form of the function which maps the auxiliary half-plane (ζ) conformally onto the (w_r) region is

$$w_r = f_1(\zeta) = -i \frac{q_r}{\pi} \arcsin \sqrt{\zeta} \quad (3.4.37a)$$

for the segment M_1M_3 where $0 \leq \zeta \leq 1$, and

$$w_r = f_1(\zeta) = -i \frac{q_r}{2} + \frac{q_r}{\pi} \operatorname{argcosh} \sqrt{\zeta} \quad (3.4.37b)$$

for the segment M_3M_4 where $1 \leq \zeta \leq \infty$

b) Region of the velocity hodograph (v_r) and the (ω) region

As we know from Section 3.4.1, in the velocity hodograph region the bottom and the sides of the canal are represented by straight lines normal to the corresponding parts of the wetted perimeter. Theoretically, at the point M_2 (and M_6) the velocity $v_r(M_2) = \infty$.

In the case of seepage without backwater, the depression curve emerges normal to the side of the canal; it is deflected downwards and at the point $y = \infty$, its tangent becomes vertical. For this case the shape of the (v_r) region is shown in Fig. 3.45 and the shape of the (ω) region in Fig. 3.46.

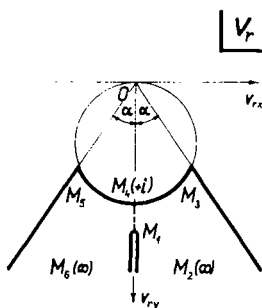


Fig. 3.45.

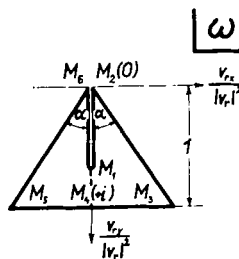


Fig. 3.46.

By assumption, in the cases of seepage with backwater close to the case of seepage without backwater, the tangent at the point $M_4(x = \infty, y = \infty)$ is horizontal. The depression curve which emerges normal to the side of the canal, is deflected to the point B where it must have an inflection in order to be able to turn laterally therefrom. Owing to the effect of this inflection, the velocity hodograph takes the shape shown in Fig. 3.47, and the (ω) region the shape shown in Fig. 3.48.

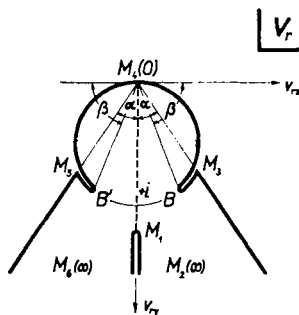


Fig. 3.47.

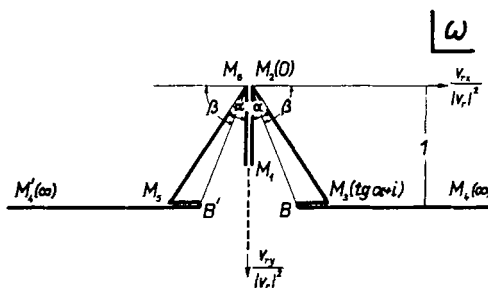


Fig. 3.48.

The characteristic quantity of the case being considered, is the constant b — the coordinate of the point B in the auxiliary half-plane (ζ) . As the value of this constant gets smaller, the segment M_3B diminishes and the point of inflection B on the depression curve approaches the point M_3 . For $b = 1$, the two points coalesce, and there is no longer a point of inflection on the depression curve.

The velocity of infiltration along the side of the canal, which is $v_r(M_2) = \infty$ at the point M_2 , decreases along the side of the canal and reaches its minimum at the point M_3 . It can be shown that for $b = 1$ when the depression curve has no longer an inflection, not only is the velocity at its minimum but also $dv_r/ds = 0$ at the point M_3 .

We have thus reached a group of seepage cases when the depression curve which no longer has an inflection, lies higher than the depression curve for the case $b = 1$. These are the cases when the minimum velocity v_r is no longer at the point M_3 but at some point A which lies on the side of the canal. The characteristic quantity of these cases is the constant a — the coordinate of the point A in the auxiliary half-plane (ζ) . The closer the point A approaches the canal bottom (the value of the constant a gets smaller), the higher lies the depression curve, until at a certain value, $a_{\min}$, the free surface of groundwater (the depression curve) becomes horizontal, on the level of the free surface of water in the canal. Fig. 3.49 shows the shape of the velocity hodograph, Fig. 3.50 the shape of the (ω) region of these cases.

Evidently, as $a \rightarrow 1$, A tends to M_3 until at $a = 1$, we have a case identical with the case from the first group for $b = 1$.

Returning to the limiting case of seepage without backwater, we see from Fig. 3.52 depicting the various cases of seepage, that the case without backwater is just

The Schwarz-Christoffel integral is of the form

$$\begin{aligned}\omega &= C' \int_{\zeta_0}^{\zeta} (\zeta - \kappa)^{(a/\pi-1)} (\zeta - a)^{(\frac{1}{2}-a/\pi-1)} (\zeta - 1)^{(\frac{1}{2}-a/\pi-1)} d\zeta + C_1 = \\ &= C \int_0^{\zeta} \frac{a - \zeta}{(\kappa - \zeta)^{(1-a/\pi)} (1 - \zeta)^{(\frac{1}{2}+a/\pi)}} d\zeta + C_1\end{aligned}\quad (3.4.38)$$

For the point M_1 , $\zeta = 0$ and $\omega = i/v_r(M_1)$. On substitution we obtain

$$C_1 = \frac{i}{v_r(M_1)} \quad (3.4.38a)$$

For the point M_2 , $\zeta = \kappa$ and $\omega = 0$. This leads to

$$C \int_0^{\kappa} \frac{a - \zeta}{(\kappa - \zeta)^{(1-a/\pi)} (1 - \zeta)^{(\frac{1}{2}+a/\pi)}} d\zeta + \frac{i}{v_r(M_1)} = 0 \quad (3.4.38b)$$

and introducing

$$J_{a1} = \int_0^{\kappa} \frac{a - \zeta}{(\kappa - \zeta)^{(1-a/\pi)} (1 - \zeta)^{(\frac{1}{2}+a/\pi)}} d\zeta \quad (3.4.39)$$

we have

$$CJ_{a1} + \frac{i}{v_r(M_1)} = 0 \quad (3.4.40)$$

or

$$\frac{i}{v_r(M_1)} = -CJ_{a1} \quad (3.4.40a)$$

We pass around the point $M_2(\zeta = \kappa)$ in the auxiliary half-plane along an infinitely small semi-circle from the segment M_1M_2 to the segment M_2A (Fig. 3.51).

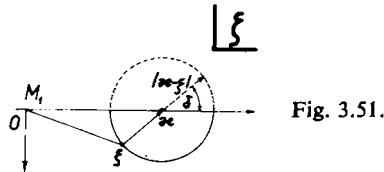


Fig. 3.51.

The expression $(\kappa - \zeta)^{(1-a/\pi)}$ will become

$$(e^{-i\pi})^{(1-a/\pi)} (\zeta - \kappa)^{(1-a/\pi)} = -e^{i\alpha} (\zeta - \kappa)^{(1-a/\pi)}$$

For $\kappa \leq \zeta \leq a$ we then have

$$\omega = 0 - Ce^{-i\alpha} \int_{\kappa}^{\zeta} \frac{a - \zeta}{(\zeta - \kappa)^{(1-a/\pi)} (1 - \zeta)^{(\frac{1}{2}+a/\pi)}} d\zeta \quad (3.4.38c)$$

For the point A , $\omega = \Re(\operatorname{tg} \alpha + i)$, $\zeta = a$. Consequently

$$\Re(\operatorname{tg} \alpha + i) = -Ce^{-i\alpha} \int_x^a \frac{a - \zeta}{(\zeta - \kappa)^{(1-\alpha/\pi)} (1 - \zeta)^{(\frac{1}{2} + \alpha/\pi)}} d\zeta \quad (3.4.38d)$$

and introducing

$$J_{a2} = \int_x^a \frac{a - \zeta}{(\zeta - \kappa)^{(1-\alpha/\pi)} (1 - \zeta)^{(\frac{1}{2} + \alpha/\pi)}} d\zeta \quad (3.4.41)$$

we obtain

$$\Re(\operatorname{tg} \alpha + i) = -Ce^{-i\alpha} J_{a2} \quad (3.4.40b)$$

By passing around the point $A(\zeta = a)$ along an infinitely small semi-circle in the (ζ) half-plane, the expression $(a - \zeta)$ changes to

$$-(\zeta - a)$$

and in the segment $AM_3(a \leq \zeta \leq 1)$ we obtain

$$\omega = -Ce^{-i\alpha} J_{a2} + Ce^{i\alpha} \int_a^\zeta \frac{\zeta - a}{(\zeta - \kappa)^{(1-\alpha/\pi)} (1 - \zeta)^{(\frac{1}{2} + \alpha/\pi)}} d\zeta \quad (3.4.38e)$$

By also introducing

$$J_{a3} = \int_a^1 \frac{\zeta - a}{(\zeta - \kappa)^{(1-\alpha/\pi)} (1 - \zeta)^{(\frac{1}{2} + \alpha/\pi)}} d\zeta \quad (3.4.42)$$

we have for the point $M_3(\omega = \operatorname{tg} \alpha + i, \zeta = 1)$

$$\begin{aligned} (\operatorname{tg} \alpha + i) &= -Ce^{-i\alpha} J_{a2} + Ce^{-i\alpha} \int_a^1 \frac{\zeta - a}{(\zeta - \kappa)^{(1-\alpha/\pi)} (1 - \zeta)^{(\frac{1}{2} + \alpha/\pi)}} d\zeta = \\ &= -Ce^{-i\alpha} (J_{a2} - J_{a3}) \end{aligned}$$

It follows from the above that

$$C = -\frac{\operatorname{tg} \alpha + i}{e^{-i\alpha}(J_{a2} - J_{a3})} = -\frac{i}{\cos \alpha (J_{a2} - J_{a3})} \quad (3.4.40c)$$

and from eqn. (3.4.40b) that

$$\Re = \frac{J_{a2}}{J_{a2} - J_{a3}} \quad (3.4.43)$$

We can now determine the function $\omega = f_2(\zeta)$ and the coordinates of the points on the boundary of the region of seepage (z) for all the segments of the boundary.

α) The segment M_1M_2 (the canal bottom): $0 \leq \zeta \leq \kappa$.

According to eqns. (3.4.38a) and (3.4.40a, c)

$$w = f_2(\zeta) = \frac{i}{(J_{a2} - J_{a3}) \cos \alpha} \left[J_{a1} - \int_0^\zeta \frac{a - \zeta}{(\kappa - \zeta)^{(1-\alpha/\pi)} (1 - \zeta)^{(\frac{1}{2} + \alpha/\pi)}} d\zeta \right] \quad (3.4.44a)$$

according to eqn. (3.4.37a)

$$w_r = f_1(\zeta) = -i \frac{d\zeta}{\pi} \arcsin \sqrt{\zeta}$$

$$f_1'(\zeta) = -i \frac{q_r}{\pi} \frac{d\zeta}{\sqrt{[\zeta(1 - \zeta)]}}$$

and substituting in eqn. (3.4.36) we have

$$\begin{aligned} z = & \frac{1}{(J_{a2} - J_{a3}) \cos \alpha} \frac{q_r}{\pi} \int_0^\zeta \left[J_{a1} - \int_0^\zeta \frac{a - \zeta}{(\kappa - \zeta)^{(1-\alpha/\pi)} (1 - \zeta)^{(\frac{1}{2} + \alpha/\pi)}} d\zeta \right] \times \\ & \times \frac{d\zeta}{\sqrt{[\zeta(1 - \zeta)]}} + iH = \frac{1}{(J_{a2} - J_{a3}) \cos \alpha} \frac{q_r}{\pi} \times \\ & \times \left[\arcsin \sqrt{\zeta} \times \left(J_{a1} - \int_0^\zeta \frac{a - \zeta}{(\kappa - \zeta)^{(1-\alpha/\pi)} (1 - \zeta)^{(\frac{1}{2} + \alpha/\pi)}} d\zeta \right) + \right. \\ & \left. + \int_0^\zeta \arcsin \sqrt{\zeta} \times \frac{a - \zeta}{(\kappa - \zeta)^{(1-\alpha/\pi)} (1 - \zeta)^{(\frac{1}{2} + \alpha/\pi)}} d\zeta \right] + iH \quad (3.4.45a) \end{aligned}$$

For the point M_2 , $\zeta = \kappa$, $z = B_0/2 + iH$. After substitution, the width of the canal on the bottom is given by

$$\frac{B_0}{2} = \frac{1}{(J_{a2} - J_{a3}) \cos \alpha} \frac{q_r}{\pi} \int_0^\kappa \arcsin \sqrt{\zeta} \times \frac{a - \zeta}{(\kappa - \zeta)^{(1-\alpha/\pi)} (1 - \zeta)^{(\frac{1}{2} + \alpha/\pi)}} d\zeta$$

and noting that

$$F_{a1} = \int_0^\kappa \arcsin \sqrt{\zeta} \times \frac{a - \zeta}{(\kappa - \zeta)^{(1-\alpha/\pi)} (1 - \zeta)^{(\frac{1}{2} + \alpha/\pi)}} d\zeta \quad (3.4.46a)$$

this becomes

$$\frac{B_0}{2} = \frac{1}{(J_{a2} - J_{a3}) \cos \alpha} \frac{q_r}{\pi} F_{a1} \quad (3.4.47)$$

β) The segment M_2A (the side of the canal): $\kappa \leq \zeta \leq a$.

According to eqns.(3.4.38c), (3.4.40c), (3.4.37a) and (3.4.36)

$$\omega = f_2(\zeta) = \frac{ie^{-i\alpha}}{(J_{a2} - J_{a3}) \cos \alpha} \int_{\kappa}^{\zeta} \frac{a - \zeta}{(\zeta - \kappa)^{(1-\alpha/\pi)} (1 - \zeta)^{(\frac{1}{2} + \alpha/\pi)}} d\zeta \quad (3.4.44b)$$

$$f_1'(\zeta) = -i \frac{q_r}{\pi} \frac{d\zeta}{\sqrt{[\zeta(1 - \zeta)]}}$$

$$\begin{aligned} z &= \frac{B_0}{2} + iH + \frac{e^{-i\alpha}}{(J_{a2} - J_{a3}) \cos \alpha} \frac{q_r}{\pi} \int_{\kappa}^{\zeta} \left[\int_{\kappa}^{\zeta} \frac{a - \zeta}{(\zeta - \kappa)^{(1-\alpha/\pi)} (1 - \zeta)^{(\frac{1}{2} + \alpha/\pi)}} d\zeta \right] \times \\ &\times \frac{d\zeta}{\sqrt{[\zeta(1 - \zeta)]}} = \frac{B_0}{2} + iH + \frac{e^{-i\alpha}}{(J_{a2} - J_{a3}) \cos \alpha} \frac{q_r}{\pi} \times \\ &\times \left[\arcsin \sqrt{\zeta} \int_{\kappa}^{\zeta} \frac{a - \zeta}{(\zeta - \kappa)^{(1-\alpha/\pi)} (1 - \zeta)^{(\frac{1}{2} + \alpha/\pi)}} d\zeta - \right. \\ &\left. - \int_{\kappa}^{\zeta} \arcsin \sqrt{\zeta} \times \frac{a - \zeta}{(\zeta - \kappa)^{(1-\alpha/\pi)} (1 - \zeta)^{(\frac{1}{2} + \alpha/\pi)}} d\zeta \right] \quad (3.4.45b) \end{aligned}$$

Writing

$$F_{a2} = \int_{\kappa}^a \arcsin \sqrt{\zeta} \times \frac{a - \zeta}{(\zeta - \kappa)^{(1-\alpha/\pi)} (1 - \zeta)^{(\frac{1}{2} + \alpha/\pi)}} d\zeta \quad (3.4.46b)$$

we obtain for the point $A(\zeta = a)$

$$z(A) = \frac{B_0}{2} + iH + \frac{e^{-i\alpha}}{(J_{a2} - J_{a3}) \cos \alpha} \frac{q_r}{\pi} (\arcsin \sqrt{a} \times J_{a2} - F_{a2}) \quad (3.4.48)$$

γ) The segment AM_3 (the side of the canal): $a \leq \zeta \leq 1$.

According to eqns. (3.4.38e), (3.4.40c), (3.4.37a) and (3.4.36)

$$\omega = f_2(\zeta) = \frac{ie^{-i\alpha}}{(J_{a2} - J_{a3}) \cos \alpha} \left[J_{a2} - \int_a^{\zeta} \frac{\zeta - a}{(\zeta - \kappa)^{(1-\alpha/\pi)} (1 - \zeta)^{(\frac{1}{2} + \alpha/\pi)}} d\zeta \right] \quad (3.4.44c)$$

$$f_1'(\zeta) = -i \frac{q_r}{\pi} \frac{d\zeta}{\sqrt{[\zeta(1 - \zeta)]}}$$

$$\begin{aligned} z &= z(A) + \frac{e^{-i\alpha}}{(J_{a2} - J_{a3}) \cos \alpha} \frac{q_r}{\pi} \int_a^{\zeta} \left[J_{a2} - \int_a^{\zeta} \frac{\zeta - a}{(\zeta - \kappa)^{(1-\alpha/\pi)} (1 - \zeta)^{(\frac{1}{2} + \alpha/\pi)}} d\zeta \right] \times \\ &\times \frac{d\zeta}{\sqrt{[\zeta(1 - \zeta)]}} = z(A) + \frac{e^{-i\alpha}}{(J_{a2} - J_{a3}) \cos \alpha} \frac{q_r}{\pi} \times \end{aligned}$$

$$\times \left[\arcsin \sqrt{\zeta} \times \left(J_{a2} - \int_a^\zeta \frac{\zeta - a}{(\zeta - \kappa)^{(1-\alpha/\pi)} (1 - \zeta)^{(\frac{1}{2} + \alpha/\pi)}} d\zeta \right) - \right. \\ \left. - \arcsin \sqrt{a} \times J_{a2} + \int_a^\zeta \arcsin \sqrt{\zeta} \times \frac{\zeta - a}{(\zeta - \kappa)^{(1-\alpha/\pi)} (1 - \zeta)^{(\frac{1}{2} + \alpha/\pi)}} d\zeta \right]$$

and using eqn. (3.4.48) we obtain

$$z = \frac{B_0}{2} + iH + \frac{e^{-i\alpha}}{(J_{a2} - J_{a3}) \cos \alpha \pi} \frac{q_r}{\pi} \times \\ \times \left[\arcsin \sqrt{\zeta} \times \left(J_{a2} - \int_a^\zeta \frac{\zeta - a}{(\zeta - \kappa)^{(1-\alpha/\pi)} (1 - \zeta)^{(\frac{1}{2} + \alpha/\pi)}} d\zeta \right) - \right. \\ \left. - \left(F_{a2} - \int_a^\zeta \arcsin \sqrt{\zeta} \times \frac{\zeta - a}{(\zeta - \kappa)^{(1-\alpha/\pi)} (1 - \zeta)^{(\frac{1}{2} + \alpha/\pi)}} d\zeta \right) \right] \quad (3.4.45c)$$

Just as before we write

$$F_{a3} = \int_a^1 \arcsin \sqrt{\zeta} \times \frac{\zeta - a}{(\zeta - \kappa)^{(1-\alpha/\pi)} (1 - \zeta)^{(\frac{1}{2} + \alpha/\pi)}} d\zeta \quad (3.4.46c)$$

For the point M_3 , $\zeta = 1$, $z = B_1/2$. We substitute these values in eqn. (3.4.45e) and rearrange the resulting equation in accordance with eqn. (3.4.46c). Recalling that

$$H = \frac{B_1 - B_0}{2} \operatorname{tg} \alpha \quad (3.4.49)$$

we obtain the relation

$$\frac{B_1 - B_0}{2} = \frac{1}{J_{a2} - J_{a3}} \frac{q_r}{\pi} \left[\frac{\pi}{2} (J_{a2} - J_{a3}) - (F_{a2} - F_{a3}) \right] \quad (3.4.50)$$

δ) The segment M_3M_4 (the depression curve): $1 \leq \zeta \leq \infty$

Passing around the point $\zeta = 1$ along an infinitely small semi-circle in the (ζ) half-plane, the expression $(1 - \zeta)^{(\frac{1}{2} + \alpha/\pi)}$ becomes

$$-i e^{-i\alpha} (\zeta - 1)^{(\frac{1}{2} + \alpha/\pi)};$$

the function $\omega = f_2(\zeta)$ will take the form

$$\omega = f_2(\zeta) = (\operatorname{tg} \alpha + i) + \frac{1}{(J_{a2} - J_{a3}) \cos \alpha} \int_1^\zeta \frac{\zeta - a}{(\zeta - \kappa)^{(1-\alpha/\pi)} (\zeta - 1)^{(\frac{1}{2} + \alpha/\pi)}} d\zeta \quad (3.4.44d)$$

according to eqn. (3.4.37b)

$$w_r = f_1(\zeta) = -i \frac{q_r}{2} + \frac{q_r}{2} \operatorname{argcosh} \sqrt{\zeta}$$

$$f_1'(\zeta) = \frac{q_r}{\pi} \frac{d\zeta}{\sqrt{[\zeta(\zeta - 1)]}}$$

and according to eqn. (3.4.36)

$$\begin{aligned} z &= \frac{B_1}{2} + \frac{q_r}{\pi} \int_1^\zeta \left[(\operatorname{tg} \alpha + i) + \right. \\ &+ \frac{1}{(J_{a2} - J_{a3}) \cos \alpha} \int_1^\zeta \frac{\zeta - a}{(\zeta - \kappa)^{(1-\alpha/\pi)} (\zeta - 1)^{(\frac{1}{2} + \alpha/\pi)}} d\zeta \left. \right] \frac{d\zeta}{\sqrt{[\zeta(\zeta - 1)]}} = \\ &= \frac{B_1}{2} + (\operatorname{tg} \alpha + i) \frac{q_r}{\pi} \operatorname{argcosh} \sqrt{\zeta} + \\ &+ \frac{1}{(J_{a2} - J_{a3}) \cos \alpha} \frac{q_r}{\pi} \left[\operatorname{argcosh} \sqrt{\zeta} \times \int_1^\zeta \frac{\zeta - a}{(\zeta - \kappa)^{(1-\alpha/\pi)} (\zeta - 1)^{(\frac{1}{2} + \alpha/\pi)}} d\zeta - \right. \\ &\quad \left. - \int_1^\zeta \operatorname{argcosh} \sqrt{\zeta} \times \frac{\zeta - a}{(\zeta - \kappa)^{(1-\alpha/\pi)} (\zeta - 1)^{(\frac{1}{2} + \alpha/\pi)}} d\zeta \right] \quad (3.4.45d) \end{aligned}$$

ε) Conditions for the determination of the calculation constants

The values which depend on the shape of the canal are defined by eqns. (3.4.47), (3.4.49) and (3.4.50). For the purpose at hand we rearrange those equations as follows:

$$\frac{B_0}{2} = \frac{1}{(J_{a2} - J_{a3}) \cos \alpha} \frac{q_r}{\pi} F_{a1} \quad (3.4.47)$$

$$\frac{B_1}{2} = \frac{1}{(J_{a2} - J_{a3}) \cos \alpha} \frac{q_r}{\pi} \left[F_{a1} + \left(\frac{\pi}{2} (J_{a2} - J_{a3}) - (F_{a2} - F_{a3}) \right) \cos \alpha \right] \quad (3.4.51)$$

$$H = \frac{\operatorname{tg} \alpha}{(J_{a2} - J_{a3})} \frac{q_r}{\pi} \left[\frac{\pi}{2} (J_{a2} - J_{a3}) - (F_{a2} - F_{a3}) \right] \quad (3.4.52)$$

Two of the above equations are always independent of one another and thus constitute two conditions for the determination of the unknown constants of the calculation. There are, of course, three constants appearing in the calculation — κ , a and q_r . As we recall, the third condition is that we must know the position of a point of the depression curve; hence we can solve our problem. The final form of eqns. (3.4.47),

(3.4.51) and (3.4.52) is as follows:

$$\frac{B_0}{H} = \frac{2}{\sin \alpha} \frac{\pi}{2} \frac{F_{a1}}{(J_{a2} - J_{a3}) - (F_{a2} - F_{a3})} \quad (3.4.53)$$

$$\frac{B_1}{H} = \frac{2}{\sin \alpha} \frac{F_{a1} + \left[\frac{\pi}{2} (J_{a2} - J_{a3}) - (F_{a2} - F_{a3}) \right] \cos \alpha}{\frac{\pi}{2} (J_{a2} - J_{a3}) - (F_{a2} - F_{a3})} \quad (3.4.54)$$

$$\frac{q_r}{H} = \frac{\pi}{\operatorname{tg} \alpha} \frac{\pi}{2} \frac{J_{a2} - J_{a3}}{(J_{a2} - J_{a3}) - (F_{a2} - F_{a3})}, \quad q = kq_r \quad (3.4.55)$$

The solution outlined above is valid for the following values of a : $a_{\min} \leq a \leq 1$, where $a_{\min}$ refers to the limiting case of the horizontal depression curve coinciding with the original surface of groundwater lying at the level of the water surface in the canal. The value of $a_{\min}$ can be determined from eqn. (3.4.55) for $q_r = 0$. Hence for $a_{\min}$ we must have

$$J_{a2} - J_{a3} = 0 \quad \text{or} \quad J_{a2} = J_{a3}. \quad (3.4.56)$$

The case of a triangular canal will be discussed towards the end of this section.

d) Mapping the auxiliary half-plane (ζ) onto the (ω) region and establishing the equations of the coordinates of the points on the boundary of the region of seepage (z) for the cases when the depression curve has inflections. Deriving the relations necessary for the determination of the unknown constants of the calculation

The whole procedure as well as the various operations are the same as those outlined in Part c). All we have to do is to appreciate the differences between the two problems in order that we may know when to expect changes in the final expressions.

The (ω) region is shown in Fig. 3.48, the auxiliary half-plane (ζ) in Fig. 3.44. The Schwarz-Christoffel integral of the form

$$\begin{aligned} \omega &= C' \int_{\zeta_0}^{\zeta} (\zeta - \kappa)^{(\alpha/\pi - 1)} (\zeta - 1)^{(\pm - \alpha/\pi - 1)} (\zeta - b) d\zeta + C_1 = \\ &= C \int_0^{\zeta} \frac{b - \zeta}{(\kappa - \zeta)^{(1 - \alpha/\pi)} (1 - \zeta)^{(\pm + \alpha/\pi)}} d\zeta + C_1 \end{aligned} \quad (3.4.57)$$

will be set up with the help of the values listed in the table which follows.

M_k	ω_k	α_k/π	ζ_k
M_1	$\frac{i}{v_r(M_1)}$	1	0
M_2	0	π/α	κ
M_3	$\operatorname{tg} \alpha + i$	$\frac{1}{2} - \frac{\alpha}{\pi}$	1
B	$\operatorname{cotg} \beta + i$	2	b
M_4	∞	$-\frac{1}{2}$	∞

Comparing eqn. (3.4.57) with eqn. (3.4.38) we see that the term $(a - \zeta)$ in eqn. (3.4.57) is replaced by the term $(b - \zeta)$. Recalling the meaning and the magnitude of the values of a and b , we see why in the solution of the case d) the side of the canal — instead of being divided into two segments — forms a single segment while the depression curve is divided into two parts. Bearing in mind the differences just stated, we can write down directly the resulting relations for the functions $\omega = f_2(\zeta)$ and $z = f(\zeta)$ (the function $w_r = f_1(\zeta)$ does not alter), applicable to the various segments.

α) The segment M_1M_2 (the canal bottom): $0 \leq \zeta \leq \kappa$

$$\omega = \frac{i}{J_{b2} \cos \alpha} \left[J_{b2} - \int_0^\zeta \frac{b - \zeta}{(\kappa - \zeta)^{(1-\alpha/\pi)} (1 - \zeta)^{(\frac{1}{2} + \alpha/\pi)}} d\zeta \right] \quad (3.4.58a)$$

$$z = \frac{1}{J_{b2} \cos \alpha} \frac{q_r}{\pi} \left[\arcsin \sqrt{\zeta} \times \left(J_{b1} - \int_0^\zeta \frac{b - \zeta}{(\kappa - \zeta)^{(1-\alpha/\pi)} (1 - \zeta)^{(\frac{1}{2} + \alpha/\pi)}} d\zeta \right) + \right. \\ \left. + \int_0^\zeta \arcsin \sqrt{\zeta} \times \frac{b - \zeta}{(\kappa - \zeta)^{(1-\alpha/\pi)} (1 - \zeta)^{(\frac{1}{2} + \alpha/\pi)}} d\zeta \right] + iH \quad (3.4.59a)$$

Substituting the values relating to the point M_2 ($\zeta = \kappa$, $z = B_0/2 + iH$) and performing some manipulations we obtain

$$\frac{B_0}{2} = \frac{1}{J_{b2} \cos \alpha} \frac{q_r}{\pi} F_{b1} \quad (3.4.60a)$$

In the above equations

$$J_{b1} = \int_0^\kappa \frac{b - \zeta}{(\kappa - \zeta)^{(1-\alpha/\pi)} (1 - \zeta)^{(\frac{1}{2} + \alpha/\pi)}} d\zeta \quad (3.4.61a)$$

$$J_{b2} = \int_{\kappa}^1 \frac{b - \zeta}{(\zeta - \kappa)^{(1-\alpha/\pi)} (1 - \zeta)^{(\frac{1}{2} + \alpha/\pi)}} d\zeta \quad (3.4.61b)$$

$$F_{b1} = \int_0^{\kappa} \arcsin \sqrt{\zeta} \times \frac{b - \zeta}{(\kappa - \zeta)^{(1-\alpha/\pi)} (1 - \zeta)^{(\frac{1}{2} + \alpha/\pi)}} d\zeta \quad (3.4.62a)$$

β) The segment M_2M_3 (the side of the canal): $\kappa \leq \zeta \leq 1$

$$\omega = \frac{i e^{-i\alpha}}{J_{b2} \cos \alpha} \int_{\kappa}^{\zeta} \frac{b - \zeta}{(\zeta - \kappa)^{(1-\alpha/\pi)} (1 - \zeta)^{(\frac{1}{2} + \alpha/\pi)}} d\zeta \quad (3.4.58b)$$

$$z = \frac{B_0}{2} + iH + \frac{e^{-i\alpha}}{J_{b2} \cos \alpha} \frac{q_r}{\pi} \left[\arcsin \sqrt{\zeta} \int_{\kappa}^{\zeta} \frac{b - \zeta}{(\zeta - \kappa)^{(1-\alpha/\pi)} (1 - \zeta)^{(\frac{1}{2} + \alpha/\pi)}} d\zeta - \right. \\ \left. - \int_{\kappa}^{\zeta} \arcsin \sqrt{\zeta} \times \frac{b - \zeta}{(\zeta - \kappa)^{(1-\alpha/\pi)} (1 - \zeta)^{(\frac{1}{2} + \alpha/\pi)}} d\zeta \right] \quad (3.4.59b)$$

Substituting the values appertaining to the point M_3 ($\zeta = 1$, $z = B_1/2$) and performing some manipulations we obtain the relation

$$\frac{B_1 - B_0}{2} = \frac{q_r}{\pi} \left(\frac{\pi}{2} J_{b2} - F_{b2} \right) \quad (3.4.60b)$$

where

$$F_{b2} = \int_{\kappa}^1 \arcsin \sqrt{\zeta} \times \frac{b - \zeta}{(\zeta - \kappa)^{(1-\alpha/\pi)} (1 - \zeta)^{(\frac{1}{2} + \alpha/\pi)}} d\zeta \quad (3.4.62b)$$

Naturally, eqn. (3.4.49) also applies.

γ) The segment M_3B (the depression curve): $1 \leq \zeta \leq b$

$$\omega = (\operatorname{tg} \alpha + i) - \frac{1}{J_{b2} \cos \alpha} \int_1^{\zeta} \frac{b - \zeta}{(\zeta - \kappa)^{(1-\alpha/\pi)} (\zeta - 1)^{(\frac{1}{2} + \alpha/\pi)}} d\zeta \quad (3.4.58c)$$

$$z = \frac{B_1}{2} + (\operatorname{tg} \alpha + i) \frac{q_r}{\pi} \operatorname{argcosh} \sqrt{\zeta} - \\ - \frac{1}{J_{b2} \cos \alpha} \frac{q_r}{\pi} \left[\operatorname{argcosh} \sqrt{\zeta} \int_1^{\zeta} \frac{b - \zeta}{(\zeta - \kappa)^{(1-\alpha/\pi)} (\zeta - 1)^{(\frac{1}{2} + \alpha/\pi)}} d\zeta - \right. \\ \left. - \int_1^{\zeta} \operatorname{argcosh} \sqrt{\zeta} \times \frac{b - \zeta}{(\zeta - \kappa)^{(1-\alpha/\pi)} (\zeta - 1)^{(\frac{1}{2} + \alpha/\pi)}} d\zeta \right] \quad (3.4.59c)$$

For the point $B(\zeta = b)$, we have

$$\omega(B) = (\operatorname{tg} \alpha + i) - \frac{J_{b4}}{J_{b2} \cos \alpha} = \cotg \beta + i \quad (3.4.60c)$$

$$z(B) = \frac{B_1}{2} + (\operatorname{tg} \alpha + i) \frac{q_r}{\pi} \operatorname{argcosh} \sqrt{b} - \frac{1}{J_{b2} \cos \alpha} \frac{q_r}{\pi} (\operatorname{argcosh} \sqrt{b} \times J_{b4} - F_{b4}) \quad (3.4.60d)$$

where

$$J_{b4} = \int_1^b \frac{b - \zeta}{(\zeta - \kappa)^{(1-\alpha/\pi)} (\zeta - 1)^{(\frac{1}{2} + \alpha/\pi)}} d\zeta \quad (3.4.61c)$$

$$F_{b4} = \int_1^b \operatorname{argcosh} \sqrt{\zeta} \times \frac{b - \zeta}{(\zeta - \kappa)^{(1-\alpha/\pi)} (\zeta - 1)^{(\frac{1}{2} + \alpha/\pi)}} d\zeta \quad (3.4.62c)$$

δ) The segment BM_5 (the depression curve): $b \leq \zeta \leq \infty$

By passing around the point $B(\zeta = b)$ along an infinitely small semi-circle in the (ζ) half-plane, the expression $(b - \zeta)$ changes to $-(\zeta - b)$. Then

$$\omega = \omega(B) + \frac{1}{J_{b2} \cos \alpha} \int_b^\zeta \frac{\zeta - b}{(\zeta - \kappa)^{(1-\alpha/\pi)} (\zeta - 1)^{(\frac{1}{2} + \alpha/\pi)}} d\zeta \quad (3.4.58d)$$

$$z = z(B) + \frac{q_r}{\pi} \int_b^\zeta \left[\omega(B) + \frac{1}{J_{b2} \cos \alpha} \int_b^\zeta \frac{\zeta - b}{(\zeta - \kappa)^{(1-\alpha/\pi)} (\zeta - 1)^{(\frac{1}{2} + \alpha/\pi)}} d\zeta \right] \frac{d\zeta}{\sqrt{[\zeta(\zeta - 1)]}}$$

Following integration by parts, substitution in accordance with eqns. (3.4.60c) and (3.4.60d) and some manipulations we obtain

$$\begin{aligned} z = & \frac{B_1}{2} + \frac{q_r}{\pi} (\operatorname{tg} \alpha + i) \operatorname{argcosh} \sqrt{\zeta} + \\ & + \frac{1}{J_{b2} \cos \alpha} \frac{q_r}{\pi} \left[\operatorname{argcosh} \sqrt{\zeta} \left(\int_b^\zeta \frac{\zeta - b}{(\zeta - \kappa)^{(1-\alpha/\pi)} (\zeta - 1)^{(\frac{1}{2} + \alpha/\pi)}} d\zeta - J_{b4} \right) - \right. \\ & \left. - \left(\int_b^\zeta \operatorname{argcosh} \sqrt{\zeta} \times \frac{\zeta - b}{(\zeta - \kappa)^{(1-\alpha/\pi)} (\zeta - 1)^{(\frac{1}{2} + \alpha/\pi)}} d\zeta - F_{b4} \right) \right] \quad (3.4.59d) \end{aligned}$$

ε) Conditions for the determination of the calculation constants

Eqns. (3.4.60a), (3.4.60b) and (3.4.49) define the quantities dependent on the shape and dimensions of the canal. Of those three equations two are always independent of one another and thus provide two conditions for the determination of the unknown

constants — κ , b , q_r . The third condition is that of knowing the position of a general point on the depression curve. The three equations rearranged directly in their final form are

$$\frac{B_0}{H} = \frac{2}{\sin \alpha} \frac{F_{b1}}{\frac{\pi}{2} J_{b2} - F_{b2}} \quad (3.4.63)$$

$$\frac{B_1}{H} = \frac{2}{\sin \alpha} \frac{F_{b1} + \left(\frac{\pi}{2} J_{b2} - F_{b2} \right) \cos \alpha}{\frac{\pi}{2} J_{b2} - F_{b2}} \quad (3.4.64)$$

$$\frac{q_r}{H} = \frac{\pi}{\operatorname{tg} \alpha} \frac{J_{b2}}{\frac{\pi}{2} J_{b2} - F_{b2}}, \quad q = kq_r \quad (3.4.65)$$

The solution obtained for the case d) also applies to seepage without back-water, i.e. to the limiting case when $b = \infty$. In this case the term $(b - \zeta)$ can be factored out and does not, therefore, appear in the integral expressions of the resulting equations.

e) Solution of a triangular-shaped canal

In this case, the constant κ ($\kappa = 0$) and eqns. (3.4.47) or (3.4.53) and eqns. (3.4.60a) or (3.4.63) will not figure in the solution, and eqns. (3.4.54), (3.4.64) become eqn. (3.4.49). The constants a and b are determined as they depend on the position of the depression curve, and the quantity of seepage is obtained using eqns. (3.4.55) and (3.4.65).

We shall now adopt a somewhat different procedure. We choose the values of the constants a and b , and calculate the reduced quantity of seepage q_r and the shape of the depression curve corresponding to the chosen values. We then plot the position of the depression curves. In each particular case we draw only the position of the known point of the depression curve, and determine by interpolation the reduced quantity of seepage q_r , the position of the depression curve, and if required, also the constants a and b for the calculation of the velocities.

For a trapezoidal cross-section this procedure would be extremely complicated because we would have to repeat it for various values of κ ($0 \leq \kappa \leq 1$), determine the dimensions of the canal corresponding to the various values by the reverse procedure, and establish the values always for constant ratios B_0/H or B_1/H .

We have obtained numerical results for a canal of a depth $H = 1$ and an angle of the sides $\alpha = \pi/4$. The integrals were evaluated with the help of Gauss' formula for dividing an interval into ten subintervals. The values of the integrals and the reduced quantity of seepage q_r as well as the coordinates of the depression curves are set out in the tables below. The depression curves are drawn in Fig. 3.52, and the reduced quantity of seepage q_r as a function of the constant a or b is shown in Fig. 3.53.

Reduced Quantity of Seepage

a	J_{a2}	J_{a3}	F_{a2}	F_{a3}	q_r/H
0.5	1.044 84	1.044 84	0.261 28	1.380 02	0.00
0.6	1.335 52	0.778 14	0.311 16	1.051 83	1.083 44
0.7	1.651 13	0.535 29	0.513 04	0.741 07	1.769 76
0.8	1.995 91	0.318 15	0.684 18	0.452 21	2.193 03
0.9	2.377 93	0.132 10	0.900 40	0.194 06	2.500 70
1.0	2.830 00	—	1.221 66	—	2.757 93

b	J_{b2}	F_{b2}	q_r/H
1.0	2.830 00	1.221 66	2.757 93
1.1	3.396 00	1.670 02	2.911 49
1.5	5.659 99	3.463 45	3.276 32
2.0	8.489 99	5.705 24	3.495 32
5.0	25.469 96	19.155 99	3.837 32
10.0	53.769 92	41.573 89	3.938 73
100.0	563.169 20	445.096 18	4.025 34

∞	5.659 99	4.483 58	4.034 70

Coordinates of the Depression Curve

<i>a, b</i>	$\zeta =$	1.1	1.5	2.0	5.0	10.0	100.0
0.5	<i>x</i>	1.00	1.202 66	2.067 93	4.046 70	6.275 19	22.828 97
	<i>y</i>	0.00	0.00	0.00	0.00	0.00	0.00
0.6	<i>x</i>	1.00	1.220 22	1.921 17	3.332 44	4.886 91	16.429 23
	<i>y</i>	0.00	0.107 32	0.303 96	0.497 87	0.627 13	1.032 16
0.7	<i>x</i>	1.00	1.245 08	1.900 55	3.086 28	4.364 52	13.789 34
	<i>y</i>	0.00	0.175 30	0.496 50	0.813 25	1.024 39	1.686 01
0.8	<i>x</i>	1.00	1.256 32	1.866 20	2.872 42	3.934 58	11.709 48
	<i>y</i>	0.00	0.217 22	0.615 25	1.007 75	1.269 39	2.089 25
0.9	<i>x</i>	1.00	1.265 32	1.845 45	2.728 55	3.641 59	10.273 68
	<i>y</i>	0.00	0.247 70	0.701 57	1.149 14	1.447 48	2.382 36
1.0	<i>x</i>	1.00	1.274 87	1.838 49	2.636 89	3.445 07	9.263 22
	<i>y</i>	0.00	0.273 18	0.773 73	1.267 34	1.596 38	2.627 42
1.1	<i>x</i>	1.00	1.277 80	1.819 96	2.542 00	3.259 10	8.383 73
	<i>y</i>	0.00	0.288 39	0.816 81	1.337 90	1.685 26	2.773 70
1.5	<i>x</i>	1.00	1.284 96	1.776 67	2.317 48	2.817 65	6.283 30
	<i>y</i>	0.00	0.324 52	0.919 17	1.505 55	1.896 44	3.121 27
2.0	<i>x</i>	1.00	1.288 94	1.749 54	2.181 20	2.552 03	5.039 78
	<i>y</i>	0.00	0.346 22	0.980 61	1.606 19	2.023 20	3.329 91
5.0	<i>x</i>	1.00	1.295 46	1.704 52	1.969 86	2.137 84	3.080 98
	<i>y</i>	0.00	0.380 09	1.076 56	1.763 34	2.221 16	3.655 72
10.0	<i>x</i>	1.00	1.297 40	1.696 05	1.907 19	2.015 02	2.500 12
	<i>y</i>	0.00	0.390 14	1.105 01	1.809 95	2.279 86	3.752 34
100.0	<i>x</i>	1.00	1.299 05	1.685 60	1.853 67	1.910 14	2.004 08
	<i>y</i>	0.00	0.398 72	1.129 30	1.849 74	2.329 99	3.834 84
∞	<i>x</i>	1.00	1.299 23	1.684 47	1.847 88	1.898 80	1.950 45
	<i>y</i>	0.00	0.399 64	1.131 93	1.854 05	2.335 41	3.843 76

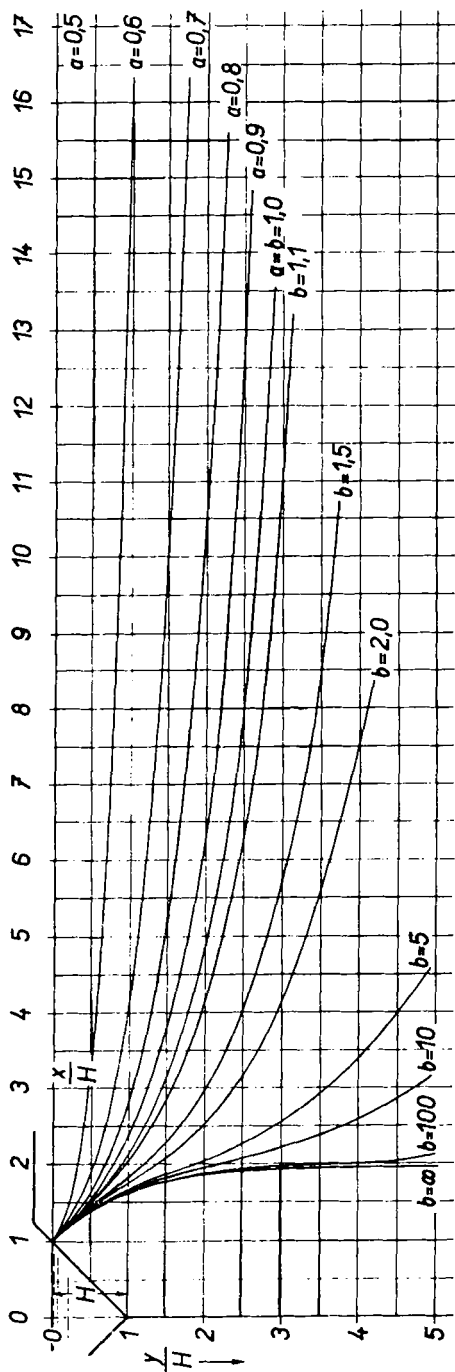


Fig. 3.52. Depression curve for seepage from a triangular canal.

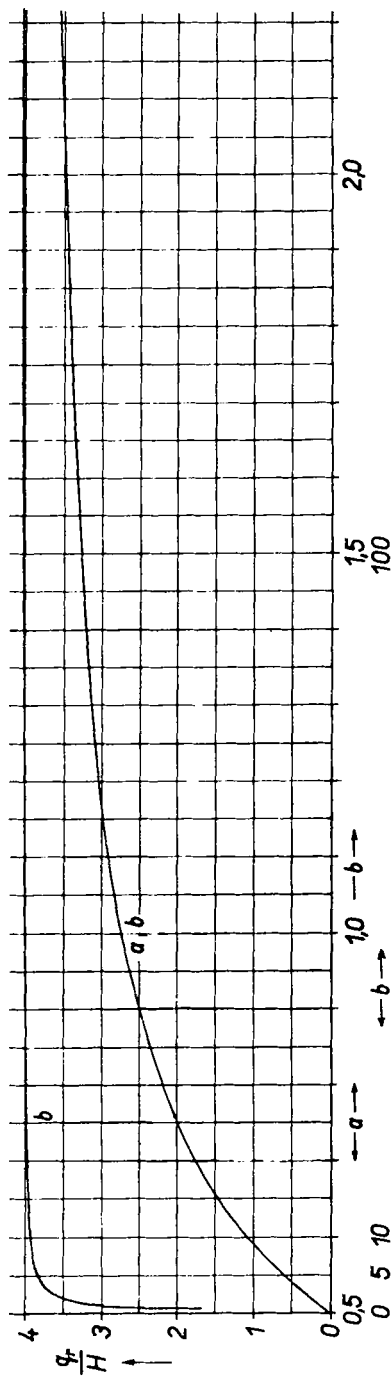


Fig. 3.53. Reduced seepage curve from a triangular canal.

3.4.4. Seepage through an earth dam on permeable subsoil

This example will illustrate the application of the indirect form of the first variant of the velocity hodograph method. As noted in Section 3.4.2, the indirect method requires neither the permeable nor the impervious segments to be linear. We shall specify exactly a portion of the boundary of the region of seepage, and establish the shape of the remaining segments by the reverse procedure in dependence on the accepted shape of the velocity hodograph which we shall select so as to make the mathematical solution as simple as possible. If the shape of the segments established in this manner corresponds, approximately at least, to the specified shape, the solution is considered to be satisfactory.

In the example, the dam is built on a permeable subsoil of a large — theoretically infinite — thickness. The coefficient of permeability of the dam material is the same as that of the permeable subsoil. It is assumed that the contours of the upstream and the downstream face, and the contours of the bottom of the reservoir upstream of the dam and of the bottom of the river bed downstream of the dam, are curvilinear; further, that a rockfill is placed at the downstream toe of the dam, and that the segment of free seepage to the surface is missing.

The region of seepage is shown schematically in Fig. 3.54.

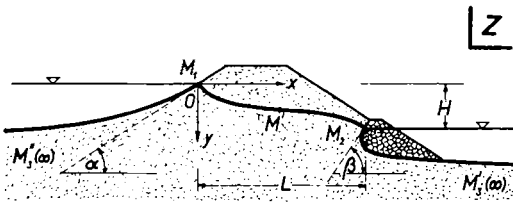


Fig. 3.54. Seepage through an earth dam on permeable subsoil.

If we place the reference plane for the determination of h at the level of the surface of water in the reservoir, we have for the (w_r) region of the reduced complex potential:

for the segment $M_3''M_1$ — the bottom of the upstream reservoir and the upstream face of the dam:

$$\varphi_r = -h = 0$$

for the depression curve M_1M_2 which is a streamline ($\psi_r = \text{const.}$) we choose

$$\psi_r = 0$$

for the segment M_2M_3' — the boundary between the dam structure and the rockfill, and the river bed downstream of the dam

$$\varphi_r = -h = H$$

where H is the total head.

v_{rx} , v_{ry} and — the velocity at the point M_3'' being zero — has a vertical tangent at that point; in the immediate vicinity of the point M_3'' the direction of the velocity is vertical, $v_{ry} \geq 0$. To simplify, we shall again choose to represent this curve by a segment of a circle centred on the Ov_{rx} axis, passing through the point M_1 and the origin of the coordinates.

If we know the shape of the velocity hodograph, i.e. the (v_r) region (Fig. 3.56), we can find the shape of the $(v_{rx} - iv_{ry})$ region by revolving the (v_r) region around the Ov_{rx} axis; the shape of the (ω) region corresponding to the latter, is obtained by

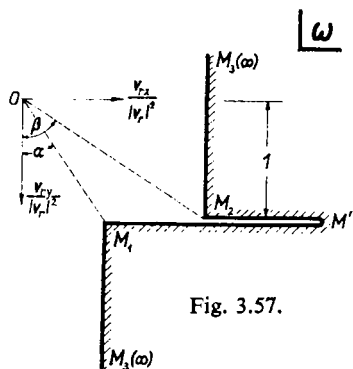


Fig. 3.57.

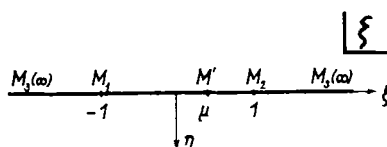


Fig. 3.58.

using eqn. (3.4.7). The (ω) region is shown in Fig. 3.57. It is a polygon with sides parallel to the coordinate axes. We shall use the Schwarz-Christoffel integral to map this polygon onto the auxiliary half-plane (ζ) (Fig. 3.58) and this in turn onto other regions.

The values required for setting up the Schwarz-Christoffel integral which gives the relation $\omega = f_2(\zeta)$, are reviewed in the table below. In this particular case the relation is

$$\omega = C \int_{\zeta_0}^{\zeta} [(\zeta - 1)(\zeta + 1)]^{-\frac{1}{2}} (\zeta - \mu) d\zeta + C_1 = C' \int_{\zeta_0}^{\zeta} \frac{(\zeta - \mu)}{\sqrt{(1 - \zeta^2)}} d\zeta + C_1 =$$

$$= -C' \sqrt{(1 - \zeta^2)} - \mu C' \arcsin \zeta + C_1$$

M_k	ω_k	α_k/π	ζ_k
M_1	$\operatorname{tg} \alpha + i$	$\frac{1}{2}$	-1
M_2	$\operatorname{tg} \beta + i$	$\frac{1}{2}$	$+1$
M_3	∞	-1	∞
M'	$\operatorname{tg} \gamma + i$	2	μ

Substituting the values relating to the point M_1 :

$$\omega = \operatorname{tg} \alpha + i, \quad \zeta = -1$$

and to the point M_2 :

$$\omega = \operatorname{tg} \beta + i, \quad \zeta = +1$$

we obtain

$$C' = -\frac{\operatorname{tg} \beta - \operatorname{tg} \alpha}{\mu\pi}, \quad C_1 = \frac{\operatorname{tg} \beta + \operatorname{tg} \alpha}{2} + i$$

so that the resulting form of the $\omega = f_2(\zeta)$ dependence is

$$\omega = \frac{\operatorname{tg} \beta - \operatorname{tg} \alpha}{\mu\pi} \sqrt{(1 - \zeta^2)} + \frac{\operatorname{tg} \beta - \operatorname{tg} \alpha}{\pi} \arcsin \zeta + \frac{\operatorname{tg} \beta + \operatorname{tg} \alpha}{2} + i \quad (3.4.66a)$$

The relation $w_r = f_1(\zeta)$ is determined by successive conformal mappings of the (w_r) region onto the (ζ) region:

$$\begin{aligned} w_1 &= \frac{w_r}{H} \pi \\ w_2 &= w_1 - \frac{\pi}{2} = \pi \left(\frac{w_r}{H} - \frac{1}{2} \right) \\ \zeta &= \sin w_2 = \sin \pi \left(\frac{w_r}{H} - \frac{1}{2} \right) \end{aligned} \quad (3.4.66b)$$

On substituting from eqn. (3.4.66b) in eqn. (3.4.66a) we obtain the relation (3.4.7)

$$\begin{aligned} \omega &= \frac{dz}{dw_r} = \frac{\operatorname{tg} \beta - \operatorname{tg} \alpha}{\mu\pi} \cos \pi \left(\frac{w_r}{H} - \frac{1}{2} \right) + \\ &+ (\operatorname{tg} \beta - \operatorname{tg} \alpha) \left(\frac{w_r}{H} - \frac{1}{2} \right) + \frac{\operatorname{tg} \beta + \operatorname{tg} \alpha}{2} + i = \\ &= \frac{\operatorname{tg} \beta - \operatorname{tg} \alpha}{\mu\pi} \sin \frac{\pi w_r}{H} + (\operatorname{tg} \beta - \operatorname{tg} \alpha) \frac{w_r}{H} + \operatorname{tg} \alpha + i \end{aligned}$$

and separating the variables and integrating we have the resulting relation (3.4.9):

$$\begin{aligned} \frac{z}{H} &= \int \frac{\omega dw_r}{H} = \iint \left[\frac{\operatorname{tg} \beta - \operatorname{tg} \alpha}{\mu\pi} \sin \frac{\pi w_r}{H} + (\operatorname{tg} \beta - \operatorname{tg} \alpha) \frac{w_r}{H} + \operatorname{tg} \alpha + i \right] \frac{dw_r}{H} = \\ &= -\frac{\operatorname{tg} \beta - \operatorname{tg} \alpha}{\mu\pi^2} \cos \frac{\pi w_r}{H} + \frac{\operatorname{tg} \beta - \operatorname{tg} \alpha}{2} \left(\frac{w_r}{H} \right)^2 + (\operatorname{tg} \alpha + i) \frac{w_r}{H} + C \end{aligned} \quad (3.4.67a)$$

For the point M_1 , $z = 0$, $w_r = 0$; on substituting in eqn. (3.4.67a)

$$C = \frac{\operatorname{tg} \beta - \operatorname{tg} \alpha}{\mu \pi^2}$$

so that eqn. (3.4.67a) may be rewritten as follows:

$$\begin{aligned} \frac{z}{H} &= \frac{\operatorname{tg} \beta - \operatorname{tg} \alpha}{\mu \pi^2} \left(1 - \cos \frac{\pi w_r}{H} \right) + \frac{\operatorname{tg} \beta - \operatorname{tg} \alpha}{2} \left(\frac{w_r}{H} \right)^2 + (\operatorname{tg} \alpha + i) \frac{w_r}{H} = \\ &= \frac{2(\operatorname{tg} \beta - \operatorname{tg} \alpha)}{\mu \pi^2} \frac{\sin^2 \frac{\pi w_r}{2H}}{2H} + \frac{\operatorname{tg} \beta - \operatorname{tg} \alpha}{2} \left(\frac{w_r}{H} \right)^2 + (\operatorname{tg} \alpha + i) \frac{w_r}{H} \quad (3.4.67b) \end{aligned}$$

The dimensions of the dam are known. For the point M_2 , $z = (L + iH)$, $w_r = H$. On substituting these values in eqn. (3.4.67b) we determine the hitherto unknown constant μ :

$$\frac{L + iH}{H} = \frac{2(\operatorname{tg} \beta - \operatorname{tg} \alpha)}{\mu \pi^2} \sin^2 \frac{\pi}{2} + \frac{\operatorname{tg} \beta - \operatorname{tg} \alpha}{2} + \operatorname{tg} \alpha + i$$

or

$$\frac{2(\operatorname{tg} \beta - \operatorname{tg} \alpha)}{\mu \pi^2} = \frac{L}{H} - \frac{\operatorname{tg} \beta + \operatorname{tg} \alpha}{2}$$

On substituting in eqn. (3.4.67b) we obtain the final form of the dependence $z = z(w_r)$ which determines the seepage through an earth dam on a permeable subsoil:

$$\begin{aligned} \frac{z}{H} &= \left(\frac{L}{H} - \frac{\operatorname{tg} \beta + \operatorname{tg} \alpha}{2} \right) \sin^2 \frac{\pi w_r}{2H} + \frac{\operatorname{tg} \beta - \operatorname{tg} \alpha}{2} \left(\frac{w_r}{H} \right)^2 + (\operatorname{tg} \alpha + i) \frac{w_r}{H} \\ 0 &\leq \varphi_r \leq H, \quad 0 \leq \psi_r \leq \infty \quad (3.4.68) \end{aligned}$$

The equation of the depression curve in the dam is found from eqn. (3.4.68) after substituting the values of the reduced complex potential on the depression curve $w_r = y$:

$$\frac{x}{H} = \left(\frac{L}{H} - \frac{\operatorname{tg} \beta + \operatorname{tg} \alpha}{2} \right) \sin^2 \frac{\pi y}{2H} + \frac{\operatorname{tg} \beta - \operatorname{tg} \alpha}{2} \left(\frac{y}{H} \right)^2 + \operatorname{tg} \alpha \frac{y}{H} \quad (3.4.68a)$$

The equation of the segment $M_2 M'_3$ is obtained by substituting the values of w_r on this segment, i.e. $w_r = H + i\psi_r$, in eqn. (3.4.68) and carrying out some manipulations

$$\begin{aligned} \frac{z}{H} &= \left(\frac{L}{H} - \frac{\operatorname{tg} \beta + \operatorname{tg} \alpha}{2} \right) \sinh^2 \frac{\pi \psi_r}{2H} - \frac{\operatorname{tg} \beta - \operatorname{tg} \alpha}{2} \left(\frac{\psi_r}{H} \right)^2 + \\ &+ (i \operatorname{tg} \beta - 1) \frac{\psi_r}{H} + \frac{L}{H} + i \quad (3.4.68b) \end{aligned}$$

or, separating the equation into the real and imaginary parts

$$\begin{aligned}\frac{x}{H} &= \left(\frac{L}{H} - \frac{\operatorname{tg} \beta + \operatorname{tg} \alpha}{2} \right) \sinh^2 \frac{\pi \psi_r}{2H} - \frac{\operatorname{tg} \beta - \operatorname{tg} \alpha}{2} \left(\frac{\psi_r}{H} \right)^2 - \frac{\psi_r}{H} + \frac{L}{H} \\ \frac{y}{H} &= \operatorname{tg} \beta \frac{\psi_r}{H} + 1\end{aligned}\quad (3.4.68c)$$

The value ψ_r/H can be determined from the second of eqns. (3.4.68c) and substituted in the first to give

$$\begin{aligned}\frac{x}{H} &= \left(\frac{L}{H} - \frac{\operatorname{tg} \beta + \operatorname{tg} \alpha}{2} \right) \sinh^2 \left[\frac{\pi}{2} \left(\frac{y}{H} - 1 \right) \cotg \beta \right] - \\ &- \frac{\operatorname{tg} \beta - \operatorname{tg} \alpha}{2} \left(\frac{y}{H} - 1 \right)^2 \cotg^2 \beta - \left(\frac{y}{H} - 1 \right) \cotg \beta + \frac{L}{H}\end{aligned}\quad (3.4.68d)$$

The equation of the segment $M_3''M_1$ is obtained on substituting $w_r = i\psi_r$ in eqn. (3.4.68) and making some manipulations

$$\frac{z}{H} = - \left(\frac{L}{H} - \frac{\operatorname{tg} \beta + \operatorname{tg} \alpha}{2} \right) \sinh^2 \frac{\pi \psi_r}{2H} - \frac{\operatorname{tg} \beta - \operatorname{tg} \alpha}{2} \left(\frac{\psi_r}{H} \right)^2 + (i \operatorname{tg} \alpha - 1) \frac{\psi_r}{H}\quad (3.4.68e)$$

or

$$\frac{x}{H} = - \left(\frac{L}{H} - \frac{\operatorname{tg} \beta + \operatorname{tg} \alpha}{2} \right) \sinh^2 \frac{\pi \psi_r}{2H} - \frac{\operatorname{tg} \beta - \operatorname{tg} \alpha}{2} \left(\frac{\psi_r}{H} \right)^2 - \frac{\psi_r}{H}\quad (3.4.68f)$$

$$\frac{y}{H} = \operatorname{tg} \alpha \frac{\psi_r}{H}$$

from which it follows that

$$\begin{aligned}\frac{x}{H} &= - \left(\frac{L}{H} - \frac{\operatorname{tg} \beta + \operatorname{tg} \alpha}{2} \right) \sinh^2 \left[\frac{\pi y}{2H} \cotg \alpha \right] - \\ &- \frac{\operatorname{tg} \beta - \operatorname{tg} \alpha}{2} \left(\frac{y}{H} \right)^2 \cotg^2 \alpha - \frac{y}{H} \cotg \alpha\end{aligned}\quad (3.4.68g)$$

Two constants, the angles α and β , not known in advance, appear in all of the eqns. (3.4.68). We shall find them from two additional conditions, for example, those relating to the positions of two points on the boundary of the (z) region. For convenience, we use the points lying on the bottom of the upstream reservoir and the points on the river bed downstream of the dam. On substituting the coordinates of these points in eqns. (3.4.68d) and (3.4.68g) we obtain a system of equations from which the values of α and β can be determined.

There remains the calculation of the reduced quantity of seepage through the dam and subsoil. The quantity of seepage through the dam and the subsoil between the dam and a chosen point M_2 , is obtained as the difference between the values of the stream function at the points M_q and M_2 , viz.

$$q_r = \psi_r(M_q) - \psi_r(M_2) = \psi_r(M_q), \quad q = kq_r \quad (3.4.69)$$

where $\psi_r(M_q)$ is established from eqn. (3.4.68c) by substituting in it the coordinates of the chosen point, which must, of course, satisfy eqn. (3.4.68d).

By analyzing the problem in detail, Numerov has shown that the solution is acceptable only for $L/H > (\operatorname{tg} \beta + \operatorname{tg} \alpha)/2$.

3.4.5. Seepage into a vertical drainage cut

This simple example will illustrate the application of the second variant of the velocity hodograph method.

The seepage region is shown schematically in Fig. 3.59.

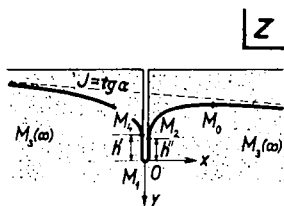


Fig. 3.59. Seepage into a vertical drainage cut.

The case we are going to solve, is a theoretical case because the width of the cut as well as the depth of water in the cut are assumed to be infinitely small. The permeable layer is of a great (theoretically infinite) thickness, and the inclination of the level of the original groundwater $J = \operatorname{tg} \alpha$.

We shall establish the region of Zhukovsky's function (w_{zh}) and the region of the velocity hodograph (v_r) by referring to Section 3.4.2b; we shall obtain the (ω) region using eqn. (3.4.11), further the relation (3.4.12) $\omega = f_1(w_{zh})$, and finally, eqn. (3.4.13) $z = f_2(w_{zh})$.

As to the boundary of the region of Zhukovsky's function: the pressure p is equal to atmospheric on the segments M_3M_4 and M_2M_3 , as well as on the segments M_4M_1 and M_1M_2 , i.e. on the free surface and on the segments of free seepage. If we place the reference plane for the determination of the piezometric heads and of the reduced potential at the level of the x axis, we have $h = -y$, $\varphi_r = -h = +y$, and according to eqns. (3.4.10) and (3.3.1)

$$v_{zh} = y - \varphi_r = y - y = 0$$

If we choose $\psi_r(M_1) = 0$, we obtain the following values of $u_{zh} = x + \psi_r$ for the

various points

$$u_{zh}(M_1) = 0, \quad u_{zh}(M_2) = +q''_r, \quad u_{zh}(M_3) = \infty, \quad u_{zh}(M_4) = -q'_r$$

where q'_r and q''_r are successively the reduced quantities of seepage into the drainage cut through the segment M_4M_1 and the segment M_1M_2 ; the total quantity of seepage into the cut is

$$q_r = q'_r + q''_r \quad (3.4.70)$$

The region of Zhukovsky's function is shaped as shown in Fig. 3.60. The figure also indicates the disposition of the various points corresponding to the similarly designated points in the (z) region.

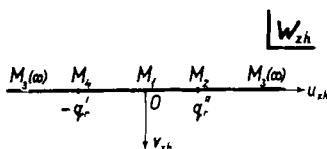


Fig. 3.60.

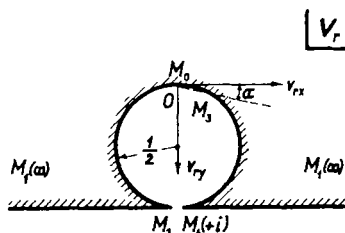


Fig. 3.61.

The region of the velocity hodograph is shown in Fig. 3.61. Its shape is obtained by using the procedure outlined in Section 3.4.1. In keeping with that section, there must correspond to the vertical segments M_4M_1 and M_1M_2 a straight line passing in the velocity hodograph (v_r) through the point $v_r = +i$, parallel to the horizontal axis Ov_{rx} . To the segment M_4M_1 there must correspond that portion of the straight line where $v_{rx} \geq 0$, to the segment M_1M_2 that with $v_{rx} \leq 0$. At the point M_1 the velocity $v_r(M_1) = \infty$. In the (z) region the points M_4 and M_1 lie on the sides of the cut as well as on the depression curve. The figure which corresponds to the depression curve in the velocity hodograph, is a circle centred on the Ov_{ry} axis, passing through the origin of the coordinates v_{rx}, v_{ry} and the point $v_r = +i$. The points M_4 and M_2 lie at the points of intersection of this circle and the straight lines corresponding to the segments M_4M_1 and M_1M_2 , i.e. at the point $v_r = +i$. To the left-hand branch of the depression curve M_3M_4 corresponds the portion of the circle between the points M_3 and M_4 , the point M_3 being defined by the direction of the velocity at M_3 . At the point M_3 the value of the velocity is the same as that of the original flow undisturbed by the drainage cut, i.e. its direction makes an angle α with the horizontal. From the point M_3 to the point M_4 the velocity vector is increasingly deflected downwards in the positive direction. To the right-hand branch of the depression curve corresponds the portion of the circle between the points M_3 and M_2 . On this branch of the depression curve lies the point M_0 at which the tangent to the depression curve is horizontal and the velocity is zero. The point in the velocity hodograph corresponding to the point M_0 , lies at the origin of the coordinates. To the portion of the

branch of the depression curve M_0M_2 corresponds a portion of the circle where $v_{rx} \leq 0$. To the portion of the branch M_0M_3 corresponds that portion of the segment M_2M_3 of the circle in the velocity hodograph where $v_{rx} \geq 0$. Similarly as before, the value and the direction of the velocity at the point M_3 are the same as in the original flow. This establishes the boundary of the (v_r) region.

The point M_0 is significant in that it represents a point on the depression curve towards which proceeds the streamline ($\psi_r = q_r''$ in our case) which separates the quantity of percolating water flowing into the cut from the rest of the percolating water which in fact flows around the cut.

We shall now establish the $(v_{rx} - iv_{ry})$ region to the (v_r) region and the (ω) region according to eqn. (3.4.11). The (ω) region is shown in Fig. 3.62.

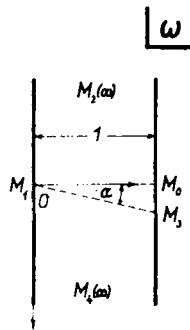


Fig. 3.62.

By successive conformal mapping of the (ω) region onto the (w_{zh}) region we shall obtain eqn. (3.4.12), using the function

$$w_1 = e^{i\pi\omega}$$

as well as the bilinear function, so that

$$\omega = \frac{i}{\pi} \ln \frac{q_r'(q_r'' - w_{zh})}{q_r''(q_r' + w_{zh})} \quad (3.4.71)$$

Substituting the values relating to the point $M_3[\omega(M_3) = (1 + i \operatorname{tg} \alpha), w_{zh}(M_3) = \infty]$ in the above, we obtain

$$J = \operatorname{tg} \alpha = \frac{1}{\pi} \ln \frac{q_r'}{q_r''} \quad (3.4.72a)$$

from which it follows — in view of eqn. (3.4.70) — that

$$q_r' = \frac{e^{\pi J}}{e^{\pi J} + 1} q_r, \quad q_r'' = \frac{1}{e^{\pi J} + 1} q_r \quad (3.4.72b)$$

The resulting equation is obtained on substituting from eqn. (3.4.71) in (3.4.13) and integrating, paying heed to the fact that for the point M_1 we have $z(M_1) = 0$,

$w_{zh}(M_1) = 0$. In fact:

$$z = \frac{i}{\pi} \{ q'_r \ln q'_r + q''_r \ln q''_r - (q'_r + w_{zh}) \ln (q'_r + w_{zh}) - (q''_r - w_{zh}) \ln (q''_r - w_{zh}) + \pi J w_{zh} \} \quad (3.4.73a)$$

The equation of the left-hand branch of the depression curve M_3M_4 is found by substituting the values of w_{zh} on the segment M_3M_4 in eqn. (3.4.73a). For the M_3M_4 segment, $v_{zh} = 0$, $\psi_r = -q'_r$, $w_{zh} = u_{zh} = x - q'_r$, $x \leq 0$. Substituting in eqn. (3.4.73a) and recalling eqn. (3.4.72b) and the fact that

$$\ln x = \ln \left[(-q_r) \frac{-x}{q_r} \right] = \ln (q_r e^{i\pi}) + \ln \frac{-x}{q_r}$$

we obtain

$$\frac{y}{q_r} = -\frac{1}{\pi} \ln (e^{\pi J} + 1) + \frac{Jx}{q_r} - \frac{1}{\pi} \frac{x}{q_r} \ln \frac{-x}{q_r} - \frac{1}{\pi} \left(1 - \frac{x}{q_r} \right) \ln \left(1 - \frac{x}{q_r} \right) - \infty < x \leq 0. \quad (3.4.73b)$$

Noting that for the right-hand branch of the depression curve M_2M_3 , $v_{zh} = 0$, $\psi_r = q''_r$, $w_{zh} = u_{zh} = x + q''_r$, $0 \leq x$, we obtain, on substituting in eqn. (3.4.73a) and making some manipulations

$$\frac{y}{q_r} = -\frac{1}{\pi} \ln (1 + e^{-\pi J}) + \frac{Jx}{q_r} + \frac{1}{\pi} \frac{x}{q_r} \ln \frac{x}{q_r} - \frac{1}{\pi} \left(1 + \frac{x}{q_r} \right) \ln \left(1 + \frac{x}{q_r} \right) \quad (3.4.73c)$$

By differentiating eqn. (3.4.73c) with respect to x and setting $dy/dx = 0$ we find the position of the point M_0 which defines the range from which the water may flow to the drainage cut:

$$\frac{x(M_0)}{q_r} = \frac{1}{e^{\pi J} - 1} \quad (3.4.74a)$$

The height of the surface of seepage on the sides of the drainage cut, i.e. the height of the point from which the depression curve emerges, is determined by substituting $x = 0$ in eqns. (3.4.73b) and (3.4.73c).

For the segment M_4M_1 :

$$\frac{h'}{q_r} = \frac{1}{\pi} \ln (e^{\pi J} + 1) \quad (3.4.74b)$$

for the segment M_1M_2 :

$$\frac{h''}{q_r} = \frac{1}{\pi} \ln (1 + e^{-\pi J}) \quad (3.4.74c)$$

3.4.6. Flow of water into a drainage ditch

As noted in Section 3.4.2, the second variant of the velocity hodograph method can also be used in the semi-reverse procedure. We shall now show how to apply this method to a practical case.

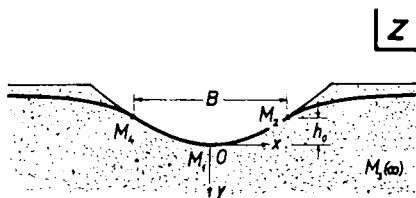


Fig. 3.63. Flow of water into a drainage ditch.

We are to determine the flow into a drainage ditch of parabolic cross-section, assuming a very small (theoretically infinitesimal) height of water in the ditch. The filtration region is shown schematically in Fig. 3.63. Since the considered case of seepage is symmetrical about the y axis, it is enough to obtain a solution for a half of the region of seepage (e.g. for $x \geq 0$).

As to the boundary of the region of Zhukovsky's function (w_{zh}): we have (just as in the case discussed in Section 3.4.5) that $v_{zh} = 0$ for the canal cross-section — the segment of free seepage — as well as for the depression curve. The reference plane for the determination of h and φ_r is again placed at the level of the free surface, i.e. at the level of the point M_1 in our case. Since the flow is symmetrical about the y axis, the axis must be a streamline. We shall choose for it the value of the stream function $\psi_r = 0$; the stream function $\psi_r = +q_r/2$ on the right-hand branch, and $\psi_r = -q_r/2$ on the left-hand branch of the depression curve. On the strength of this, we have: for the point $M_1 - u_{zh}(M_1) = 0$, for the point $M_2 - u_{zh}(M_2) = (B + q_r)/2$, for the point $M_4 - u_{zh}(M_4) = -(B + q_r)/2$, for the point $M_3 - u_{zh}(M_3) = \infty$. The region of Zhukovsky's function is the half-plane shown in Fig. 3.64.

Next, we determine the shape of the region of the velocity hodograph (v_r) (Fig. 3.65). The only segment that we can determine exactly, is the portion of the boundary of the (v_r) region which corresponds to the depression curve, i.e. the segment of the circle centred at the point $v_r = i/2$, having radius $r = 1/2$. We know that the direction of the velocity on the depression curve, defined at a selected point by the tangent to the depression curve, changes continuously toward the ditch from horizontal to a certain maximum inclination. The velocity at the point M_3 is zero and increases in the direction of the ditch.

In the velocity hodograph the two branches of the depression curve which are symmetrical, are represented by the segments M_3M_2 (the right-hand branch) and M_3M_4 (the left-hand branch) which are also symmetrical.

We do not know how to determine exactly the shape of the segment of the boundary of the (v_r) hodograph corresponding to the ditch cross-section $M_4M_1M_2$. Therefore, we choose it in a way which makes a straightforward matter of the solution

without being at odds with the conditions specific to the given seepage problem. Thus, for example, we definitely know that the velocity vector at the point M_1 is vertical, $v_{ry} < 0$. Therefore, in the velocity hodograph the point M_1 must lie on the negative part of the Ov_r axis. Further, the points M_2 and M_4 which lie on the segment $M_2M_1M_4$, must also lie on the circle corresponding to the depression curve. In the

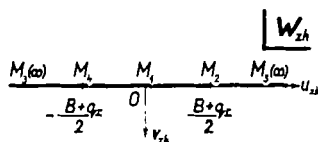


Fig. 3.64.

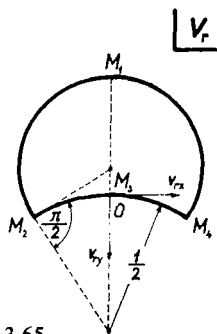


Fig. 3.65.

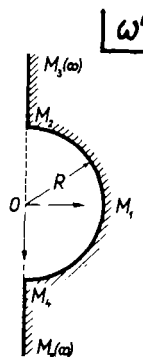


Fig. 3.66.

(v_r) region we shall choose the shape of the segment $M_2M_1M_4$ to be a segment of a circle centred on the Ov_r axis and intersecting the circle corresponding to the depression curve at right angles. This assumption introduced in the solution may but need not be close to reality. Only after we have completed all the operations and compared the shape of the ditch cross-section established by the reverse procedure with the specified shape, will we be able to decide whether the solution is acceptable or whether another shape of the segment $M_2M_1M_4$ must be chosen in the (v_r) hodograph.

In the course of the subsequent solution we shall find the ($v_{rx} - iv_{ry}$) region, and using the bilinear mapping function according to (3.4.15), establish the relation

$$\omega' = \frac{i}{v_{rx} - iv_{ry}} + 1 = 1 + i \frac{dz}{dw_r}$$

The (ω') region is shown in Fig. 3.66. Recalling that a mapping effected by a bilinear function will represent circles as circles — a straight line being simply a special case of a circle — we can readily see that the shape of the boundary is correct.

The dependence $w_{zh} = f(\omega')$ is determined by successive conformal mappings

$$\begin{aligned} w_1 &= \frac{i\omega'}{R} \\ w_2 &= \frac{1}{2} \left(w_1 + \frac{1}{w_1} \right) = \frac{1}{2} \left(\frac{i\omega'}{R} + \frac{R}{i\omega'} \right) \\ w_{zh} &= \frac{B + q_r}{2} w_2 = \frac{B + q_r}{2} \cdot \frac{1}{2} \left(\frac{i\omega'}{R} + \frac{R}{i\omega'} \right) \end{aligned}$$

which, after some manipulations, leads to

$$\omega' = -iR \left\{ \frac{2w_{zh}}{B + q_r} + \sqrt{\left[\left(\frac{2w_{zh}}{B + q_r} \right)^2 - 1 \right]} \right\} \quad (3.4.75)$$

The value of ω' from eqn. (3.4.75) is substituted in eqn. (3.4.16). Following integration and noting that for the point M_1 , $w_{zh}(M_1) = 0$, $z(M_1) = 0$, we obtain the resulting relation defining the flow in the case considered:

$$z = w_{zh} - \frac{1}{(B + q_r)R} \left\{ iw_{zh}^2 + w_{zh} \sqrt{\left[\left(\frac{B + q_r}{2} \right)^2 - w_{zh}^2} \right] + \left(\frac{B + q_r}{2} \right)^2 \arcsin \frac{2w_{zh}}{B + q_r} \right\} \quad (3.4.76a)$$

The unknown constant R is found by substituting the known values of another point, say the point M_2 : $z(M_2) = B/2 - ih_0$, $w_{zh} = (B + q_r)/2$. Separation of the equation into the real and imaginary parts gives

$$\frac{1}{(B + q_r)R} = \frac{q_r}{\pi} \left(\frac{2}{B + q_r} \right)^2 \quad (3.4.76b)$$

At the same time we can determine the height above the ditch bottom of the point M_2 towards which the depression curve aims

$$h_0 = \frac{q_r}{\pi} \quad (3.4.76c)$$

Hence the final form of eqn. (3.4.76a) is

$$z = w_{zh} - \frac{q_r}{\pi} \left\{ i \left(\frac{2w_{zh}}{B + q_r} \right)^2 + \frac{2w_{zh}}{B + q_r} \sqrt{\left[1 - \left(\frac{2w_{zh}}{B + q_r} \right)^2 \right]} + \arcsin \frac{2w_{zh}}{B + q_r} \right\} \quad (3.4.77a)$$

After substitution of the respective values of w_{zh} , eqn. (3.4.77a) will yield the coordinates of the various points of the ditch cross-section and of the depression curve, corresponding to the chosen shape of the velocity hodograph.

For the ditch cross-section $M_4M_1M_2$: $w_{zh} = x + \psi_r$, where $-(B + q_r)/2 \leq w_{zh} \leq (B + q_r)/2$. On substituting in eqn. (3.4.77a) and separating the equation into the real and imaginary parts we have

$$x = x + \psi_r - \frac{q_r}{\pi} \left\{ \frac{2(x + \psi_r)}{B + q_r} \sqrt{\left[1 - \left(\frac{2(x + \psi_r)}{B + q_r} \right)^2 \right]} + \arcsin \frac{2(x + \psi_r)}{B + q_r} \right\}$$

$$y = - \frac{q_r}{\pi} \left[\frac{2(x + \psi_r)}{B + q_r} \right]^2 \quad (3.4.77b)$$

Eliminating the expression $2(x + \psi_r)/(B + q_r)$ and paying heed to eqn. (3.4.76c) we obtain for the right-hand half of the cross-section

$$x = \frac{B}{2} \sqrt{\left(-\frac{y}{h_0}\right)} - q_r \left\{ \sqrt{\left(-\frac{y}{h_0}\right)} \left[-\frac{1}{2} + \frac{1}{\pi} \sqrt{\left(1 + \frac{y}{h_0}\right)} \right] + \frac{1}{\pi} \arcsin \sqrt{\left(-\frac{y}{h_0}\right)} \right\} \\ -h_0 \leq y \leq 0 \quad (3.4.77c)$$

As a detailed analysis suggests, on the segment $M_4M_1M_2$ the value of the expression in the square brackets of eqn. (3.4.77c) is very small for $0 \geq y \geq -h_0$, never in fact exceeding 0.06. Hence the whole term in the square brackets of eqn. (3.4.77c) is negligible in comparison with the first term, and the equation of the right-hand half of the ditch cross-section is approximately

$$x \doteq \frac{B}{2} \sqrt{\left(-\frac{y}{h_0}\right)} \quad (3.4.77d)$$

hence the cross-section determined by the reverse procedure is roughly parabolic.

The values to be substituted in eqn. (3.4.77a) for the right-hand branch of the depression curve are

$$w_{zh} = x + \frac{q_r}{2}, \quad \frac{B}{2} \leq x < \infty.$$

The equation of the depression curve then turns out to be

$$y = -h_0 - \frac{q_r}{\pi} \left[\sqrt{\left[\left(\frac{2x + q_r}{B + q_r} \right)^2 - 1 \right]} \left\{ \sqrt{\left[\left(\frac{2x + q_r}{B + q_r} \right)^2 - 1 \right]} - \frac{2x + q_r}{B + q_r} \right\} + \right. \\ \left. + \operatorname{argcosh} \frac{2x + q_r}{B + q_r} \right] \quad (3.4.77e) \\ \frac{B}{2} \leq x \leq \infty$$

The unknown value appearing in the resulting equations is the quantity of seepage $-q_r$ into the drainage ditch. To determine it, we have to know the co-ordinates of a point lying on the depression curve. After substituting these coordinates in eqn. (3.4.77e) we find q_r either by successive approximations or graphically; $q = kq_r$.

3.4.7. Seepage through the central core of an earth dam

Our next example will show how to use the third variant of the velocity hodograph method. As we mentioned in Section 3.4.2, this variant is best suited for problems in which the filtration flow has no free surface and all the remaining segments of the boundary of the region of seepage (z) are linear.

The case of seepage which we are going to solve is shown schematically in Fig. 3.67. The central core has a triangular cross-section, the water surface upstream of the dam just reaches to the vertex of the core. Compared with the coefficient of permeability of the core, the coefficient of permeability of the dam is very large so that the dam material offers virtually no resistance to the flow. The subsoil of the dam is impervious and the height of water downstream of the dam is infinitely small.

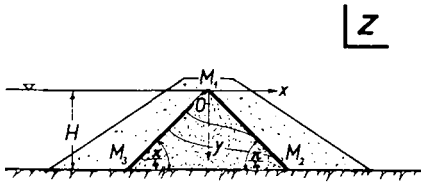


Fig. 3.67. Seepage through an earth dam core.

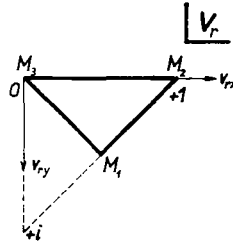


Fig. 3.68.

We first construct the region of the hodograph. To the impervious segment M_3M_2 corresponds a linear segment lying on the Ov_{rx} axis. To the permeable segment M_3M_1 making an angle of $\pi/4$ with the horizontal, corresponds a linear segment lying on the straight line which passes through the origin of the coordinates v_{rx}, v_{ry} and makes an angle of $\pi/4$ in the same sense with the Ov_{ry} axis. To the segment of free seepage, M_1M_2 , which also makes an angle of $\pi/4$ with the horizontal axis corresponds a linear segment lying on the straight line which passes through the point $v_r = i$ and makes an angle of $\pi/4$ with the Ov_{ry} axis. It is clear that the shape of the (v_r) region of the velocity hodograph is as shown in Fig. 3.68.

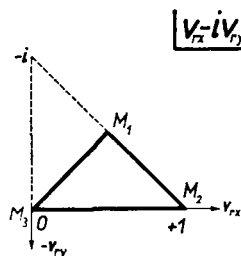


Fig. 3.69.

In the next step we establish the $(v_{rx} - iv_{ry})$ region corresponding to the (v_r) region by revolving the latter about the Ov_{rx} axis. The $(v_{rx} - iv_{ry})$ region is shown in Fig. 3.69.

The application of the third variant consists of finding the relation $(v_{rx} - iv_{ry}) = f(z)$ directly according to eqn. (3.4.17). Since in our case the $(v_{rx} - iv_{ry})$ region and the region of seepage (z) are of similar shape, the conformal mapping will be very simple:

$$(v_{rx} - iv_{ry}) = f(z) = \frac{1}{2} \left(\frac{z}{H} + 1 - i \right) \quad (3.4.78)$$

The expression from eqn. (3.4.78) is substituted in eqn. (3.4.18a). Recalling that for the point M_1 , $z(M_1) = 0$ and that the value of the reduced complex potential was chosen to be $w_r(M_1) = 0$, integrating the equation gives

$$w_r = \frac{1-i}{2} z + \frac{z^2}{4H} \quad (3.4.79a)$$

Separation of the equation into the real and imaginary parts yields the equations of the equipotential lines and the streamlines

$$\left. \begin{aligned} \varphi_r &= \frac{1}{2}(x+y) + \frac{x^2-y^2}{4H} = \text{const.} \\ \psi_r &= \frac{1}{2}(y-x) + \frac{xy}{2H} = \text{const.} \end{aligned} \right\} \quad (3.4.79b)$$

The total reduced quantity of seepage, q_r , through the dam is determined from eqn. (3.4.79a) by substitution of the values for the point M_3 for which $z(M_3) = H(-1+i)$, $w_r(M_3) = 0 + iq_r$:

$$q_r = \frac{H}{2}, \quad q = \frac{kH}{2} \quad (3.4.80)$$

What is interesting about this result obtained by a theoretically exact method, is that it is in complete agreement with the result arrived at by the routine method of calculation which is only approximate.

The seepage through the core considered in our case is usually calculated as follows:

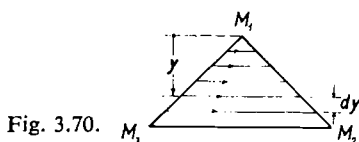


Fig. 3.70.

The core is divided into elementary horizontal strips of height dy and the flow is assumed to be horizontal. The flow in a strip lying at a depth y below the core vertex and of length $l = 2y$ takes place under the total head $\Delta h = y$. Under these conditions the reduced velocity of seepage flow in the strip is

$$v_r = J = \frac{\Delta h}{l} = \frac{y}{2y} = \frac{1}{2}$$

and the total reduced quantity of seepage through the core

$$q_r = \int_0^H v_r dy = \frac{1}{2} \int_0^H dy = \frac{H}{2}, \quad q = \frac{kH}{2}$$

This agreement seems to be a good enough reason for us to assume that a simple method is capable of yielding correct results even in cases of more intricate shapes of the central core, e.g. when the angle between the core sides and the horizontal is other than $\pi/4$, or when the core is not symmetrical, i.e. whenever the application of the exact method becomes too complicated mathematically.

3.5. Some examples of conformal mapping

The relations between the various quantities set out in the preceding sections, were obtained with the help of conformal mapping. As pointed out at the beginning of Chapter 3, the book contains no explanation of the principles of conformal mapping, for this is the province of the specialist publications. However, it may serve a useful purpose to review here some examples of conformal mappings, especially the simple basic types which, when applied successively, enable the designer to map even comparatively complicated regions onto one another, and mappings used repeatedly in the solution of problems of seepage.

The principle which must always be borne in mind when mapping one region onto another is that under the mapping, the sense of the distribution of the points on the boundaries of the regions must remain unchanged relative to both the original and the mapped regions. This principle will frequently help us to determine on which side of the mapped boundary the region itself lies.

1. $w = z + a$ translates the (z) region to the (w) region; the point $z = 0$ is represented by the point $w = a$ (Fig. 3.71);

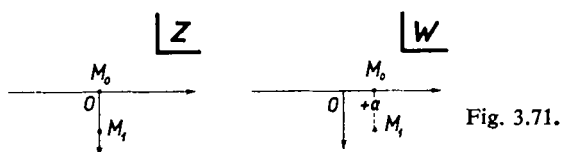


Fig. 3.71.

2. $w = az$ extends the (z) region to the (w) region; the point $z = 0$ is represented by the point $w = 0$ (Fig. 3.72);

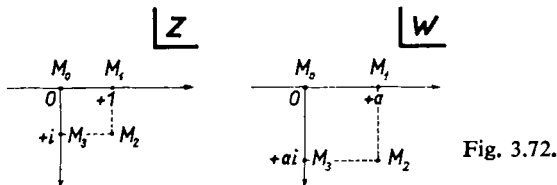


Fig. 3.72.

3. $w = ze^{i\varphi}$ rotates the (z) region through the angle φ in the positive direction (Fig. 3.73). Special cases:

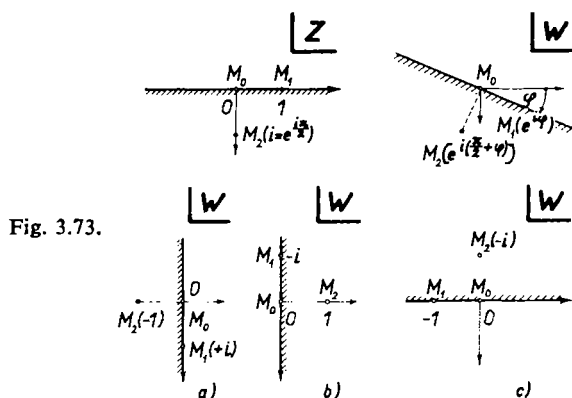


Fig. 3.73.

$$w = ze^{i\frac{\pi}{2}} = iz \quad (\text{Fig. 3.73a})$$

$$w = ze^{-i\frac{\pi}{2}} = -iz \quad (\text{Fig. 3.73b})$$

$$w = ze^{i\pi} = -z \quad (\text{Fig. 3.73c})$$

$$w = ze^{-i\pi} = -z$$

$$4. w = \frac{1}{z}$$

a reciprocal function maps the points interior to the unit circle into points exterior to the unit circle, and vice versa (Fig. 3.74). If $z = re^{i\vartheta}$ we have

$$w = \frac{1}{r} e^{-i\vartheta} = Re^{i\Theta}$$

$$R = \frac{1}{r}, \quad \Theta = -\vartheta$$

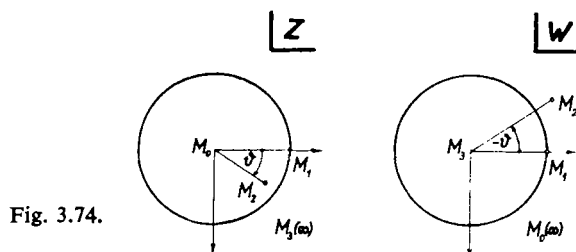


Fig. 3.74.

The point $z = 0$ is mapped to $w = \infty$, the point $z = \infty$ to $w = 0$.

$$5. w = \frac{az + b}{cz + d}$$

a bilinear function maps the region interior to a circle or a region exterior to a circle or a half-plane, to a region

interior to a circle, or to a region exterior to a circle or to a half-plane. Three points on the boundary of the (z) region can be mapped to three points specified in advance on the boundary of the (w) region by choosing one of the constants a, b, c, d and determining the remaining ones by substituting successively the coordinates of three pairs of corresponding points.

A special case:

$$\frac{w - w_1}{w - w_2} \frac{w_3 - w_2}{w_3 - w_1} = \frac{z - z_1}{z - z_2} \frac{z_3 - z_2}{z_3 - z_1}$$

maps the half-plane (z) onto the half-plane (w) so that the points M_1, M_2, M_3 on the boundary of the (z) half-plane go over to the points M_1, M_2, M_3 on the boundary of the (w) half-plane (Fig. 3.75).

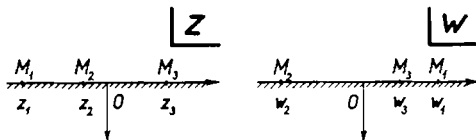


Fig. 3.75.

If one of the points M_1, M_2, M_3 in the (z) region or in the (w) region lies at infinity, the term in which the coordinate of this infinitely distant point appears, is omitted from the function which maps the (z) region onto the (w) region. If, e.g. $z(M_2) = \infty$ and $w(M_3) = \infty$ (Fig. 3.76), then

$$\frac{w - w_1}{w - w_2} = \frac{z - z_1}{z_3 - z_1}$$

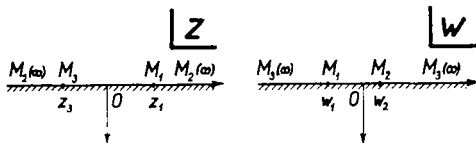


Fig. 3.76.

It is easy to determine on which side of the boundary (the boundary consists of a straight line) the mapped region will lie: the distribution of the corresponding points on the boundary must have the same sense relative to both the original and the mapped region.

6. $w = z^n$

multiplies the angle at the origin of the coordinates (Fig. 3.77) by a factor n ; the Ox axis is mapped onto the Ou axis.

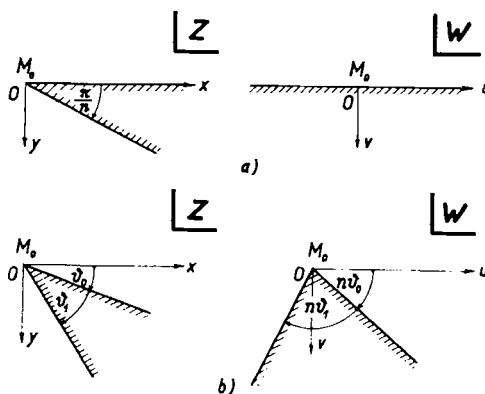


Fig. 3.77.

$$7. w^2 = \left(\frac{z}{s}\right)^2 + 1$$

maps a half-plane with a slit along the imaginary axis onto the half-plane (Fig. 3.78)

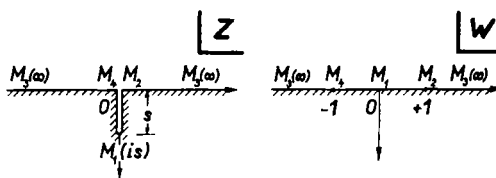


Fig. 3.78.

$$8. w = \frac{1}{2} \left(\frac{z}{R} + \frac{R}{z} \right)$$

maps a positive half-plane with a circular slit of radius R onto the positive half-plane (Fig. 3.79);

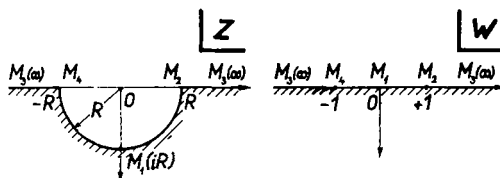


Fig. 3.79.

the points

$$M_1(z = iR), \quad M_2(z = +R), \quad M_3(z = \infty), \quad M_4(z = -R)$$

go over to the points

$$M_1(w = 0), \quad M_2(w = 1), \quad M_3(w = \infty), \quad M_4(w = -1)$$

$$9. w = \frac{1}{2} \left(z + \frac{1}{z} \right)$$

maps the positive half-plane with a circular slit of radius $R > 1$ onto the positive half-plane with an elliptical slit

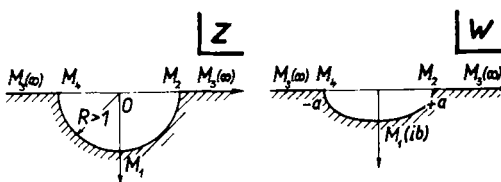


Fig. 3.80.

(Fig. 3.80), the semi-axes of the ellipse being

$$a = \frac{1}{2} \left(R + \frac{1}{R} \right) > 1, \quad b = \frac{1}{2} \left(R - \frac{1}{R} \right)$$

10. $w = \frac{1}{2} \left(\frac{z}{R} + \frac{R}{z} \right)$ maps the interior of a semi-circle of radius R in the positive half-plane onto the negative half-plane (Fig. 3.81)

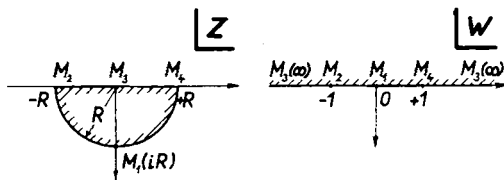


Fig. 3.81.

11. $w = \sin z$ maps a half-strip of width π onto the half-plane (Fig. 3.82)

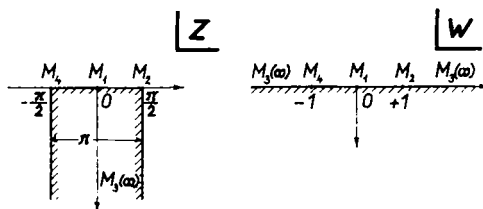


Fig. 3.82.

12. $w = \cosh z$ maps a half-strip of width π onto the half-plane (Fig. 3.83)

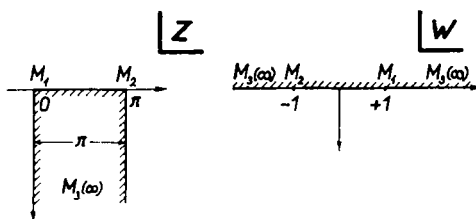


Fig. 3.83.

13. $w = e^z$ maps a strip of width π onto the half-plane (Fig. 3.84)

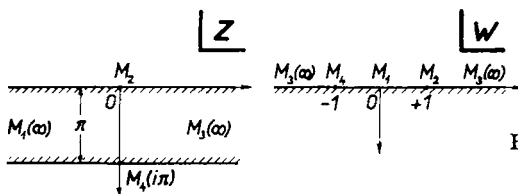


Fig. 3.84.

14. $w = \tanh z$ maps a strip of width $\pi/2$ onto the half-plane (Fig. 3.85)

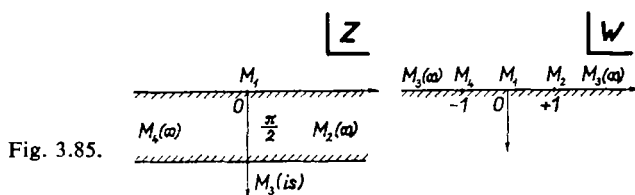


Fig. 3.85.

$$15. \cosh w = \frac{\cosh z}{\cos s}$$

maps a strip with a slit along the imaginary axis onto a strip without a slit (Fig. 3.86)

$$\alpha = \ln \cotg \left(\frac{\pi}{4} - \frac{s}{2} \right)$$

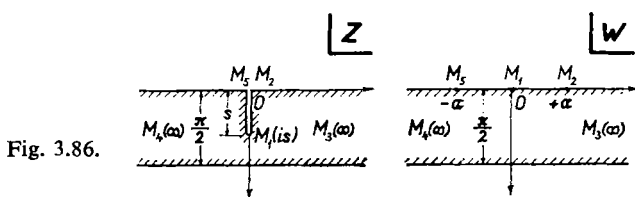


Fig. 3.86.

$$16. w = \operatorname{sn}(z; \lambda)$$

maps a rectangle onto the half-plane. (Fig. 3.87). The modulus λ of the elliptic functions is determined for the value of the ratio between the complete elliptic integral of the first kind and its complement

$$\frac{K'}{K} = 2 \frac{\omega_2}{\omega_1}$$

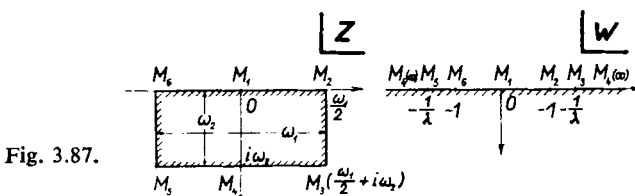


Fig. 3.87.

$$17. w = \frac{1}{z}$$

— Fig. 3.88

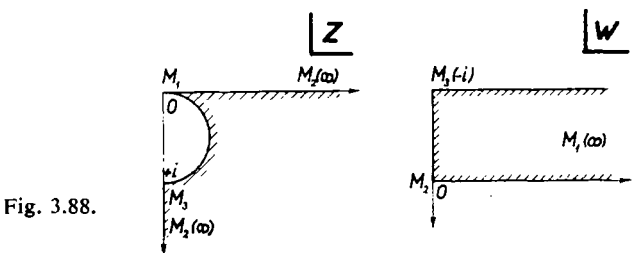


Fig. 3.88.

$$18. w = \frac{1}{z}$$

— Fig. 3.89

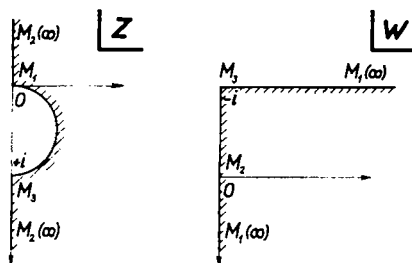


Fig. 3.89.

19. The Schwarz-Christoffel integral

$$w = C \int_{z_0}^z (z - z_1)^{\alpha_1 - 1} (z - z_2)^{\alpha_2 - 1} \dots (z - z_n)^{\alpha_n - 1} dz + C_1$$

maps the positive half-plane (z) onto an n -gon in the (w) region (Fig. 3.90). The values of $z_1, z_2, \dots, z_n$ are the coordinates of the points $M_1, M_2, \dots, M_n$ on the real axis

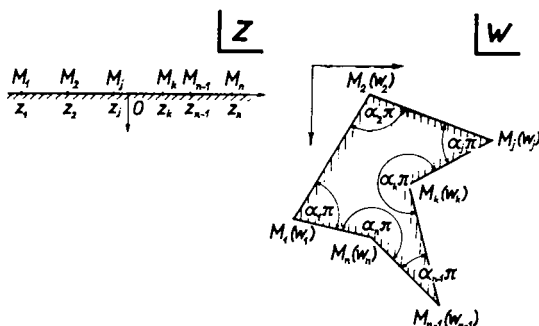


Fig. 3.90.

of the (z) half-plane which are mapped to the vertices of the n -gon with coordinates $w_1, w_2, \dots, w_n$; $\alpha_1 \pi, \alpha_2 \pi, \dots, \alpha_n \pi$ are the interior angles of the n -gon at the respective vertices;

$$\alpha_1 + \alpha_2 + \dots + \alpha_n = n - 2$$

In practical problems the vertices of the n -gon in the (w) region are usually specified while the position of the points on the real axis of the (z) half-plane is not known beforehand. The position of three of them (e.g. z_1, z_2, z_3) can be chosen at will, the position of the remaining points must be determined, simultaneously with the mapping constants C and C_1 , from the conditions of the mapping. The position of the three selected points must be chosen so as to make the sense of the distribution of the points M_i ($i = 1$ to n) on the boundaries of the (z) and (w) regions the same relative to the region.

If one of the vertices of the polygon in the (w) region corresponds to a point at infinity in the (z) region, the factor corresponding to it will not appear in the Schwarz-Christoffel integral. This fact (provided that the coordinates z_i have been chosen properly) leads to a simplification of the Schwarz-Christoffel integral which can be used to advantage in some of the problems. Thus, for example, for $z_1 = \infty$

$$w = C \int_{z_0}^z (z - z_2)^{\alpha_2 - 1} (z - z_3)^{\alpha_3 - 1} \dots (z - z_n)^{\alpha_n - 1} dz + C_1$$

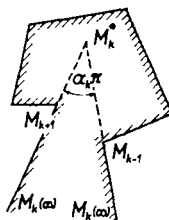


Fig. 3.91.

If one of the vertices of the n -gon, e.g. M_k , is at infinity, the Schwarz-Christoffel integral will not change its form if the angle $\alpha_k \pi$ of two straight lines intersecting at infinity is replaced by the angle at their finite point of intersection with an opposite sign (Fig. 3.91).

CHAPTER 4

APPROXIMATE METHODS FOR SOLVING TWO-DIMENSIONAL PROBLEMS OF GROUNDWATER HYDRAULICS

The methods for exact analytical solution discussed in the preceding chapter, are of great theoretical significance because they enable us to make detailed studies of groundwater problems and to gain deeper insight into the mechanism of groundwater flow. However, they are laborious and exacting in interpretation of the results. To get around these difficulties in practical applications, a number of approximate methods were developed capable of supplying solutions of adequate accuracy to a broad range of problems. The approximate methods, though naturally based on the exact analytical procedures, allow us to make various assumptions and properly chosen approximations, and with their help, to obtain simply formulated conclusions which are of considerable value to the practising engineer.

The range of problems which lend themselves to approximate solutions, is not confined to cases of two-dimensional flow. In the interest of continuity with the material of the following chapters, we shall continue to use the three-dimensional Cartesian system of coordinates and to represent two-dimensional flow in its vertical x, z plane.

4.1. Method of successive conformal mapping

We start from a purely geometrical interpretation of conformal mapping. Fig. 4.1a shows a rectangular grid in the complex plane (y); at the origin O , the grid is provided with a slit of length s (for simplicity, $s = 1.0$ in Fig. 4.1a). We effect conformal mapping of the (y) plane onto the (ζ) plane by means of the function

$$\zeta = \sqrt{(s^2 + y^2)}, \quad y = x + iz, \quad \zeta = \xi + i\eta \quad (4.1.1)$$

Separating the above into the real and imaginary parts we obtain

$$\xi^2 - \eta^2 = s^2 + x^2 - z^2$$

$$\xi\eta = xz$$

which leads to

$$\xi^2 = \frac{1}{2}[\sqrt{(s^2 + x^2 - z^2)^2 + 4x^2z^2}] + s^2 + x^2 - z^2 \quad (4.1.2)$$

$$\eta^2 = \frac{1}{2}[\sqrt{(s^2 + x^2 - z^2)^2 + 4x^2z^2}] - s^2 - x^2 + z^2 \quad (4.1.3)$$

Considering eqns. (4.1.2), (4.1.3) we find that the rectangular grid in the (y) plane maps into a curvilinear grid in the (ζ) plane — Fig. 4.1b. The slit in the (y) plane becomes a line segment on the real axis of the (ζ) plane. Examining the transformed grid we see that at greater distances from the origin the effect of the transformation diminishes, for there the curvilinear grid resembles the rectangular grid in the (y) plane. This is because

$$\left. \frac{d\zeta}{dy} \right|_{\zeta \rightarrow \infty} = 1$$

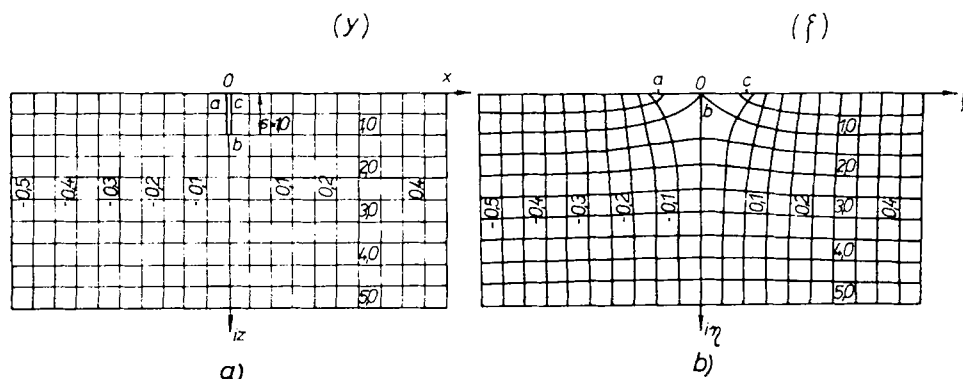


Fig. 4.1. Transformation of a half-plane with slit.

Evidently, the transformation function (4.1.1) can be used to map any configuration, such as a pattern of streamlines representing the trajectories of the motion of water particles in a flow, from the (y) plane onto the (ζ) plane. If necessary, several such transformations can be made successively. By using a proper procedure we can thus prepare the ground for the application of the analytical solution to a modified and considerably simplified region.

We shall illustrate the principle of successive conformal mapping by an example calculation of the main parameters of groundwater flow in the foundation of a weir. The hydraulic structure is provided with an upstream blanket of length d , and two narrow, relatively impervious cut-off walls. The depth of embedment of the upstream wall is s_1 , of the downstream one, s_2 (Fig. 4.2a). The cut-off walls are at a distance b apart.

In the solution we shall make use of a function of the type (4.1.1). Suppose that the origin of the (y) plane lies at points 2, 4, and that the upstream cut-off wall coincides with the slit on the imaginary axis of the (y) plane. Under transformation, the slit maps into a line segment, and the geometry of the flow region changes at the same time. However, the changes will not be too drastic, provided that the distance b between the cut-off walls is greater than the sum of their depths. Nor will the surface of the impervious subsoil undergo substantial deformation if the longer of the cut-off walls passes through less than half the thickness of the permeable layer of the weir

foundation. As these requirements are satisfied by the scheme shown in Fig. 4.2a, we may according to (4.1.1) use the transformation function

$$y' = \sqrt{(s_1^2 + y^2)} \quad (4.1.4)$$

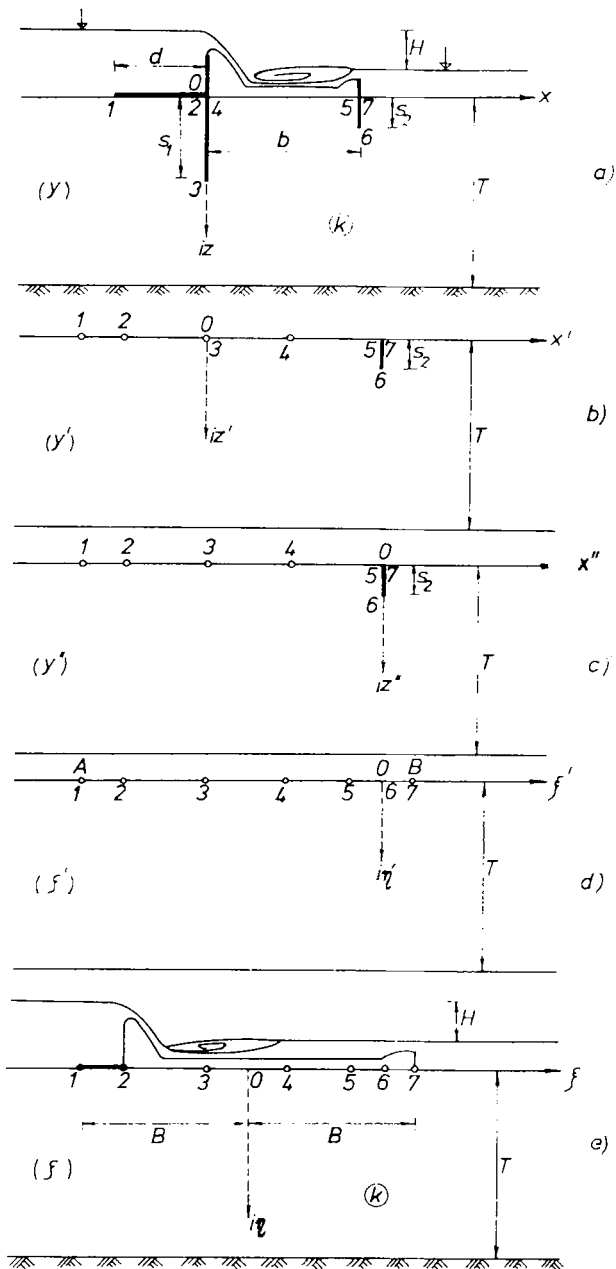


Fig. 4.2. Transformation of the region of flow in the weir foundation.

We see from Fig. 4.2b that the transformation eliminated the upstream cut-off wall. To simplify, assume that the transformation has a limited range of action. According to our suppositions, the cut-off wall on the downstream side will shift from its place, but without changing its depth. The surface of the relatively impermeous subsoil remains in its original position.

In the next step, we move the origin of the coordinate system to points 5, 7, i.e. to the fixing of the downstream wall (Fig. 4.2c), and transform the (y'') plane so defined into the (ζ') plane with the help of the function

$$\zeta' = \sqrt{[s_2^2 + (y'')^2]} \quad (4.1.5)$$

The scheme in Fig. 4.2d shows a favourable flattening of the whole foundation which no longer has any vertical cut-off wall.

To facilitate the next step, we move the origin of the coordinate system to the centre of the flat foundation. According to Fig. 4.2e, we obtain new coordinates ξ of the points of the foundation in the (ζ) plane. The half-width of the foundation is B

$$B = \frac{\zeta'_B - \zeta'_A}{2} \quad (4.1.6)$$

hence

$$\zeta = \zeta' - \zeta'_A - B \quad (4.1.7)$$

	POINT						
	1	2	3	4	5	6	7
(y)	$-d, 0$	$0, 0$	$0, is_1$	$0, 0$	$b, 0$	b, is_2	$b, 0$
(y')	$-\sqrt{s_1^2 + d^2}, 0$	$-s_1, 0$	$0, 0$	$s_1, 0$	$\sqrt{b^2 + s_1^2}, 0$	$\sqrt{b^2 + s_1^2}, is_2$	$\sqrt{b^2 + s_1^2}, 0$
(y'')	$-\left(\sqrt{b^2 + s_1^2} + \sqrt{s_1^2 + d^2}\right), 0$	$-\left(\sqrt{b^2 + s_1^2} + s_1\right), 0$	$-\sqrt{b^2 + s_1^2}, 0$	$-\left(\sqrt{b^2 + s_1^2} - s_1\right), 0$	$0, 0$	$0, is_2$	$0, 0$
(ζ')	$-\left(\sqrt{\left(\sqrt{b^2 + s_1^2} + \sqrt{s_1^2 + d^2}\right)^2 + s_2^2}, 0\right)$	$-\left(\sqrt{\left(\sqrt{b^2 + s_1^2} + s_1\right)^2 + s_2^2}, 0\right)$	$-\sqrt{b^2 + s_1^2 + s_2^2}, 0$	$-\left(\sqrt{\left(\sqrt{b^2 + s_1^2} - s_1\right)^2 + s_2^2}, 0\right)$	$-s_2, 0$	$0, 0$	$s_2, 0$

Fig. 4.3. Formulae for calculating the coordinates of transformed points.

$$B = \frac{s_2 + \sqrt{\left(\sqrt{b^2 + s_1^2} + \sqrt{s_1^2 + d^2}\right)^2 + s_2^2}}{2}$$

The table shown in Fig. 4.3 summarizes the formulae for calculating the positions of the points in successive transformations. The result is a pattern of flow with no cut-off wall underneath a flat foundation on permeable subsoil of finite thickness. This problem has already been solved analytically in Section 3.2.3. Using the appropriate formulae, we can calculate the uplift pressure at any point of the flat foundation, and hence also at the points corresponding to points 1, 2, 3, 4, 5, 6, 7 of the original scheme shown in Fig. 4.2a.

Of particular significance is knowledge of the hydrodynamic component of the uplift pressure in the toe of the vertical cut-off wall on the downstream side of the weir. This value is needed in calculations of the structure stability and resistance to deformations by seepage [3].

The method of successive conformal mapping was worked out in detail by Filchakov [7] who studied seepage of water under the foundations of hydraulic structures. It can, however, also be applied to other problems.

4.2. Method of drawn flow nets

A shortcoming of the method of successive conformal mapping is that it allows no simple or rapid determination of pressures or velocities at an arbitrary point of the field. More detailed information is provided by a flow net determined experimentally [14]. In some cases, a trial-and-error graphical method will do as well.

We start from the properties of the stream function $\psi(x, z)$ and the potential function $\varphi(x, z)$ which satisfy the condition (3.1.12). There are clearly two possible ways of calculating the volume flow dq moving per unit width of the field through the element ds (Fig. 4.4a)

$$dq = v_x dz - v_z dx = \frac{\partial \psi}{\partial z} dz + \frac{d\psi}{dx} dx = d\psi \quad (4.2.1)$$

The value of dq can also be obtained with the help of the potential function. We have

$$\begin{aligned} v &= v_x \cos \alpha + v_z \cos \beta = \frac{\partial \varphi}{\partial x} \cos \alpha + \frac{\partial \varphi}{\partial z} \cos \beta = \\ &= \frac{\partial \varphi}{\partial x} \frac{dx}{ds} + \frac{\partial \varphi}{\partial z} \frac{dz}{ds} = \frac{1}{ds} \left(\frac{\partial \varphi}{\partial x} dx + \frac{\partial \varphi}{\partial z} dz \right) = \frac{d\varphi}{ds} \end{aligned} \quad (4.2.2)$$

Through the differential segment dn measured in the direction of the potential line will flow the quantity

$$dq = \frac{d\varphi}{ds} dn \quad (4.2.3)$$

Compare eqn. (4.2.1) with eqn. (4.2.3):

$$\frac{d\varphi}{d\psi} = \frac{ds}{dn} \quad (4.2.4)$$

and change from the differential to the difference values:

$$\frac{\Delta\varphi}{\Delta\psi} = \frac{\Delta s}{\Delta n} \quad (4.2.5)$$

If we choose the same ratio between the increments of the functions $\varphi(x, z)$ and $\psi(x, z)$, we obtain

$$\frac{\Delta s}{\Delta n} = 1 \quad (4.2.6)$$

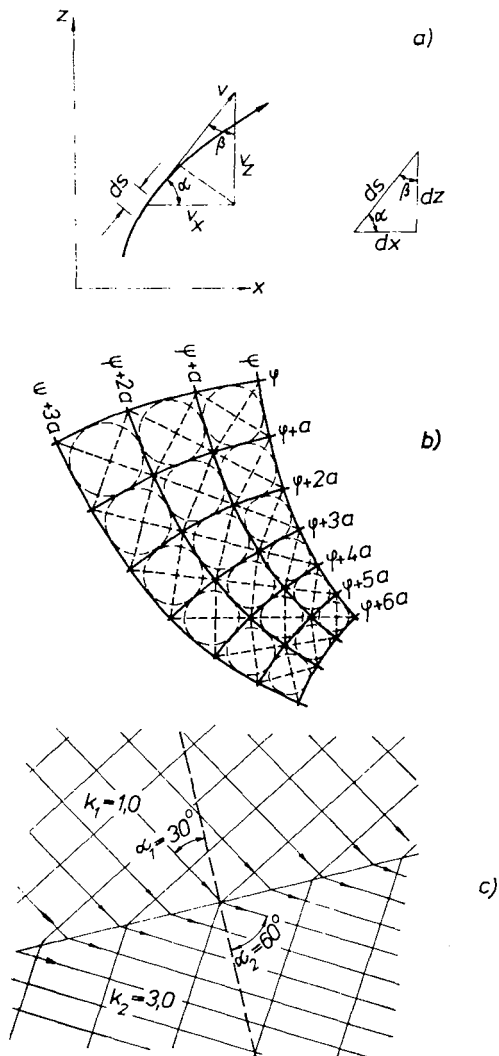


Fig. 4.4. Basic properties of flow nets.

A graphical representation of the family of lines $\varphi = \text{const.}$ and $\psi = \text{const.}$ is a net of curvilinear squares (Fig. 4.4b) with definite properties. The lines $\varphi = \text{const.}$ and $\psi = \text{const.}$ intersect each other at right angles. The diagonals of the curvilinear squares should intersect each other at right angles. A circle can be inscribed in each curvilinear square but usually, the centres of the inscribed circles will not be the same as the points of intersection of the diagonals.

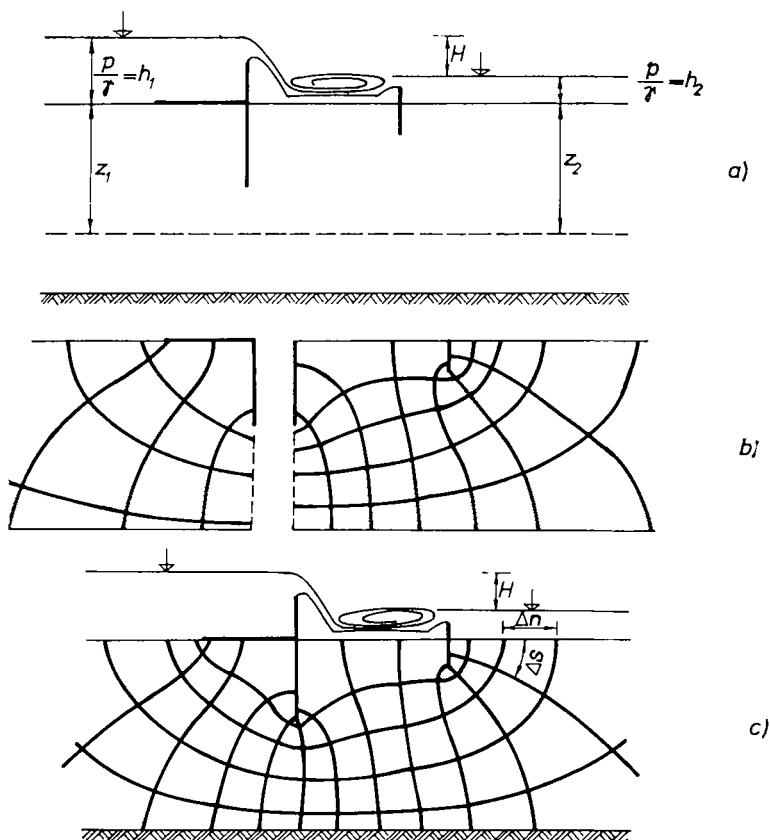


Fig. 4.5. Graphical construction of a flow net.

In a trial-and-error graphical construction of a flow net we respect the prescribed boundary conditions. The surfaces of the impervious portions of the boundaries of the flow region are streamlines to which the equipotential lines must be orthogonal. On the remaining boundary lines, the values of the potential are prescribed. As an example, consider the flow region underneath the weir (Fig. 4.5a). The function φ is generally defined there as follows:

$$\varphi = -kh = -k \left(\frac{p}{\gamma} + z \right) \quad (4.2.7)$$

p/γ is the pressure head at a given point of the flow field,

z is the elevation head of that point measured from a chosen reference plane.

According to (4.2.7), the value of φ along the bottom of the upstream reservoir and along the bottom of the downstream reservoir is constant. The images of the bottoms of the two reservoirs at the same time represent the boundary equipotential lines of the whole flow net. We have thus bounded the region geometrically and are now required to draw in it a flow net whose boundary lines are known. The task will be made easier if the region is divided into characteristic segments. The dividing line is usually drawn at a place where the probable course of the equipotential line can be estimated most accurately (Fig. 4.5b): we shall be satisfied with an accuracy that satisfies practical demands.

The result is a flow net having m flow channels and n channels between the main equipotential lines in the boundaries. In the sense of (4.2.3) through one of the channels flows

$$\Delta q = \frac{\Delta \varphi}{\Delta s} \Delta n \quad (4.2.8)$$

Therefore, for $\Delta s = \Delta n$ and m flow channels we obtain in the whole flow region

$$q = m \Delta \varphi \quad (4.2.9)$$

The difference $\Delta \varphi$ relating to two adjacent equipotential lines is determined from the formula

$$\Delta \varphi = \frac{\varphi_{\max} - \varphi_{\min}}{n} \quad (4.2.10)$$

$\varphi_{\max}$, $\varphi_{\min}$ being the maximum and the minimum value of the potential on the boundaries. In the case considered

$$\Delta \varphi = \frac{\varphi_1 - \varphi_2}{n} = - \frac{k(h_1 - h_2)}{n} = - \frac{kH}{n} \quad (4.2.11)$$

Hence

$$q = -kH \frac{m}{n} \quad (4.2.12)$$

The values of k , H are defined by the conditions of the problem, while m/n depends on the geometry of the flow region. It is sometimes expedient to generalize the result with the help of the so-called reduced volume flow q_r . In Section 3.1.7 we outlined the possibility of reduction with respect to the permeability coefficient k . In the exposition that follows we shall make use of a somewhat broader definition, viz.

$$q_r = \frac{q}{\varphi_{\max} - \varphi_{\min}} = - \frac{q}{k(h_1 - h_2)} = - \frac{q}{kH} \quad (4.2.13)$$

$$q_r = \frac{m}{n} \quad (4.2.14)$$

At a general point of the flow field

$$\varphi = \varphi_{\min} + (\varphi_{\max} - \varphi_{\min}) \frac{n'}{n} \quad (4.2.15)$$

where n' is the number of channels between the equipotential line, from the boundary (which corresponds to $\varphi_{\min}$) to the point considered

According to (4.2.15) in the case being considered we have that

$$h = h_2 + (h_1 - h_2) \frac{n'}{n} \quad (4.2.16)$$

The values of the reduced potential are defined as follows

$$\varphi_r = \frac{\varphi - \varphi_{\min}}{\varphi_{\max} - \varphi_{\min}} \quad (4.2.17)$$

and similarly

$$h_r = \frac{h - h_{\min}}{h_{\max} - h_{\min}} = \frac{h - h_2}{h_1 - h_2} \quad (4.2.18)$$

Accordingly

$$h_r = \varphi_r = \frac{n'}{n} \quad (4.2.19)$$

The above formulae make it possible to determine from a flow net not only the quantity of seepage but also the pressure at any point of the flow field.

Somewhat more complicated is the construction of flow nets in regions in which there exist two or more soils with different coefficients of permeability. On the interface between two layers the streamlines experience a break, according to the laws proposed in Section 3.1.6, and the equipotential lines become discontinuous. The informative diagram in Fig. 4.4c shows a flow net for a parallel flow which enters the interface, suffers a break and continues through the second layer at a different angle. A more general example of such a flow net is shown in Fig. 4.6. It refers to seepage from a half of an infiltration tank which passes through a relatively impervious overburden and reaches into a permeable subsoil composed of two layers. We see from the diagram that under the tank bottom the water flows nearly vertically through the upper layer, and in a predominantly horizontal direction in the lower, more permeable layer.

Complications arise when drawing a flow net for a region with a free surface. As we know from Section 1.4.5, the free surface is a streamline on the assumption that the flow is steady, i.e. ignoring the effect of evaporation or of precipitations. The distribution of the potential on this streamline follows from the requirement that the values of the potential at the points of intersection of the equipotential lines and the free surface should be proportional to the elevation heads of those points

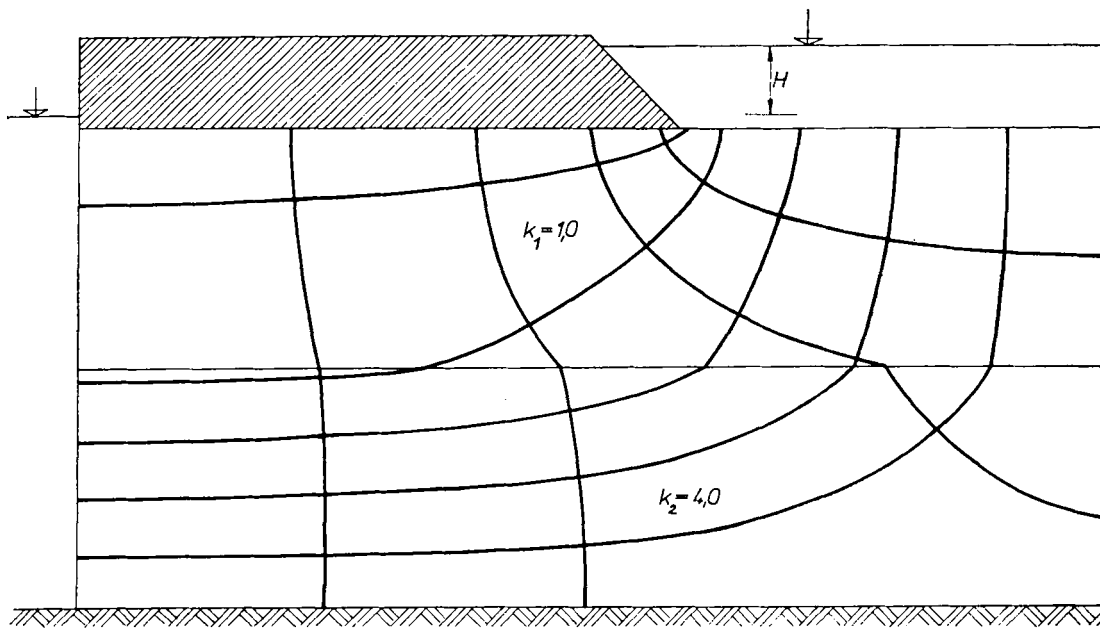


Fig. 4.6. Flow net in a non-homogeneous region.

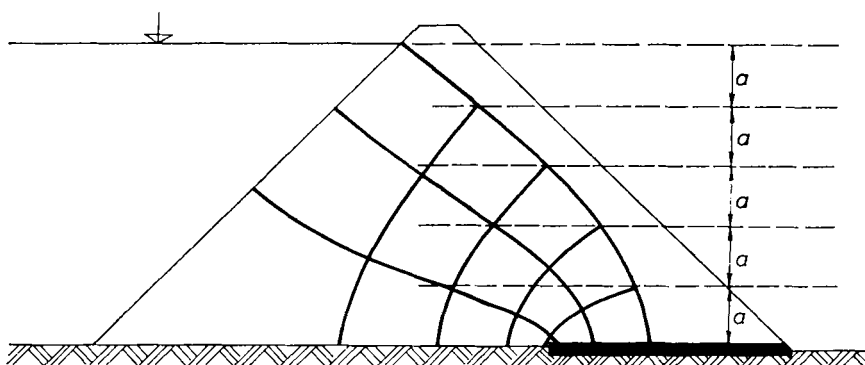


Fig. 4.7. Flow net in a region with a free surface.

(eqn. (1.4.31)). Therefore, the free surface is established by trial and error simultaneously with the whole flow net.

Fig. 4.7 shows a flow net describing the flow through an earth dam. Because the permeability of the material is anisotropic (due to the technology of the dam construction), we selected an elevated flow region (Section 1.4).

We consider a structure provided on the downstream side with a drain blanket which carries the percolated water into the hinterland. The procedure of drawing the respective flow net is as follows: The distance between the levels in the reservoir and

in the blanket is divided into a number of equal parts. The points of intersection of the equipotential lines and the free surface must lie on horizontal lines drawn at a distance a apart. The trial-and-error construction of the flow net is continued until the boundary conditions specified on the free surface as well as on the remaining boundaries of the region are satisfied (Fig. 4.7).

The quantity of percolating water can be calculated from the flow net obtained for the elevated region. However, the calculations are carried out for the substitute coefficient of permeability k

$$k = \frac{k_{\text{hor}}}{\sqrt{\Lambda}}$$

where

Λ — is the coefficient of anisotropy, and

k_{hor} — the coefficient of permeability in the horizontal direction.

4.3. Method of fragments for the solution of two-dimensional groundwater flow

Practising engineers frequently come across cases where the length of the regions in which the examined flow takes place, is several times the thickness of the permeable medium. The flow is then one-dimensional in a considerable portion of the flow field, and only becomes two-dimensional in the end segments.

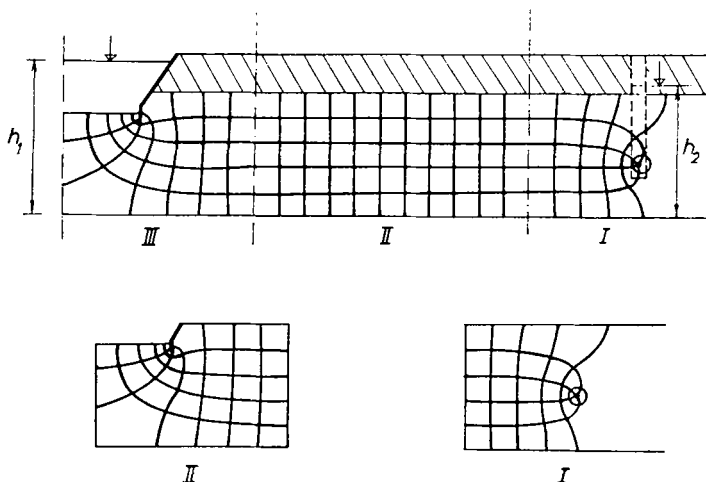


Fig. 4.8. Method of dividing a field into fragments.

To illustrate we have drawn in Fig. 4.8 a scheme of pressure seepage from an infiltration tank. The water penetrates into the ground through a sandy bottom, passes through a permeable layer and is collected by horizontal collectors. The

collecting drains are forced in at right angles to the direction of the groundwater inflow, with overhanging ends, thus forming an essentially horizontal drain. As the flow net drawn in Fig. 4.8 shows, at some distance from the inlet as well as from the outlet profile, the equipotential lines are vertical. We can therefore divide the flow region into parts and solve each of them separately.

In the interest of a generalized exposition, assume that the given flow region was divided into N parts (fragments). In each fragment we draw a flow net having m flow channels and n channels between the equipotential lines.

In each fragment the flow net processes a certain part of the total difference $\varphi_1 - \varphi_2$ of the values of the potential function. If we order the potentials successively, we obtain the sequence

$$\varphi_{I,II} = \varphi_2 + \Delta\varphi_I, \quad \varphi_{II,III} = \varphi_{I,II} + \Delta\varphi_{II}$$

For the boundary equipotential line on the edge of the last part,

$$\varphi_N = \varphi_{N-I,N} + \Delta\varphi_N = \varphi_1$$

As the water flows continuously through the whole region, we have, according to the equation of continuity,

$$\begin{aligned} \frac{m_I}{n_I} (\varphi_2 + \Delta\varphi_I - \varphi_2) &= \frac{m_{II}}{n_{II}} (\varphi_{I,II} + \Delta\varphi_{II} - \varphi_{I,II}) = \dots \\ \dots &= \frac{m_N}{n_N} (\varphi_{N-I,N} + \Delta\varphi_N - \varphi_{N-I,N}) \end{aligned}$$

Hence

$$\frac{m_I}{n_I} \Delta\varphi_I = \frac{m_{II}}{n_{II}} \Delta\varphi_{II} = \dots = \frac{m_N}{n_N} \Delta\varphi_N \quad (4.3.1)$$

Using the notation

$$\frac{m}{n} = q_i = \frac{1}{\Phi} \quad (4.3.2)$$

we can write the equalities (4.3.1) as follows:

$$\frac{\Delta\varphi_I}{\Phi_I} = \frac{\Delta\varphi_{II}}{\Phi_{II}} = \dots = \frac{\Delta\varphi_N}{\Phi_N}$$

Hence

$$\Delta\varphi_{II} = \Delta\varphi_I \frac{\Phi_{II}}{\Phi_I}, \quad \Delta\varphi_{III} = \Delta\varphi_I \frac{\Phi_{III}}{\Phi_I}, \quad \dots, \quad \Delta\varphi_N = \Delta\varphi_I \frac{\Phi_N}{\Phi_I}$$

Since it simultaneously holds that

$$\sum_{i=I}^N \Delta\varphi_i = \Delta\varphi_I \left(1 + \frac{\Phi_{II}}{\Phi_I} + \dots + \frac{\Phi_N}{\Phi_I} \right) = \varphi_1 - \varphi_2$$

we find that

$$\Delta\varphi_I = \frac{(\varphi_1 - \varphi_2) \Phi_I}{\sum_{i=I}^N \Phi_i} \quad (4.3.3)$$

Similar relations can also be written for the remaining fragments. Consequently, we can write generally for the i -th fragment that

$$\Delta\varphi_i = \frac{(\varphi_1 - \varphi_2) \Phi_i}{\sum_{i=I}^N \Phi_i} \quad (4.3.4)$$

Accordingly, on the dividing lines between the fragments

$$\varphi_{i-I,i} = \varphi_2 + (\varphi_1 - \varphi_2) \frac{\sum_{i=I}^{i-I} \Phi_i}{\sum_{i=I}^N \Phi_i} \quad (4.3.5)$$

The value of Φ_i in each fragment characterizes the resistance which the respective fragment offers to flowing water, and this is why Φ_i is called the modulus of resistance. The meaning of the modulus of resistance immediately becomes clear when considering the formula for the volume flow

$$q = \frac{\varphi_1 - \varphi_2}{\sum_{i=I}^N \Phi_i} \quad (4.3.6)$$

If we know the values of Φ_i , we can easily determine the potentials on the boundaries of the various fragments by using eqn. (4.3.5). The potential at any point inside a fragment is established from partial flow nets.

A combination in which the flow nets are solved graphically in some fragments and a numerical procedure is used in others, is also feasible.

Theoretically, our exposition is built on the method of conformal mapping which enables us to set forth the principles of correct division of a flow field into fragments. We shall illustrate the principle of the solution by way of an example. In the exposition we shall need some results presented in the following preliminary theoretical analysis.

Consider the Gauss plane (ζ) (Fig. 4.9, left), and the points A, B, C lying on its real axis. We map the (ζ) plane onto the (ζ') using the transformation function

$$\zeta' = \sqrt{(\zeta^2 - \zeta_C^2)} \quad (4.3.7)$$

The mapping of the (ζ') plane into a rectangle in the (y) plane is effected by the transformation function

$$\zeta' = \operatorname{sn} \left(\frac{\pi y}{2T}, \sigma \right) \quad (4.3.8)$$

where $\operatorname{sn}(\pi y/2T, \sigma)$ is the so-called sine amplitude (argument $\pi y/2T$, modulus σ). For further information on this function as well as on the allied functions cn (cosine amplitude), dn (delta amplitude), the reader is referred to the relevant sections of textbooks on mathematics, dealing with the Jacobian elliptic functions.

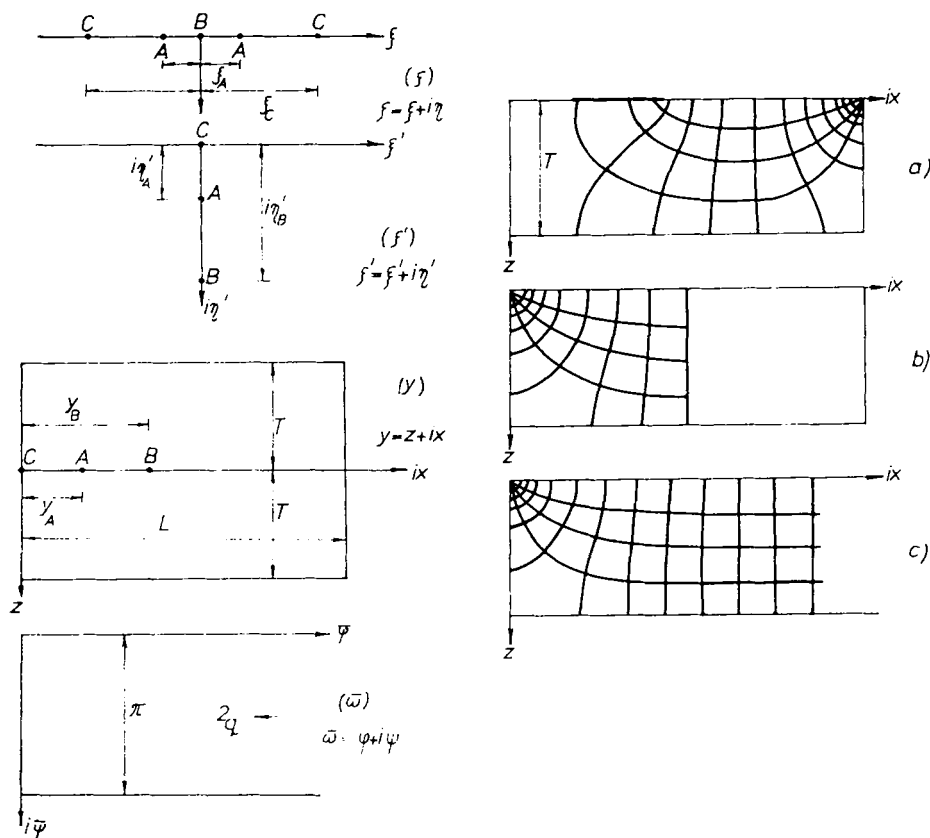


Fig. 4.9. Solution of water motion inside fragments.

The modulus σ in the transformation function (4.3.8) is determined using the ratio

$$\frac{T}{L} = \frac{K(\sigma)}{K(\sigma')} \quad (4.3.9)$$

where $K(\sigma)$, $K(\sigma')$ are the complete elliptic integrals of the first kind of modulus σ and σ' , respectively. In eqn. (4.3.9)

$$\sigma' = \sqrt{1 - \sigma^2} \quad (4.3.10)$$

Further analysis will be made easier by the substitution

$$\bar{y} = \frac{\pi y}{2T} \quad (4.3.11)$$

upon which the function (4.3.8) changes to

$$\zeta' = \operatorname{sn}(\bar{y}, \sigma) \quad (4.3.12)$$

We sum up the fundamental relationships applicable to the points A, B, C :

$$\zeta'_B = \operatorname{sn}(i\bar{x}_B, \sigma) = i \frac{\operatorname{sn}(\bar{x}_B, \sigma')}{\operatorname{cn}(\bar{x}_B, \sigma')}, \quad \eta'_B = \frac{\operatorname{sn}(\bar{x}_B, \sigma')}{\operatorname{cn}(\bar{x}_B, \sigma')} \quad (4.3.13)$$

$$\zeta'_C = 0, \quad \zeta_C = \eta'_B \quad (4.3.14)$$

$$\zeta_B = \sqrt{[(\zeta'_B)^2 + \zeta_C^2]} = \sqrt{\left[-\frac{\operatorname{sn}^2(\bar{x}_B, \sigma')}{\operatorname{cn}^2(\bar{x}_B, \sigma')} + \frac{\operatorname{sn}^2(\bar{x}_B, \sigma')}{\operatorname{cn}^2(\bar{x}_B, \sigma')}\right]} = 0 \quad (4.3.15)$$

$$\zeta'_A = \operatorname{sn}(i\bar{x}_A, \sigma) \quad (4.3.16)$$

$$\zeta_A = \sqrt{[(\zeta'_A)^2 + \zeta_C^2]} = \sqrt{\left[-\frac{\operatorname{sn}^2(\bar{x}_A, \sigma')}{\operatorname{cn}^2(\bar{x}_A, \sigma')} + \frac{\operatorname{sn}^2(\bar{x}_B, \sigma')}{\operatorname{cn}^2(\bar{x}_B, \sigma')}\right]} \quad (4.3.17)$$

In the $(\bar{\omega})$ plane, assume a parallel flow in the direction of the real axis. Denote by the symbol q the half-volume flow through a strip of width π . The flow is defined by the potential function φ and the stream function ψ . The values of φ, ψ form a co-ordinate system in the plane $\bar{\omega} = \varphi + i\psi$. The $(\bar{\omega})$ plane is mapped into (ζ) by means of the function

$$\bar{\omega} = \frac{2q}{\pi} \operatorname{argcosh} \frac{\zeta}{\zeta_A} + c \quad (4.3.18)$$

where c is a constant. Choosing $c = 0$ in the case being considered, we have

$$\bar{\omega} = \frac{2q}{\pi} \operatorname{argcosh} \sqrt{\frac{(\zeta')^2 + \zeta_C^2}{(\zeta'_A)^2 + \zeta_C^2}} \quad (4.3.19)$$

and substituting eqn. (4.3.12)

$$\bar{\omega} = \frac{2q}{\pi} \operatorname{argcosh} \sqrt{\frac{\operatorname{sn}^2(\bar{y}, \sigma) + \zeta_C^2}{(\zeta'_A)^2 + \zeta_C^2}} \quad (4.3.20)$$

According to eqn. (3.1.23) of Section 3.1.4, the complex velocity in the $(\bar{y})$ plane is obtained by differentiating eqn. (4.3.20) with respect to $\bar{y}$:

$$\bar{w} = \bar{v}_z - i\bar{v}_x = \frac{d\bar{\omega}}{d\bar{y}} = \frac{2q}{\pi} \frac{\operatorname{sn}(\bar{y}, \sigma) \operatorname{cn}(\bar{y}, \sigma) \operatorname{dn}(\bar{y}, \sigma)}{\sqrt{\{[\operatorname{sn}^2(\bar{y}, \sigma) - (\zeta'_A)^2][\operatorname{sn}^2(\bar{y}, \sigma) + \zeta_C^2]\}}} \quad (4.3.21)$$

The resulting formulae describe the situation in the field shown schematically in Fig. 4.9a. We first note the simplified case when $\bar{x}_B = \bar{x}_A = 0$ (Fig. 4.9b). In place of eqn. (4.3.21) we obtain

$$\bar{w} = \frac{2q \operatorname{cn}(\bar{y}, \sigma) \operatorname{dn}(\bar{y}, \sigma)}{\pi \operatorname{sn}(\bar{y}, \sigma)} \quad (4.3.22)$$

On the $i\bar{x}$ axis where $\bar{z} = 0$

$$\begin{aligned} \bar{w} &= -i\bar{v}_x = \frac{2q [1/\operatorname{cn}(\bar{x}, \sigma')] [\operatorname{dn}(\bar{x}, \sigma')/\operatorname{cn}(\bar{x}, \sigma')]}{\pi i[\operatorname{sn}(\bar{x}, \sigma')/\operatorname{cn}(\bar{x}, \sigma')]} = \\ &= -i \frac{2q}{\pi} \frac{\operatorname{dn}(\bar{x}, \sigma')}{\operatorname{sn}(\bar{x}, \sigma') \operatorname{cn}(\bar{x}, \sigma')} \end{aligned} \quad (4.3.23)$$

$$\bar{v}_x = \frac{2q}{\pi} \frac{\operatorname{dn}(\bar{x}, \sigma')}{\operatorname{sn}(\bar{x}, \sigma') \operatorname{cn}(\bar{x}, \sigma')} \quad (4.3.24)$$

Fig. 4.9c shows the flow net for the case where L is very large compared with T , and where eqn. (4.3.9) yields $\sigma = 0$, $\sigma' = 1.0$. The complex velocity in the $(\bar{y})$ plane will be

$$\bar{w} = \frac{2q \cos \bar{y}}{\pi \sin \bar{y}} = \frac{2q}{\pi \operatorname{tg} \bar{y}} \quad (4.3.25)$$

and on the $i\bar{x}$ axis where $\bar{z} = 0$

$$\bar{w} = \frac{2q \cosh \bar{x}}{\pi i \sinh \bar{x}} = -i \frac{2q}{\pi \operatorname{tgh} \bar{x}} \quad (4.3.26)$$

$$\bar{v}_x = \frac{2q}{\pi \operatorname{tgh} \bar{x}} \quad (4.3.27)$$

Comparing the three solutions outlined above, we can draw the following conclusions:

a) The relationships set forth above, describe a flow between two boundaries with prescribed values of the potential function. If the boundaries are sufficiently remote from one another, a parallel flow determined by nearly vertical potential lines takes place in the intervening part of the field.

b) The potential lines may be assumed to be vertical in the part between the two boundaries, if the distance between them is greater than (2 to 3) T .

c) If the boundaries are a very great distance apart, we may expect the potential lines to be vertical and the parallel flow to take place at a distance of (1 to 1.5) T from the origin.

Knowledge of the basic trends of development of the equipotential lines will enable us to divide the flow field into fragments. Each fragment is then solved separately.

The above facts will be of assistance in the solution of the following practical example: consider a pressure flow in the field between an infiltration object and a horizontal intercepting drain (Fig. 4.10). A permeable layer is overlain by an overburden of poor permeability. We divide the flow field into three fragments. We know theoretical solution of the situation in the first fragment, for we can make use of the results obtained previously for the lower part of the (y) plane (Fig. 4.9, left). The

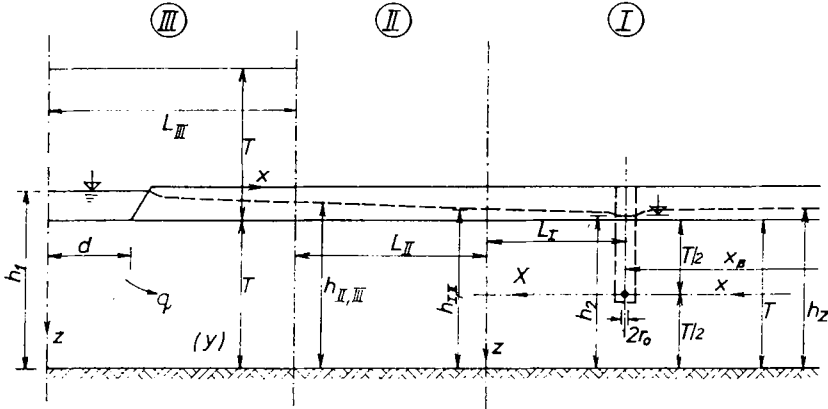


Fig. 4.10. Application of the method of fragments.

quantity flowing through fragment *III* per unit time is q . As the dividing line between the fragments is at a sufficient distance from the inlet profile, we set $\sigma = 0$, $\sigma' = 1.0$ in the theoretical solution. Moreover, taking into account the symmetry of the infiltration tank bottom as well as the symmetry of the flow, $y_A = 0$. The abscissa $y_B = ix_B$ is replaced by the simpler symbol id . According to eqn. (4.3.16), in the geometrically reduced ($\bar{y}$) region

$$\zeta'_A = 0$$

By eqns. (4.3.13) and (4.3.14)

$$\zeta_c = \frac{\operatorname{sn}(\bar{d}, \sigma')}{\operatorname{cn}(\bar{d}, \sigma')} = \frac{\operatorname{tgh} \bar{d}}{1/\cosh \bar{d}} = \sinh \bar{d}$$

In eqn. (4.3.21) we have

$$\operatorname{sn}(\bar{y}, \sigma) = \sin \bar{y}, \quad \operatorname{cn}(\bar{y}, \sigma) = \cos \bar{y}, \quad \operatorname{dn}(\bar{y}, \sigma) = 1$$

Accordingly

$$\bar{w} = \frac{2q}{\pi} \frac{\sin \bar{y} \cos \bar{y}}{\sqrt{[\sin^2 \bar{y} (\sin^2 \bar{y} + \sinh^2 \bar{d})]}} = \frac{2q}{\pi} \frac{\cos \bar{y}}{\sqrt{(\sin^2 \bar{y} + \sinh^2 \bar{d})}}$$

Noting eqn. (4.3.11), in the actual flow region we have

$$\begin{aligned}\bar{w} &= \frac{2q}{\pi} \frac{\cos(\pi y/2T)}{\sqrt{[\sin^2(\pi y/2T) + \sinh^2(\pi d/2T)]}} \cdot \frac{\pi}{2T} = \\ &= \frac{q}{T} \frac{\cos(\pi y/2T)}{\sqrt{[\sin^2(\pi y/2T) + \sinh^2(\pi d/2T)]}}\end{aligned}\quad (4.3.28)$$

Along the ix axis where $z = 0$

$$\begin{aligned}\sin \frac{\pi ix}{2T} &= i \sinh \frac{\pi x}{2T} \\ \cos \frac{\pi ix}{2T} &= \cosh \frac{\pi x}{2T}\end{aligned}$$

Eqn. (4.3.28) may therefore be rearranged as follows:

$$w = -iv_x = \frac{q}{Ti} \frac{\cosh(\pi x/2T)}{\sqrt{[\sinh^2(\pi x/2T) - \sinh^2(\pi d/2T)]}}; \quad (4.3.29)$$

this leads to

$$v_x = \frac{q}{T} \frac{\cosh(\pi x/2T)}{\sqrt{[\sinh^2(\pi x/2T) - \sinh^2(\pi d/2T)]}} \quad (4.3.30)$$

The distribution of the potential function φ along the ix axis is obtained by integrating eqn. (4.3.30)

$$\varphi = \int v_x dx = \frac{q}{T} \int \frac{\cosh(\pi x/2T)}{\sqrt{[\sinh^2(\pi x/2T) - \sinh^2(\pi d/2T)]}} dx \quad (4.3.31)$$

We introduce the notation

$$\sinh \frac{\pi x}{2T} = \tau$$

$$dx = d\tau \frac{2T}{\pi \cosh(\pi x/2T)}$$

and on substitution in eqn. (4.3.31) we have

$$\varphi = \frac{2q}{\pi} \int \frac{d\tau}{\sqrt{[\tau^2 - \sinh^2(\pi d/2T)]}} = \frac{2q}{\pi} \operatorname{argcosh} \frac{\sinh(\pi x/2T)}{\sinh(\pi d/2T)} + c \quad (4.3.32)$$

where c is the constant of integration.

With reference to Fig. 4.10, consider at $x = d$, the value $\varphi = \varphi_1$, and at $x = L_{III}$, the value $\varphi_{II,III}$. On substituting in eqn. (4.3.32) we have $\varphi_1 = c$,

$$\varphi_{II,III} = \frac{2q}{\pi} \operatorname{argcosh} \left[\sinh (\pi L_{III}/2T) / \sinh (\pi d/2T) \right] + \varphi_1$$

$$\varphi_{II,III} = 2q\pi \operatorname{argcosh} \left[\sinh (\pi L_{III}/2T) / \sinh (\pi d/2T) \right] + \varphi_1$$

Accordingly

$$\Delta\varphi_{III} = -\frac{2q}{\pi} \operatorname{argcosh} \left[\sinh (\pi L_{III}/2T) / \sinh (\pi d/2T) \right] \quad (4.3.33)$$

$$q = \frac{\pi(\varphi_{II,III} - \varphi_1)}{2 \operatorname{argcosh} \left[\sinh (\pi L_{III}/2T) / \sinh (\pi d/2T) \right]} \quad (4.3.34)$$

The reduced quantity of seepage is

$$q_r = \frac{\pi}{2 \operatorname{argcosh} \left[\sinh (\pi L_{III}/2T) / \sinh (\pi d/2T) \right]} \quad (4.3.35)$$

and the modulus of resistance of fragment III is

$$\Phi_{III} = \frac{1}{q_r} = \frac{2}{\pi} \operatorname{argcosh} \frac{\sinh (\pi L_{III}/2T)}{\sinh (\pi d/2T)} \quad (4.3.36)$$

In fragment I, the horizontal drain of radius r_0 is in its most effective position, i.e. in the middle of the permeable stratum (Fig. 4.10). We shall again make use of the previous solution and modify it to suit the changed boundary conditions. We choose $y_A = y_B$ (i.e. we work with a sink). According to eqn. (4.3.16)

$$\zeta'_A = i \frac{\operatorname{sn}(\bar{x}_A, \sigma')}{\operatorname{cn}(\bar{x}_A, \sigma')} = i\eta'_B = i\zeta_C, \quad \bar{y} = \frac{\pi y}{T}$$

Because q flows through the entire range, we obtain, using eqn. (4.3.21), that

$$\bar{w} = \frac{q}{\pi} \frac{\operatorname{sn}(\bar{y}, \sigma) \operatorname{cn}(\bar{y}, \sigma) \operatorname{dn}(\bar{y}, \sigma)}{\operatorname{sn}^2(\bar{y}, \sigma) + \zeta_C^2} \quad (4.3.37)$$

Since the sink lies on the imaginary axis, we have, for $\sigma = 0$, $\sigma' = 1.0$,

$$\zeta'_B = \sin i\bar{x}_B = i \sinh \bar{x}_B$$

Accordingly,

$$\zeta_C = \sinh \bar{x}_B$$

$$\bar{w} = \frac{q}{\pi} \frac{\sin \bar{y} \cos \bar{y}}{\sin^2 \bar{y} + \sinh^2 \bar{x}_B} \quad (4.3.38)$$

In the axis passing through the horizontal drain, we have for $z = 0$,

$$\bar{w} = -i \frac{q}{\pi} \frac{\sinh \bar{x} \cosh \bar{x}}{\sinh^2 \bar{x} - \sinh^2 \bar{x}_B}$$

Taking into consideration the actual dimensions of the flow region, we obtain

$$w = -i \frac{q}{T} \frac{\sinh (\pi x/T) \cosh (\pi x/T)}{\sinh^2 (\pi x/T) - \sinh^2 (\pi x_B/T)} \quad (4.3.39)$$

Assume that both $\bar{x}_B$ and $\bar{x}$ are very large and that we have

$$x = x_B + X$$

$$\sinh \frac{\pi x}{T} = \frac{\text{Exp}[(\pi/T)(x_B + X)]}{2} = \frac{\text{Exp}(\pi x_B/T) \text{Exp}(\pi X/T)}{2}$$

$$\sinh \frac{\pi x}{T} \rightarrow \cosh \frac{\pi x}{T}$$

$$w = -i \frac{q}{T} \frac{\text{Exp}(2\pi X/T)}{\text{Exp}(2\pi X/T) - 1} \quad (4.3.40)$$

Consequently,

$$v_x = \frac{q}{T} \frac{\text{Exp}(2\pi X/T)}{\text{Exp}(2\pi X/T) - 1} \quad (4.3.41)$$

$$\varphi = \int v_x dX = \frac{q}{\pi} \int \frac{\text{Exp}(2\pi X/T) dX}{\text{Exp}(2\pi X/T) - 1} = \frac{q}{T} \frac{T}{2\pi} \ln \left(\text{Exp} \frac{2\pi X}{T} - 1 \right) + c$$

$$\varphi = \frac{q}{2\pi} \ln \left(\text{Exp} \frac{2\pi X}{T} - 1 \right) + c \quad (4.3.42)$$

where c is the constant of integration; at the point $X = r_0$ we have $\varphi = \varphi_2$

$$c = \varphi_2 - \frac{q}{2\pi} \ln [\text{Exp}(2\pi r_0/T) - 1]$$

$$\varphi - \varphi_2 = \frac{q}{2\pi} \ln \frac{\text{Exp}(2\pi X/T) - 1}{\text{Exp}(2\pi r_0/T) - 1} \quad (4.3.43)$$

Also of interest is the distribution of the potential along the interface between the overburden and the permeable stratum. In the reduced region we have there

$$\bar{y} = -\frac{\pi}{2} + i\bar{x}$$

According to eqn. (4.3.38)

$$\bar{w} = \frac{q}{\pi} \frac{-i \cosh \bar{x} \sinh \bar{x}}{\cosh^2 \bar{x} + \sinh^2 \bar{x}_B} \quad (4.3.44)$$

Following all manipulations we obtain in the actual flow region

$$v_x = \frac{q}{T} \frac{\text{Exp}(2\pi X/T)}{\text{Exp}(2\pi X/T) + 1} \quad (4.3.45)$$

at the point $X = 0$, i.e. on the axis of the drain,

$$v_x = \frac{q}{2T}$$

in the far hinterland of the drainage where X is negative, $v_x \rightarrow 0$.

Integration of eqn. (4.3.45) with respect to X yields

$$\varphi = \frac{q}{2\pi} \ln \left(\text{Exp} \frac{2\pi X}{T} + 1 \right) + c$$

at the point $X = L_I$, we have $\varphi = \varphi_{I,II}$, so that along the interface between the overburden and the permeable layer

$$\varphi - \varphi_{I,II} = \frac{q}{2\pi} \ln \frac{\text{Exp}(2\pi X/T) + 1}{\text{Exp}(2\pi L_I/T) + 1} \quad (4.3.46)$$

In the hinterland of the drainage, X is negative. For large negative X we write $\varphi = \varphi_z$,

$$\varphi_z - \varphi_{I,II} = \frac{q}{2\pi} \ln \frac{1}{\text{Exp}(2\pi L_I/T) + 1} \quad (4.3.47)$$

The volume flow into the drains is calculated from eqn. (4.3.43) after the substitution $\varphi = \varphi_{I,II}$, $X = L_I$:

$$q = \frac{2\pi(\varphi_{I,II} - \varphi_z)}{\ln \frac{\text{Exp}(2\pi L_I/T) - 1}{\text{Exp}(2\pi r_0/T) - 1}} \quad (4.3.48)$$

The reduced quantity of seepage q_r is

$$q_r = \frac{2\pi}{\ln \frac{\text{Exp}(2\pi L_I/T) - 1}{\text{Exp}(2\pi r_0/T) - 1}} \quad (4.3.49)$$

The modulus of resistance of fragment I

$$\Phi_I = \frac{1}{q_r} = \frac{1}{2\pi} \ln \frac{\text{Exp}(2\pi L_I/T) - 1}{\text{Exp}(2\pi r_0/T) - 1} \quad (4.3.50)$$

For fragment *II* where there occurs a parallel flow according to Fig. 4.10, the modulus of resistance is

$$\Phi_{II} = \frac{L_{II}}{T} \quad (4.3.51)$$

We now have available all the data necessary for an analysis of the interaction between fragments. Thus, for example, for the inter-fragment potential $\varphi_{I,II}$ we obtain from eqn. (4.3.5)

$$\varphi_{I,II} = \varphi_2 + (\varphi_1 - \varphi_2) \frac{\ln \frac{\text{Exp}(2\pi L_I/T) - 1}{\text{Exp}(2\pi r_0/T) - 1}}{\ln \frac{\text{Exp}(2\pi L_I/T) - 1}{\text{Exp}(2\pi r_0/T) - 1} + \frac{2\pi L_{II}}{T} + 4 \operatorname{argcosh} \frac{\sinh(\pi L_{III}/2T)}{\sinh(\pi d/2T)}} \quad (4.3.52)$$

Similarly, for the inter-fragment potential $\varphi_{II,III}$, eqn. (4.3.5) yields

$$\varphi_{II,III} = \varphi_2 + (\varphi_1 - \varphi_2) \frac{\ln \frac{\text{Exp}(2\pi L_I/T) - 1}{\text{Exp}(2\pi r_0/T) - 1} + \frac{2\pi L_{II}}{T}}{\ln \frac{\text{Exp}(2\pi L_I/T) - 1}{\text{Exp}(2\pi r_0/T) - 1} + \frac{2\pi L_{II}}{T} + 4 \operatorname{argcosh} \frac{\sinh(\pi L_{III}/2T)}{\sinh(\pi d/2T)}} \quad (4.3.53)$$

The advantages of the method of fragments can also be put to good use in cases involving pressure-free flows in the region being examined. In such cases we carry out several alternative solutions in the individual sections. We choose various inter-fragment potentials and construct the corresponding flow nets. Each solution yields a certain value for the modulus of resistance. We are then able to draw graphs showing the dependence of Φ on the value of the inter-fragment potential. With the help of these graphs and successive approximations we obtain values which ensure the fulfilment of the conditions of continuity of the flow.

However, in most practical cases the flow occurs at a small angle of the free surface. In problems of this sort we can obviate iteration by using the method of substitute lengths which we shall discuss in the next section.

4.4. Method of substitute lengths

The fragment shown in Fig. 4.11a contains a flow region near an infiltration reservoir. The water flows laterally into a collector system. The seepage is described by a curvilinear flow net which can be mapped conformally into a rectilinear net (Fig. 4.11b). Near the dividing line between the fragments, the curvilinear net is

almost rectilinear, and if we choose a suitable transformation, its shape will experience practically no change. The transformed flow region is characterized by the fact that its inlet profile is vertical and hence its shape is different from that of the original region. The main difference is seen to be in its length which has increased relative to the boundary of the original reservoir. The difference L_n between the increased and

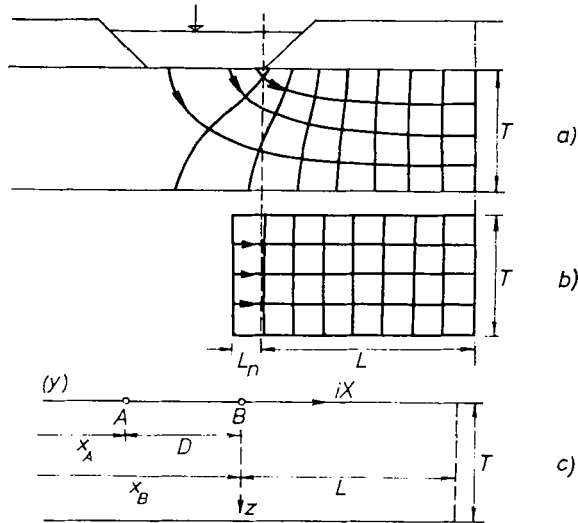


Fig. 4.11. Principle of the method of substitute lengths.

the original length is called the substitute length. The modulus of resistance of the whole fragment is readily found from the relation

$$\Phi = \frac{1}{q_r} = \frac{L + L_n}{T} \quad (4.4.1)$$

If the substitute lengths L_n of all fragments are known, the calculation of the inter-fragment potentials on the interfaces between the individual fragments becomes particularly simple. After establishing the inter-fragment potentials, we can subject each part of the flow region to a separate analysis.

For an example calculation of the substitute length, we choose the flow region shown in Fig. 4.11a. We suppose it to be lying in the (y) plane (Fig. 4.11c), and apply the solution obtained in the previous section for $\sigma = 0$, $\sigma' = 1.0$. In the geometrically reduced $(\bar{y})$ region we use the formula (4.3.21) which gives

$$\bar{w} = \frac{2q}{\pi} \frac{\sin \bar{y} \cos \bar{y}}{\sqrt{\{[\sin^2 \bar{y} - (\zeta'_A)^2] (\sin^2 \bar{y} + \zeta_C^2)\}}} \quad (4.4.2)$$

According to eqns. (4.3.14), (4.3.16)

$$\zeta_c = \sinh \bar{x}_B$$

$$\zeta'_A = i \sinh \bar{x}_A$$

$$\bar{w} = \frac{2q}{\pi} \frac{\sin \bar{y} \cos \bar{y}}{\sqrt{[(\sin^2 \bar{y} + \sinh^2 \bar{x}_A)(\sin^2 \bar{y} + \sinh^2 \bar{x}_B)]}}$$

Along the ix axis of the original (y) region

$$w = \frac{2q}{\pi} \frac{i \sinh (\pi x/2T) \cosh (\pi x/2T)}{\sqrt{\{[\sinh^2 (\pi x_A/2T) - \sinh^2 (\pi x/2T)] [\sinh^2 (\pi x_B/2T) - \sinh^2 (\pi x/2T)]\}}} \frac{\pi}{2T} \quad (4.4.3)$$

Assuming x to be very large

$$x = x_A + D + X$$

we have

$$x_B = x_A + D$$

For positive values of X we obtain along the ix axis

$$w = i \frac{q}{T} \frac{\text{Exp} [(\pi/T)(x_A + D + X)]}{\sqrt{\{[\text{Exp} [(\pi/T)(x_A + D + X)] - \text{Exp} (\pi x_A/T)] \{ \text{Exp} [(\pi/T)(x_A + D + X)] - \text{Exp} [(\pi/T)(x_A + D)] \} \}}} \quad (4.4.4)$$

$$w = i \frac{q}{T} \frac{\text{Exp} (\pi X/T)}{\sqrt{\{[\text{Exp} (\pi X/T) - \text{Exp} (-\pi D/T)] [\text{Exp} (\pi X/T) - 1]\}}} \quad (4.4.5)$$

For small values of D , eqn. (4.4.5) becomes formally identical with eqn. (4.3.40).

For very large D we have

$$v_x = \frac{q}{T} \frac{\text{Exp} (\pi X/T)}{\sqrt{\{[\text{Exp} (\pi X/T) - 1]\}}} = \frac{q}{T} \sqrt{\frac{\text{Exp} (\pi X/T)}{\text{Exp} (\pi X/T) - 1}}$$

The potential function is obtained by integrating eqn. (4.4.5) with respect to X :

$$\begin{aligned} \varphi &= \frac{q}{T} \int \frac{\text{Exp} (\pi X/T) dX}{\sqrt{\{[\text{Exp} (\pi X/T) - \text{Exp} (-\pi D/T)] [\text{Exp} (\pi X/T) - 1]\}}} \\ \varphi &= \frac{q}{\pi} \ln \left| 2 \text{Exp} \frac{\pi X}{T} - \left[1 + \text{Exp} \left(-\frac{\pi D}{T} \right) \right] \right| + \\ &\quad + 2 \sqrt{\left\{ \left[\text{Exp} \frac{\pi X}{T} - \text{Exp} \left(-\frac{\pi D}{T} \right) \right] \left[\text{Exp} \frac{\pi X}{T} - 1 \right] \right\}} + c \quad (4.4.6) \end{aligned}$$

where c is the constant of integration; at the point $X = 0$, we choose $\varphi = \varphi_1$

$$\varphi_1 = \frac{q}{\pi} \ln |1 - \text{Exp}(-\pi D/T)| + c$$

$$\varphi - \varphi_1 = \frac{q}{\pi} \ln \left| \frac{2 \text{Exp}(\pi X/T) - [1 + \text{Exp}(-\pi D/T)] + 2 \sqrt{\{[\text{Exp}(\pi X/T) - \text{Exp}(-\pi D/T)][\text{Exp}(\pi X/T) - 1]\}}}{1 - \text{Exp}(-\pi D/T)} \right| \quad (4.4.7)$$

at the point $X = L$, we choose $\varphi = \varphi_0$, so that

$$q = \frac{\pi(\varphi_0 - \varphi_1)}{\ln \left| \frac{2 \text{Exp}(\pi L/T) - [1 + \text{Exp}(-\pi D/T)] + 2 \sqrt{\{[\text{Exp}(\pi L/T) - \text{Exp}(-\pi D/T)][\text{Exp}(\pi L/T) - 1]\}}}{1 - \text{Exp}(-\pi D/T)} \right|} \quad (4.4.8)$$

The modulus of resistance Φ is

$$\Phi = \frac{1}{q_r} = \frac{1}{\pi} \ln \left| \frac{2 \text{Exp}(\pi L/T) - [1 + \text{Exp}(-\pi D/T)] + 2 \sqrt{\{[\text{Exp}(\pi L/T) - \text{Exp}(-\pi D/T)][\text{Exp}(\pi L/T) - 1]\}}}{1 - \text{Exp}(-\pi D/T)} \right| \quad (4.4.9)$$

The substitute length L_n is established using eqn. (4.4.1):

$$L_n = \Phi T - L = \frac{T}{\pi} \ln \left| \frac{2 \text{Exp}(\pi L/T) - [1 + \text{Exp}(-\pi D/T)] + 2 \sqrt{\{[\text{Exp}(\pi L/T) - \text{Exp}(-\pi D/T)][\text{Exp}(\pi L/T) - 1]\}}}{1 - \text{Exp}(-\pi D/T)} \right| - L \quad (4.4.10)$$

We note some extreme cases. Thus, for example, when D is very large, the substitute length is

$$L_n = \frac{T}{\pi} \ln \left| 2 \text{Exp} \frac{\pi L}{T} - 1 + 2 \sqrt{\left[\text{Exp} \frac{\pi L}{T} \left(\text{Exp} \frac{\pi L}{T} - 1 \right) \right]} \right| - L \quad (4.4.11)$$

For sufficiently large L

$$L_n = \frac{T}{\pi} \ln \left(4 \text{Exp} \frac{\pi L}{T} \right) - L = \frac{T}{\pi} \ln 4 \quad (4.4.12)$$

Another case is the opposite extreme, when $D \rightarrow 0$. $\text{Exp}(\pi L/T) \gg 1$. Then the substitute length is

$$L_n = \frac{T}{\pi} \ln \left| \frac{4 \text{Exp}(\pi L/T)}{1 - \text{Exp}(-\pi D/T)} \right| - L \quad (4.4.13)$$

For comparatively large values of L

$$\begin{aligned}
 L_n &= \frac{T}{\pi} \ln 4 + \frac{T}{\pi} \ln \operatorname{Exp} \frac{\pi L}{T} - \frac{T}{\pi} \ln \left[1 - \operatorname{Exp} \left(-\frac{\pi D}{T} \right) \right] - L = \\
 &= \frac{T}{\pi} \ln \frac{4}{1 - \operatorname{Exp} (-\pi D/T)}
 \end{aligned}
 \quad (4.4.14)$$

The application of the method of substitute lengths to practical problems will best be explained in an illustrative example. Fig. 4.12a shows the case of seepage from a receptacle of surface water into a system of drains. The flow takes place in the presence of a free surface the inclination of which is small because of the considerable width of the flow region. Using the method of successive conformal mappings, we can manipulate the geometry of the boundaries so as to achieve a flow from very flat receptacles (Fig. 4.12b). The flow net corresponding to this case will differ from the original in the immediate vicinity of the objects only. The whole flow region is divided into fragments. The substitute lengths of both end sections are determined on the condition that the inlet and outlet resistances should be approximately the

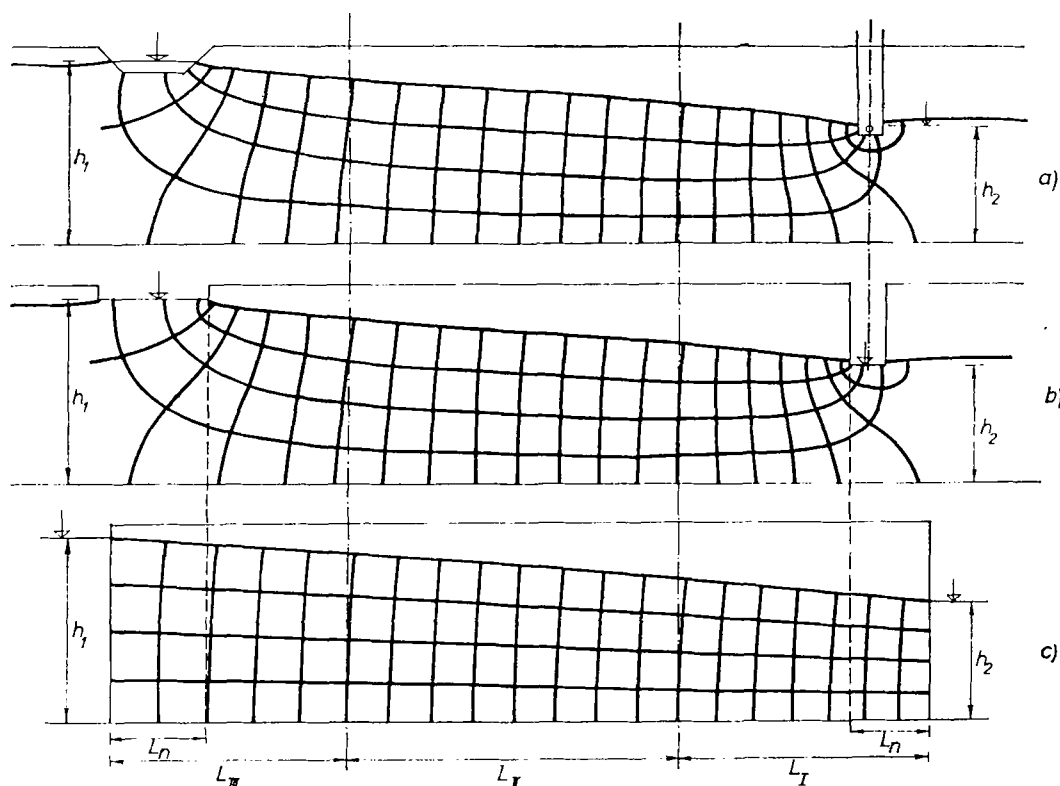


Fig. 4.12. Application of the method of substitute lengths.

same as in the case of seepage or collection occurring under pressure. The substitute lengths are added to the actual lengths of the fragments, and the problem is solved in keeping with the principles of one-dimensional flow outlined in Section 2 (Fig. 4.12c). The solution yields the required inter-fragment potentials, and having those, we can turn to an analysis of the actual conditions of flow in the individual fragments.

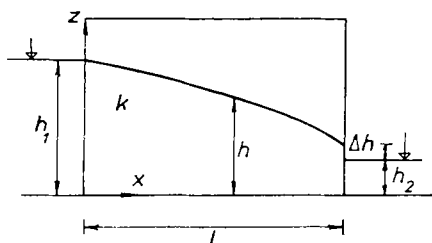


Fig. 4.13. Scheme illustrating Charny's proof.

As the graphically constructed flow nets suggest, a surface of seepage arises at the outlet into the collector object. In the simplified formulation used herein, the method of substitute lengths takes only scanty account of this effect. We must show, therefore, that the influence of a surface of seepage on the values of the inter-fragment potentials is far from being significant. A solution to the problem was obtained by Charny [5]. Essentially, we are to investigate the seepage through a rectangular massif with a surface of seepage Δh present on its downstream portion (Fig. 4.13). The quantity of water flowing per unit time through each vertical profile is given by the formula

$$q = \int_0^h v_x dz = \int_0^h \frac{\partial \varphi}{\partial x} dz \quad (4.4.15)$$

Charny introduces the following integral in the solution:

$$I = \int_0^h \varphi(x, z) dz \quad (4.4.16)$$

After differentiating the integral with respect to x we obtain

$$\frac{dI}{dx} = \int_0^h \frac{\partial \varphi}{\partial x} dz + \varphi \frac{\partial h}{\partial x} \quad (4.4.17)$$

Hence, on substituting eqn. (1.4.16) we have

$$\frac{dI}{dx} = \int_0^h \frac{\partial \varphi}{\partial x} dz - \frac{d}{dx} \left(\frac{kh^2}{2} \right) \quad (4.4.18)$$

$$q = \int_0^h \frac{\partial \varphi}{\partial x} dz = \frac{dI}{dx} + \frac{d}{dx} \left(\frac{kh^2}{2} \right) \quad (4.4.19)$$

This brings us to the differential equation

$$q \, dx = dI + d\left(\frac{kh^2}{2}\right) \quad (4.4.20)$$

which is readily solved by integration between the limits $x = 0$ and $x = L$, for the actual heights of the levels on the boundaries, $h_1, h_2 + \Delta h$:

$$qL = I|_{x=L} - I|_{x=0} + \frac{1}{2}k(h_2 + \Delta h)^2 - \frac{1}{2}kh_1^2 \quad (4.4.21)$$

In the above

$$\begin{aligned} I|_{x=L} &= -k \int_0^{h_2 + \Delta h} h \, dz = -kh_2 \int_0^{h_2} dz - k \int_{h_2}^{h_2 + \Delta h} z \, dz = \\ &= -kh_2^2 - \frac{1}{2}k(h_2 + \Delta h)^2 + \frac{1}{2}kh_2^2 = -\frac{1}{2}kh_2^2 - \frac{1}{2}k(h_2 + \Delta h)^2 \\ I|_{x=0} &= -k \int_0^{h_1} h_1 \, dz = -kh_1^2 \end{aligned}$$

On substituting in eqn. (4.4.21) we obtain

$$q = \frac{k}{2L} (h_1^2 - h_2^2); \quad (4.4.22)$$

this corresponds to eqn. (2.3.9) derived on the assumption that the flow is one-dimensional. Eqn. (2.3.9) is thus an exact solution. We explain this by the fact that the effect of the surface of seepage — largest on the boundary of the region — grows weaker in the direction against the flow, and vanishes at a certain distance. In these places apply the assumptions on which the theory of one-dimensional flow is based. Since the principle of continuity applies simultaneously, the volume flow which has passed through the region of validity of the theory of one-dimensional flow, had to pass also through the region in which the effect of the surface of seepage manifests itself markedly.

It follows from what has been said that the effect of the surface of seepage grows weaker as we move against the flow to the inside of the flow field. The differences between the levels calculated on the basis of the theories of one-dimensional and two-dimensional flow, are large in the region near the boundary. It is recommended, therefore, to place the dividing line between the fragments in locations where the effect of the surface of seepage is no longer substantial.

4.5. Superposition of velocities

Groundwater hydraulics is founded on general physical principles, which is why the laws of the mechanics of motion apply to it. In the linear region we can make use of the advantages of superposition which is of considerable importance for both theory and practice. We consider here the principle of superposition of velocities in a field defined by a continuous change of the velocity potential.

We first solve some of the basic cases, such as the symmetrical discharge from a narrow cut. Suppose that a linear source exists in the (ζ) plane (Fig. 4.14a). In view

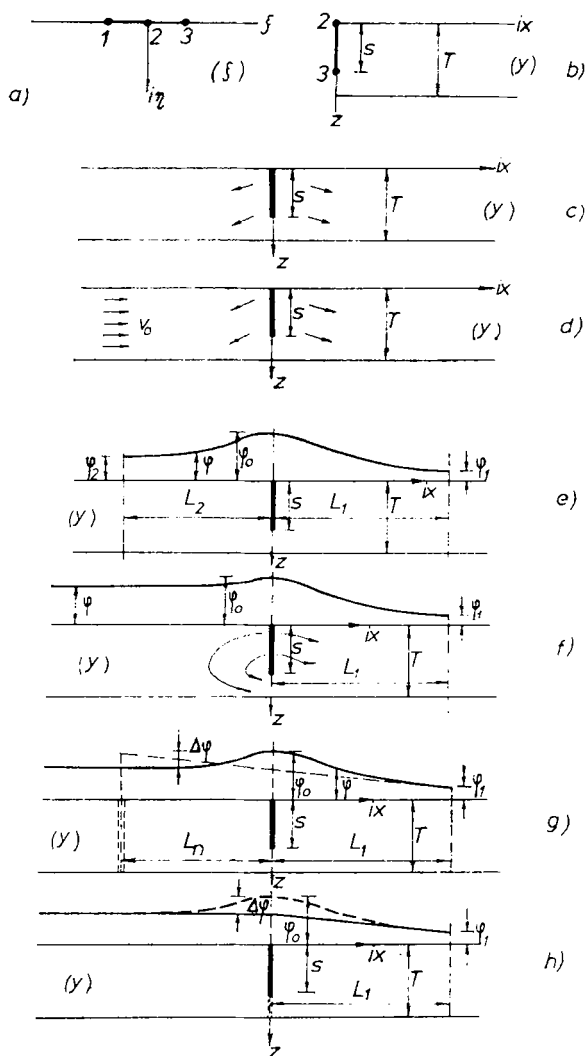


Fig. 4.14. Basic idea of the method of superposition of velocities.

of the symmetry of the discharge to both sides of the cut, we have, just as in eqn. (4.3.18), that

$$\omega = \frac{q}{\pi} \operatorname{argcosh} \frac{\zeta}{\zeta_3}$$

where ζ_3 is the coordinate of point 3 at the end of the source.

We transform the flow region shown in Fig. 4.14a into the (y) plane using the function

$$\zeta = \sin \frac{\pi y}{2T}$$

$$\omega = \frac{q}{\pi} \operatorname{argcosh} \frac{\sin (\pi y/2T)}{\sin (\pi y_3/2T)}$$

In the (y) plane, we set $y_3 = s$.

The complex velocity w at any point of the field is obtained by differentiating ω with respect to y :

$$w = \frac{q}{2T} \frac{\cos (\pi y/2T)}{\sqrt{[\sin^2 (\pi y/2T) - \sin^2 (\pi s/2T)]}}$$

We write out the functions of the complex argument $y = z + ix$:

$$w = \frac{q}{2T} \frac{\cos (\pi z/2T) \cosh (\pi x/2T) - i \sin (\pi z/2T) \sinh (\pi x/2T)}{\sqrt{\{[\sin (\pi z/2T) \cosh (\pi x/2T) + i \cos (\pi z/2T) \sinh (\pi x/2T)]^2 - \sin^2 (\pi s/2T)\}}}$$
(4.5.1)

Along the ix axis where $z = 0$

$$w = -iv_x = -i \frac{q}{2T} \frac{\cosh (\pi x/2T)}{\sqrt{[\sinh^2 (\pi x/2T) - \sin^2 (\pi s/2T)]}}$$
(4.5.2)

Along the z axis, in the interval $0 \leq z \leq s$

$$w = -iv_x = -i \frac{q}{2T} \frac{\cos (\pi z/2T)}{\sqrt{[\sin^2 (\pi s/2T) - \sin^2 (\pi z/2T)]}}$$
(4.5.3)

Along the z axis, in the interval $s \leq z \leq T$

$$w = v_z = \frac{q}{2T} \frac{\cos (\pi z/2T)}{\sqrt{[\sin^2 (\pi z/2T) - \sin^2 (\pi s/2T)]}}$$
(4.5.4)

Along the straight line $z = T = \text{const.}$

$$w = -iv_x = -i \frac{q}{2T} \frac{\sinh (\pi x/2T)}{\sqrt{[\cosh^2 (\pi x/2T) - \sin^2 (\pi s/2T)]}}$$
(4.5.5)

A close analysis of eqns. (4.5.1) and (4.5.5) reveals that the flow occurring at not too large a distance from the real axis is already parallel, and that at the end of the line segment s , there is a singular point at which the velocity is theoretically infinite. Along the line segment s , the water flows out at right angles to the real axis.

Suppose that the line segment s represents an hydraulically imperfect cut from which water flows out symmetrically to both sides (Fig. 4.14c). Because of the symmetry, the relationships derived above also apply to points in which the coordinate

ix is negative. We shall take care of this fact by treating x in the formulae as an absolute value.

We now assume that the symmetrical flow is affected by a parallel flow coming from the left. Let the velocity of the parallel flow be v_0 . According to the principle of superposition of velocities, the velocity vectors add up geometrically (vectorially) (Fig. 4.14d). A plane convenient for carrying out this operation is the Gauss plane in which a complex number can readily be represented in the trigonometrical form

$$y = r(\cos \tau + i \sin \tau)$$

where

r — the length of the radius vector (modulus, absolute value), and

τ — the angle (in radians) that the vector makes with the positive real axis (argument of the complex number).

The complex plane is an aid used to express the real flow. In the composition of flows, we are concerned with the actual motion of water and regard the mathematical description as a means of recording our ideas. This is why we choose a convention which expresses the physical nature of the process.

Along the cut the velocity vector of the basic flow is orthogonal to the wall. Since the velocity vector of the added flow is in the same direction, we have, after superposition,

$$w = \frac{q}{2T} \frac{\cos(\pi z/2T) \cosh(\pi x/2T) - i \sin(\pi z/2T) \sinh(\pi |x|/2T)}{\sqrt{[(\sin(\pi z/2T) \cosh(\pi x/2T) + i \cos(\pi z/2T) \sinh(\pi |x|/2T))^2 - \sin^2 \pi s/2T]} \pm i \frac{q_0}{T}} \quad (4.5.6)$$

The upper sign applies to the region to the right of the real axis. Suppose that

$$q_0 = v_0 T \quad (4.5.7)$$

Along the ix axis, where $z = 0$

$$w = -iv_x = -\frac{i}{T} \left(\frac{q}{2} \frac{\cosh(\pi x/2T)}{\sqrt{[\sinh^2(\pi |x|/2T) + \sin^2(\pi s/2T)]}} \pm q_0 \right) \quad (4.5.8)$$

Along the z axis, in the interval $0 \leq z \leq s$

$$w = -iv_x = -\frac{i}{T} \left(\frac{q}{2} \frac{\cos(\pi z/2T)}{\sqrt{[\sin^2(\pi s/2T) - \sin^2(\pi z/2T)]}} \pm q_0 \right) \quad (4.5.9)$$

Along the z axis, in the interval $s \leq z \leq T$

$$w = \frac{1}{T} \left(\frac{q}{2} \frac{\cos(\pi z/2T)}{\sqrt{[\sin^2(\pi z/2T) - \sin^2(\pi s/2T)]}} \pm iq_0 \right) \quad (4.5.10)$$

In the interval $s \leq z \leq T$, the velocity vector is not perpendicular to the axis of the cut.

The distribution of the potential function along the imaginary axis is obtained from eqn. (4.5.8) from which it follows that

$$v_x = \frac{1}{2T} \left(q \frac{\cosh(\pi x/2T)}{\sqrt{[\sinh^2(\pi|x|/2T) + \sin^2(\pi s/2T)]}} \pm 2q_0 \right) \quad (4.5.11)$$

$$\varphi = \frac{1}{2T} \int \left(\frac{q \cosh(\pi x/2T)}{\sqrt{[\sinh^2(\pi|x|/2T) + \sin^2(\pi s/2T)]}} \pm 2q_0 \right) dx,$$

so that on integration

$$\varphi = \frac{1}{\pi} \left(q \operatorname{argsinh} \frac{\sinh(\pi|x|/2T)}{\sin(\pi s/2T)} \pm \frac{\pi q_0 |x|}{T} \right) + c \quad (4.5.12)$$

where c is the constant of integration.

At the point $x = 0$, we choose $\varphi = \varphi_0$. Eqn. (4.5.12) then yields $\varphi_0 = c$ (Fig. 4.14e).

$$\varphi - \varphi_0 = \frac{1}{\pi} \left(q \operatorname{argsinh} \frac{\sinh(\pi|x|/2T)}{\sin(\pi s/2T)} \pm \frac{\pi q_0 |x|}{T} \right) \quad (4.5.13)$$

At the point $x = L_1$, we consider $\varphi = \varphi_1$

$$\varphi_1 - \varphi_0 = \frac{1}{\pi} \left(q \operatorname{argsinh} \frac{\sinh(\pi L_1/2T)}{\sin(\pi s/2T)} + \frac{\pi q_0 L_1}{T} \right)$$

which leads to

$$q_0 = \frac{T}{L_1} \left[(\varphi_1 - \varphi_0) - \frac{q}{\pi} \operatorname{argsinh} \frac{\sinh(\pi L_1/2T)}{\sin(\pi s/2T)} \right] \quad (4.5.14)$$

At the point $x = |L_2|$, we choose $\varphi = \varphi_2$. Eqn. (4.5.13) then gives

$$q = \frac{\pi[\varphi_2 - \varphi_0 + (|L_2|/L_1)(\varphi_1 - \varphi_0)]}{\operatorname{argsinh} \frac{\sinh(\pi|L_2|/2T)}{\sin(\pi s/2T)} + \frac{|L_2|}{L_1} \operatorname{argsinh} \frac{\sinh(\pi L_1/2T)}{\sin(\pi s/2T)}} \quad (4.5.15)$$

Note the special case shown in Fig. 4.14f when

$$q_0 = \frac{q}{2} \quad (4.5.16)$$

From eqn. (4.5.13) we have

$$\varphi - \varphi_0 = \frac{q}{\pi} \left(\operatorname{argsinh} \frac{\sinh(\pi|x|/2T)}{\sin(\pi s/2T)} \pm \frac{\pi|x|}{2T} \right) \quad (4.5.17)$$

In the region to the left of the real axis we have, for a sufficiently large $|x|$,

$$\varphi - \varphi_0 = \frac{q}{\pi} \ln \frac{1}{\sin(\pi s/2T)} \quad (4.5.18)$$

There, the value of the potential tends to a certain limiting value. This is brought about by the fact that, according to eqn. (4.5.11) the flow velocity falls off to zero with increasing $|x|$.

The volume flow proceeding from the cut is calculated from eqn. (4.5.14)

$$q = \frac{\pi(\varphi_1 - \varphi_0)}{(\pi L_1/2T) + \operatorname{argsinh} \frac{\sinh(\pi L_1/2T)}{\sin(\pi s/2T)}} \quad (4.5.19)$$

For a sufficiently large L_1

$$q = \frac{\pi(\varphi_1 - \varphi_0)}{(\pi L_1/T) + \ln [1/\sin(\pi s/2T)]} \quad (4.5.20)$$

In the extreme case being considered, all the water which has percolated into the cut, flows into the region to the right of the real axis. The auxiliary volume flow q_0 which we have introduced in the solution, produced regulation of the flow.

The substitute length L_n characteristic of the inlet resistances, can readily be calculated from eqn. (4.5.20):

$$q_r = \frac{1}{(L_1/T) + (1/\pi) \ln [1/\sin(\pi s/2T)]}$$

$$\Phi = \frac{L_1}{T} + \frac{1}{\pi} \ln \frac{1}{\sin(\pi s/2T)}$$

Hence

$$L_n = T\Phi - L_1 = \frac{T}{\pi} \ln \frac{1}{\sin(\pi s/2T)} \quad (4.5.21)$$

We shall simplify the problem by supposing that the seepage takes place from a fictitious, hydraulically perfect cut situated at a distance L_n from the original, hydraulically imperfect cut (Fig. 4.14g). This simplification will, in fact, make the problem amenable to solution by the theory of one-dimensional flow (Chapter 2).

A fact worthy of notice is that — in contrast to the exact solution of the effect of an imperfect object — we register on the boundary of the simplified scheme, a decrease of the potential, $\Delta\varphi$, corresponding to the difference $\varphi - \varphi_0$. According to eqn. (4.5.18)

$$q = \frac{\varphi - \varphi_0}{(1/\pi) \ln [1/\sin(\pi s/2T)]} = \frac{T\Delta\varphi}{L_n} \quad (4.5.22)$$

Similarly, according to eqn. (4.5.20)

$$q = \frac{\varphi_1 - \varphi_0}{(L_1/T) + (1/\pi) \ln [1/\sin (\pi s/2T)]} = \frac{(\varphi_1 - \varphi_0) T}{L_1 + L_n} \quad (4.5.23)$$

By comparing eqn. (4.5.22) with eqn. (4.5.23) we obtain the proportion

$$\frac{\Delta\varphi}{\varphi_1 - \varphi_0} = \frac{L_n}{L_1 + L_n}$$

which expresses the geometrical scheme shown in Fig. 4.14g.

4.6. Method of concentrated losses

We now reverse the problem discussed in Section 4.5. Instead of an hydraulically imperfect cut, consider a hypothetical, hydraulically perfect cut in the same location. It is required that the quantity of seepage should be the same in both cases. According to Fig. 4.14h, we must take into account the lowered level on the axis of the perfect cut. This lowering just corresponds to the value of $\Delta\varphi$. We view the situation as though all the inlet resistances were concentrated on the axis of the cut, and only a one-dimensional flow characterized by a linear pressure line, existed in the region.

The method of concentrated losses clearly represents a more consistent development of the method of substitute lengths. In complicated cases involving more general flow configurations, the method of concentrated losses is combined with other approximate methods. The flow is assumed to be composed of several partial one-dimensional flows connecting with one another. In reality, the streamlines within the limits of those partial segments are curved, and this gives rise to vertical resistance gradients. The losses produced by the vertical component of the resistance, are assumed to be concentrated in the boundaries of the segments. As a result, the pressure line ceases to be continuous and becomes a composite of several linear sections. The discontinuities which arise on the interfaces between the various parts, describe the effect of the vertical component of the resistance in the segment being considered.

We shall illustrate the idea on which the method of concentrated losses is based, by an example calculation of seepage under a protective dam. Fig. 4.15a shows a relatively impervious dam with a relatively impervious blanket on its upstream side. Downstream of the dam is a berm with a relatively impervious subsoil, and farther down, a trapezoidal canal which extends into a saturated stratum under an overburden of poor permeability. The water percolates from the river, and we assume a symmetrical seepage to both sides of the river bed.

In the calculation we start from the river and calculate the inlet parts first. The water seeps through the river bed and immediately changes the direction of the flow.

In the segment from the axis of the river bed to the beginning of the poorly permeable overburden, the vertical resistances may be characterized with the help of the substitute length L'_n . In calculating them we shall make use of eqn. (4.3.36) written as follows in terms of the approximate relations:

$$\phi = \frac{2}{\pi} \ln 2 \frac{\frac{1}{2} \text{Exp}(\pi L_{II}/2T)}{\sinh(\pi d/2T)} = \frac{L_{II}}{T} + \frac{2}{\pi} \ln \frac{1}{\sinh(\pi d/2T)} \quad (4.6.1)$$

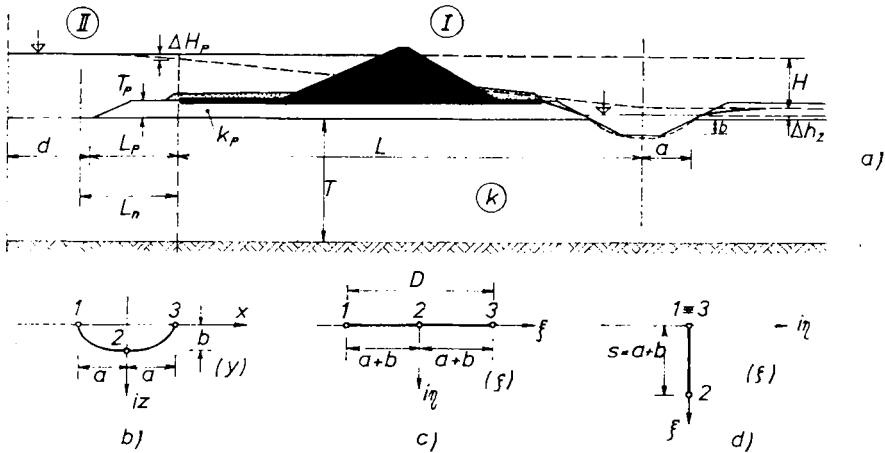


Fig. 4.15. Solution by the method of concentrated losses.

In the case being considered, the substitute length is just that to the shore line; hence, in the sense of the exposition of Section 4.4

$$L'_n = \frac{2T}{\pi} \ln \frac{1}{\sinh(\pi d/2T)} \quad (4.6.2)$$

Generally speaking, the parallel flow will produce a certain loss, ΔH_0 , along the length L'_n . We suppose it to be concentrated in the line of the shore line, and pass to the next segment, i.e. the region within the limits of the permeable portion of the overburden. There, we assume the flow to be quasi-one-dimensional. This essentially involves solving eqn. (2.6.9), the integral of which is eqn. (2.6.10). At the point $x = 0$, we assume that $\Delta H = \Delta H_0$, at the point $x = L_p$, $\Delta H = \Delta H_p$. For $\beta = \sqrt{(k_p/kTT_p)}$ we then obtain

$$\Delta H = C_1 \cosh \beta x + C_2 \sinh \beta x$$

$$C_1 = \Delta H_0$$

$$\Delta H_p = \Delta H_0 \cosh \beta L_p + C_2 \sinh \beta L_p$$

$$C_2 = \frac{\Delta H_p - \Delta H_0 \cosh \beta L_p}{\sinh \beta L_p}$$

Hence, at a general point

$$\Delta H = \Delta H_0 \left(\cosh \beta x - \frac{\sinh \beta x}{\tanh \beta L_p} \right) + \Delta H_p \frac{\sinh \beta x}{\sinh \beta L_p} \quad (4.6.3)$$

Differentiation with respect to x gives

$$\frac{\partial(\Delta H)}{\partial x} = \beta \left[\Delta H_0 \left(\sinh \beta x - \frac{\cosh \beta x}{\tanh \beta L_p} \right) + \Delta H_p \frac{\cosh \beta x}{\sinh \beta L_p} \right]$$

According to (2.6.12), the volume flow at the point $x = L_p$

$$q|_{x=L_p} = kT\beta \left[\Delta H_0 \frac{\sinh^2 \beta L_p - \cosh^2 \beta L_p}{\sinh \beta L_p} + \Delta H_p \frac{1}{\tanh \beta L_p} \right] \quad (4.6.4)$$

$$q|_{x=L_p} = kT\beta \left[\frac{\Delta H_p}{\tanh \beta L_p} - \frac{\Delta H_0}{\sinh \beta L_p} \right] \quad (4.6.5)$$

At the point $x = 0$, the volume flow is

$$q|_{x=0} = kT\beta \left[\frac{\Delta H_p}{\sinh \beta L_p} - \frac{\Delta H_0}{\tanh \beta L_p} \right] \quad (4.6.6)$$

On the vertical line passing through the point $x = 0$, the equation of continuity applies, so that

$$kT \frac{\Delta H_0}{L'_n} = kT\beta \left[\frac{\Delta H_p}{\sinh \beta L_p} - \frac{\Delta H_0}{\tanh \beta L_p} \right] \quad (4.6.7)$$

Hence

$$\Delta H_0 = \beta L'_n \left[\frac{\Delta H_p}{\sinh \beta L_p} - \frac{\Delta H_0}{\tanh \beta L_p} \right] \quad (4.6.8)$$

$$\Delta H_0 = \frac{\beta \Delta H_p L'_n}{\sinh \beta L_p + \beta L'_n \cosh \beta L_p} \quad (4.6.9)$$

If we know the loss ΔH_p at the end of the permeable portion of the overburden, we can readily calculate the concentrated loss on the level of the shore line from eqn. (4.6.9). Of very special importance for our procedure is the magnitude of $\beta L'_n$. For $\beta L'_n > 1$, our deductions would be invalid, and we would have to solve eqn. (2.6.9) for other boundary conditions. Essentially, however, such a solution would bring nothing new, and that is why we shall be satisfied when explaining the method of concentrated losses, with the assumption that $\beta L'_n \leq 1$.

We substitute eqn. (4.6.9) in eqn. (4.6.5). Then

$$\begin{aligned} q|_{x=L_p} &= kT\beta \Delta H_p \left[\frac{1}{\tanh \beta L_p} - \frac{\beta L'_n}{\sinh \beta L_p (\sinh \beta L_p + \beta L'_n \cosh \beta L_p)} \right] = \\ &= \frac{kT\beta \Delta H_p}{\sinh \beta L_p} \left[\cosh \beta L_p - \frac{\beta L'_n}{\sinh \beta L_p + \beta L'_n \cosh \beta L_p} \right] = \\ &= \frac{kT\beta \Delta H_p}{\sinh \beta L_p} \frac{\beta L'_n (\cosh^2 \beta L_p - 1) + \sinh \beta L_p \cosh \beta L_p}{\beta L'_n \cosh \beta L_p + \sinh \beta L_p} \end{aligned}$$

from which it follows that

$$q|_{x=L_p} = kT\beta \Delta H_p \frac{\beta L'_n \sinh \beta L_p + \cosh \beta L_p}{\beta L'_n \cosh \beta L_p + \sinh \beta L_p} \quad (4.6.10)$$

At the point $x = L_p$ the reduced quantity of seepage

$$q_r = \beta T \frac{\beta L'_n \sinh \beta L_p + \cosh \beta L_p}{\beta L'_n \cosh \beta L_p + \sinh \beta L_p} \quad (4.6.11)$$

For the part lying between the river and the end of the natural overburden, the modulus of resistance

$$\Phi_{II} = \frac{1}{q_r}$$

The segment between the axis of the river bed and the end of the impervious blanket offers to the flowing water a resistance which can be expressed in terms of the substitute length L_n :

$$L_n = T\Phi_{II} = \frac{1}{\beta} \frac{\beta L'_n \cosh \beta L_p + \sinh \beta L_p}{\beta L'_n \sinh \beta L_p + \cosh \beta L_p} \quad (4.6.12)$$

It is certainly of interest that if we assume larger values of the argument βL_p when $\sinh \beta L_p \doteq \cosh \beta L_p$, we obtain

$$L_n = \frac{1}{\beta} \quad (4.6.13)$$

This result is a proof that almost the entire quantity of seepage passes through the overburden, and that the river bed proper remains ineffective for otherwise the substitute length L'_n for the direct effect of seepage from the river bed would appear in eqn. (4.6.13).

The actual value of ΔH_p is determined in accordance with the situation prevailing in the remaining, first, part of the flow field. In that part, there is located a trapezoidal canal, the overburden is of poor permeability, and the water percolates into it under pressure. Since no outflow of water into the hinterland is assumed, the water flows along curved paths. We have shown in Sections 4.3, 4.4 that the groundwater level

in the hinterland must lie higher than the level in the canal. We consider the difference between the levels, Δh_z , to be a concentrated loss acting on the axis of the canal. The height Δh_z is a basic performance datum because the level in the hinterland is usually a value prescribed by documents relating to the overall solution of protection against the effects of seepage.

We realize that a canal of an exactly trapezoidal shape is impossible to construct in practice. Moreover, the effect of bunching of the equipotential lines in the proximity of the singular points will enter the picture. Of particular interest is the point in the intersection of the canal slope and the interface between the overburden and the permeable stratum in the subsoil. The slope is not generally orthogonal to the interface, which is why the last equipotential line (i.e. the canal contour) cannot be orthogonal to the relatively impervious interface, either. The singularity that arises there, though not apparent at first glance, can become the cause of sliding of the slopes and washing out of the finer fractions of the material.

In Chapter 3 we have discussed the problem of canals of trapezoidal and parabolic shape. In this section we tackle the solution of an elliptical contour with which we intend to replace the boundary of the flow region.

We shall start the theoretical examination from the scheme shown in Fig. 4.15b. We transform the (y) half-plane with a cut-out semi-ellipse into the (ζ) with no cut-out (Fig. 4.15c) using the function

$$\zeta = \frac{ay - b\sqrt{(y^2 + b^2 - a^2)}}{a - b} \quad (4.6.14)$$

From the above

$$\frac{d\zeta}{dy} = \frac{1}{a - b} \left(a - \frac{by}{\sqrt{(y^2 + b^2 - a^2)}} \right) \quad (4.6.15)$$

At points far removed from the origin

$$\frac{d\zeta}{dy} \rightarrow \frac{a - b}{a - b} = 1 \quad (4.6.16)$$

As the region mainly affected by the transformation is evidently that near the ellipse, the transformation (4.6.14) is well adapted to successive conformal mappings. We can, therefore, transform our region with the elliptical canal into a region with a flat canal (Fig. 4.14c). According to (4.6.14), the half-width of the bottom, $D/2$, will be

$$\frac{D}{2} = \frac{a^2 - b^2}{a - b} = a + b \quad (4.6.17)$$

According to Fig. 4.15d, we could similarly transform the (y) half-plane with the cut-out semi-ellipse into the (ζ) plane with a slit of length s

$$s = a + b \quad (4.6.18)$$

on the real axis.

Assume that the dimensions of the canal allow us to apply the method of successive conformal mappings. Then we do not expect that the straight line representing the course of the surface of the relatively impervious subsoil, will experience deformation under the transformation. According to eqn. (4.5.18) we can determine the concentrated loss on the canal axis as follows:

$$\Delta h_z = \frac{q}{k\pi} \ln \frac{1}{\sin(\pi s/2T)} = \frac{q}{k\pi} \ln \frac{1}{\sin[\pi(a+b)/2T]} \quad (4.6.19)$$

By Fig. 4.15a, the volume flow

$$q = k \frac{TH}{L_n + L} \quad (4.6.20)$$

so that

$$\Delta h_z = \frac{TH}{\pi(L_n + L)} \ln \frac{1}{\sin[\pi(a+b)/2T]} \quad (4.6.21)$$

On the interface between parts *I, II*, the equation of continuity leads to

$$\frac{\Delta H_p}{L_n} = \frac{H - \Delta H_p}{L} \quad (4.6.22)$$

Hence

$$\Delta H_p = H \frac{L_n}{L_n + L} \quad (4.6.23)$$

Of importance from the engineering standpoint are the velocities along the perimeter of the elliptical canal. We shall base the pertinent analysis on the results obtained in the previous sections, in which we solved the seepage into a flat-bottom canal and the seepage into an hydraulically imperfect cut.

According to eqn. (4.4.5) with the substitution $X = -D/2$, we have in the first case that at the centre of the bottom

$$w = i \frac{q}{T} \frac{\text{Exp}(-\pi D/2T)}{\sqrt{\{[\text{Exp}(-\pi D/2T) - \text{Exp}(-\pi D/T)] [\text{Exp}(-\pi D/2T) - 1]\}}}$$

$$v_z = \frac{q}{T} \frac{\text{Exp}(-\pi D/4T)}{\text{Exp}(-\pi D/2T) - 1} = - \frac{q}{T} \frac{\text{Exp}(\pi D/4T)}{\text{Exp}(\pi D/2T) - 1} \quad (4.6.24)$$

In accordance with the basic relations between the flow in the (ζ) plane and the flow in the transformed (y) plane

$$w_y = w_\zeta \left| \frac{d\zeta}{dy} \right| \quad (4.6.25)$$

The velocity in the (y) plane is equal to the velocity in the (ζ) plane multiplied by the absolute value of the ratio between the respective differentials. This ratio is

given by eqn. (4.6.15). For a point of the ellipse, lying on the imaginary axis $x = 0$, $y = ib$

$$\frac{d\zeta}{dy} = \frac{1}{a-b} \left(a - \frac{ib^2}{ia} \right) = \frac{a+b}{a} \quad (4.6.26)$$

At the centre of the elliptical canal perimeter, the water flows vertically upwards; hence for $D/2 = a + b$

$$v = \frac{q}{T} \frac{\text{Exp} [\pi(a+b)/2T]}{\text{Exp} [\pi(a+b)/T] - 1} \frac{a+b}{a} \quad (4.6.27)$$

In the horizontal direction we shall concern ourselves with the points on the contact of the ellipse with the interface between the overburden and the permeable stratum. We shall use the transformation shown in Fig. 4.15c, with the cut lying on the real axis of the (ζ) plane. We have there

$$\frac{d\zeta}{dy} = \frac{a+b}{b} \quad (4.6.28)$$

For $q_0 = q/2$ and for $x = 0$, eqn. (4.5.8) leads to

$$v_{\zeta} = \frac{q}{2T} \left(\frac{1}{\sin (\pi s/2T)} \pm 1 \right) \quad (4.6.29)$$

Hence, on the edges, the velocity will be

$$v = v_{\zeta} \left| \frac{d\zeta}{dy} \right| = \frac{q}{2T} \left(\frac{1}{\sin (\pi s/2T)} \pm 1 \right) \frac{a+b}{b} \quad (4.6.30)$$

On substituting eqn. (4.6.18) in (4.6.30) we obtain

$$v = \frac{q}{2T} \left[\frac{1}{\sin [\pi(a+b)/2T]} \pm 1 \right] \frac{a+b}{b} \quad (4.6.31)$$

The plus sign applies to the point on the elliptical canal edge facing the dam. On the opposite side, the velocity of the percolating water is noticeably smaller. In either case, however, the velocities on both sides of the canal are found to increase with decreasing b . It is desirable, therefore, that an adequately high column of water should be maintained in the canal under the level of the overburden, i.e., in the interest of stability, the canal should be cut to a sufficient depth into the permeable stratum.

4.7. Time variations of water motion along a streamline

In many groundwater problems we are interested in the time in which a particle of water will move from a certain initial point to a specified position on a streamline (the so-called time of dwell in sanitary engineering). An important factor which substantially affects the course of the phenomenon, is the dynamically effective porosity, μ_e . In a plane defined by the Cartesian coordinates x, z , the motion in the direction of the coordinate axes is described by the equations

$$\begin{aligned}\frac{dx}{dt} &= \frac{1}{\mu_e} \frac{\partial \varphi}{\partial x} = \bar{v}_x = \frac{v_x}{\mu_e} \\ \frac{dz}{dt} &= \frac{1}{\mu_e} \frac{\partial \varphi}{\partial z} = \bar{v}_z = \frac{v_z}{\mu_e}\end{aligned}\quad (4.7.1)$$

where

v_x, v_z — are the theoretical velocities in a medium regarded as a continuum, and
 $\bar{v}_x, \bar{v}_z$ — are the actual velocities of motion of a water particle in a porous medium.

The time t of the travel of a particle along a streamline defined by the curve L , is obtained from the equation

$$t = \int_L \frac{ds}{\sqrt{[(\bar{v}_x)^2 + (\bar{v}_z)^2]}} = \mu_e \int_L \frac{ds}{\sqrt{(v_x^2 + v_z^2)}} \quad (4.7.2)$$

ds — the differential of the distance,

t — the time.

It should be recalled, when carrying out practical calculations, that a permeable medium is not arranged homogeneously. Usually there exist in it horizons of greater permeability which carry water easily and at a faster rate. Localization of such horizons is difficult; hence the reference calculations are made for a homogeneous soil and the results are verified experimentally on the site.

To demonstrate the method of calculation, we shall consider the motion of a water particle from an hydraulically imperfect cut. Water percolates from the cut unsymmetrically and laterally into the region to the right of the real axis. We shall confine our considerations to the motion along the imaginary ix axis (Fig. 4.14b). According to eqn. (4.5.8), with the substitution $q_0 = q/2$, we have

$$v_x = \frac{q}{2T} \left(\frac{\cosh(\pi x/2T)}{\sqrt{[\sinh^2(\pi x/2T) + \sin^2(\pi s/2T)]}} + 1 \right) \quad (4.7.3)$$

We rearrange eqn. (4.7.3) as follows:

$$v_x = \frac{q}{2T} \frac{\cosh(\pi x/2T) \sqrt{[\sinh^2(\pi x/2T) + \sin^2(\pi s/2T)] + \sinh^2(\pi x/2T) + \sin^2(\pi s/2T)}}{\sinh^2(\pi x/2T) + \sin^2(\pi s/2T)}$$

and recall that we have

$$\cosh \frac{\pi x}{2T} = \sqrt{\left(\sinh^2 \frac{\pi x}{2T} + 1 \right)}$$

For the sake of simplification and conciseness, we adopt the notation

$$\sinh \frac{\pi x}{\pi T} = A$$

$$\sin^2 \frac{\pi s}{2T} = B$$

$$\begin{aligned} v_x &= \frac{q}{2T} \frac{\sqrt{(A^2 + 1)} \sqrt{(A^2 + B^2)} + A^2 + B}{A^2 + B} = \\ &= \frac{q}{2T(A^2 + B)} \frac{(A^2 + 1)(A^2 + B) - (A^2 + B)^2}{\{\sqrt{[(A^2 + 1)(A^2 + B)]} - (A^2 + B)\}} = \\ &= \frac{q}{2T} \frac{1 - B}{\sqrt{[(A^2 + 1)(A^2 + B)]} - (A^2 + B)} \end{aligned}$$

In the calculation of the time of dwell (eqn. (4.7.2)) we shall need the integral

$$I = \int_0^L \frac{dx}{v_x} = \frac{2T}{q(1 - B)} \left[\int_0^L \sqrt{[(A^2 + 1)(A^2 + B)]} dx - \int_0^L (A^2 + B) dx \right] \quad (4.7.4)$$

We calculate the partial integrals I_1, I_2

$$\begin{aligned} I_1 &= \int_0^L \cosh \frac{\pi x}{2T} \sqrt{\left(\sinh^2 \frac{\pi x}{2T} + \sin^2 \frac{\pi s}{2T} \right)} dx = \\ &= \frac{T}{\pi} \left[\sinh \frac{\pi x}{2T} \sqrt{\left(\sinh^2 \frac{\pi x}{2T} + \sin^2 \frac{\pi s}{2T} \right)} + \sin^2 \frac{\pi s}{2T} \operatorname{argsinh} \frac{\sinh(\pi x/2T)}{\sin(\pi s/2T)} \right]_0^L \\ I_2 &= \int_0^L \left(\sinh^2 \frac{\pi x}{2T} + \sin^2 \frac{\pi s}{2T} \right) dx = \frac{2T}{\pi} \left[\frac{\sinh(\pi x/T)}{4} - \frac{\pi x}{2T} \left(\frac{1}{2} - \sin^2 \frac{\pi s}{2T} \right) \right]_0^L \end{aligned}$$

and obtain

$$I = I_1 + I_2 = \frac{T}{\pi} \left\{ \sinh \frac{\pi L}{2T} \sqrt{\left(\sinh^2 \frac{\pi L}{2T} + \sin^2 \frac{\pi s}{2T} \right)} + \right. \\ \left. + \sin^2 \frac{\pi s}{2T} \operatorname{argsinh} \frac{\sinh(\pi L/2T)}{\sin(\pi s/2T)} - \left[\sinh \frac{\pi L}{2T} \cosh \frac{\pi L}{2T} - \frac{\pi L}{T} \left(\frac{1}{2} - \sin^2 \frac{\pi s}{2T} \right) \right] \right\}$$

Altogether, we have for the time t

$$t = \mu_e \frac{2T^2}{q\pi \cos^2(\pi s/2T)} \left\{ \frac{\pi L}{2T} - \sin^2 \frac{\pi s}{2T} \left(\frac{\pi L}{T} - \operatorname{argsinh} \frac{\sinh(\pi L/2T)}{\sin(\pi s/2T)} \right) - \right. \\ \left. - \sinh \frac{\pi L}{2T} \left[\cosh \frac{\pi L}{2T} - \sqrt{\left(\sinh^2 \frac{\pi L}{2T} + \sin^2 \frac{\pi s}{2T} \right)} \right] \right\} \quad (4.7.5)$$

For the extreme case when $s = 0$ (i.e. for a point source)

$$t = \mu_e \frac{2T^2}{q\pi} \left[\frac{\pi L}{2T} - \sinh \frac{\pi L}{2T} \left(\cosh \frac{\pi L}{2T} - \sinh \frac{\pi L}{2T} \right) \right] \quad (4.7.6)$$

$$t = \mu_e \frac{2T^2}{q\pi} \left\{ \frac{\pi L}{2T} - \frac{1}{2} \left[1 - \operatorname{Exp} \left(-\frac{\pi L}{T} \right) \right] \right\} \quad (4.7.7)$$

In the other extreme case when $s/T = 1$, we use L'Hopital's rule, and in eqn. (4.7.5) set

$$\sigma(s) = \frac{\pi L}{2T} - \sin^2 \frac{\pi s}{2T} \left[\frac{\pi L}{T} - \operatorname{argsinh} \frac{\sinh(\pi L/2T)}{\sin(\pi s/2T)} \right] - \\ - \sinh \frac{\pi L}{2T} \left[\cosh \frac{\pi L}{2T} - \sqrt{\left(\sinh^2 \frac{\pi L}{2T} + \sin^2 \frac{\pi s}{2T} \right)} \right] \\ \tau(s) = \cos^2(\pi s/2T)$$

Differentiating both expressions twice, we obtain $\sigma''(s)$, $\tau''(s)$. Then

$$\lim_{s/T \rightarrow 1} t = \lim_{s/T \rightarrow 1} \frac{\mu_e 2T^2}{q\pi} \frac{\sigma''(s)}{\tau''(s)} = \frac{\mu_e 2T^2}{q\pi} \frac{\pi L}{2T} = \frac{\mu_e T L}{q} \quad (4.7.8)$$

In the extreme case of $s/T = 1$, we obtain, in fact, a parallel flow along the distance L . There, we have

$$q = \frac{T}{L} |\Delta\phi| \quad (4.7.9)$$

where $\Delta\phi$ is the difference between the potentials at the beginning and the end of the

path. Substituting in eqn. (4.7.8) we have

$$t = \frac{\mu_e T L^2}{T |\Delta\varphi|} = \frac{\mu_e L^2}{|\Delta\varphi|} \quad (4.7.10)$$

The same result follows from eqn. (4.7.7) if the term in the square brackets is neglected. Evidently, acceleration of the process in the vicinity of the cut exerts no substantial effect on the time t . The value of L must, of course, be sufficiently large.

4.8. Superposition of velocity potential, mirror representation

In Section 4.5 we stated the conditions of the geometrical composition of velocity vectors, and on this principle we based the method of superposition of velocities. In the superposed partial flows we obtained the distribution of the velocity potential by integration. As an example we presented eqn. (4.5.12) which essentially expresses the effect of two partial potential functions on the resulting value of the velocity potential. The same result could be obtained by integrating the velocity in each partial flow separately and composing (superposing) the results. The basis of the indicated step follows directly from the linearity of the basic partial differential equations. Consider, e.g. a flow described by the equation

$$\frac{\partial^2 \varphi_1}{\partial x^2} + \frac{\partial^2 \varphi_1}{\partial z^2} = 0$$

and simultaneously, another flow satisfying the relation

$$\frac{\partial^2 \varphi_2}{\partial x^2} + \frac{\partial^2 \varphi_2}{\partial z^2} = 0$$

Assuming identity of the geometry of the flow region and also that we know with certainty the values of the functions φ_1 , φ_2 on the boundaries, we can form the linear combination

$$\varphi = a\varphi_1 + b\varphi_2 + c$$

where a , b , c are constants.

In the process we obtain the resulting distribution of the function $\varphi(x, z)$ which defines the flow according to the equation

$$\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial z^2} = 0, \quad \varphi = f(x, z)$$

The method of superposition of potentials is of considerable significance and is used in the solution of a number of practical problems.

Assume, for example, that there exists in the (y) plane a flow obtained by the conformal mapping of a parallel flow, effected with the help of the function

$$\omega = \frac{q}{2\pi} \ln y \quad (4.8.1)$$

where ω is the complex potential defined as follows:

$$\omega = \varphi + i\psi$$

Separating eqn. (4.8.1) into the real and imaginary parts:

$$\varphi = \frac{q}{2\pi} \ln r$$

$$\psi = \frac{q}{2\pi} \vartheta$$

where r, ϑ are the polar coordinates of a point in the (y) plane.

The function (4.8.1) describes the flow from a point source or the flow into a point sink in the (y) plane. The sense of the flow is determined by the sign of q . What is essential is that the flow field occupies the whole (y) plane, and that at each point we know the value of φ, ψ and also the value of the complex potential.

Consider now in the (y) plane a number of point sources of equal strength q , distributed along the real axis at equal distances b (Fig. 4.16a).

For simplicity consider a geometrically reduced flow region. Make the substitution

$$\bar{y} = y \frac{\pi}{b} \quad (4.8.2)$$

In accordance with the principle of superposition, we have in the reduced region that

$$\bar{\omega} = \frac{q}{2\pi} \left[\ln \bar{y} + \sum_{j=1}^{\infty} \ln (\bar{y} - \bar{y}_j) \right] \quad (4.8.3)$$

$$y_j = \pm j b, \quad \bar{y}_j = \pm j \pi, \quad \omega = \bar{\omega} \quad (4.8.4)$$

Eqn. (4.8.3) may also be written as follows:

$$\begin{aligned} \bar{\omega} &= \frac{q}{2\pi} \left[\ln \bar{y} + \sum_{j=1}^{\infty} \ln (\bar{y} \mp j\pi) \right] = \\ &= \frac{q}{2\pi} \left[\ln \bar{y} + \sum_{j=1}^{\infty} \ln (-j\pi) \left(1 \pm \frac{\bar{y}}{j\pi} \right) \right] = \\ &= \frac{q}{2\pi} \left[\ln \frac{\bar{y}}{j\pi} \prod_{j=1}^{\infty} \left(1 + \frac{\bar{y}}{j\pi} \right) \left(1 - \frac{\bar{y}}{j\pi} \right) \right] \end{aligned} \quad (4.8.5)$$

Hence

$$\bar{\omega} = \frac{q}{2\pi} \ln \frac{\bar{y}}{j\pi} \prod_{j=1}^{\infty} \left[1 - \left(\frac{\bar{y}}{j\pi} \right)^2 \right] \quad (4.8.6)$$

In the theory of analytic functions it is proved with the help of the theorem of Weierstrass that

$$\bar{y} \prod_{j=1}^{\infty} \left[1 - \left(\frac{\bar{y}}{j\pi} \right)^2 \right] = \sin \bar{y}$$

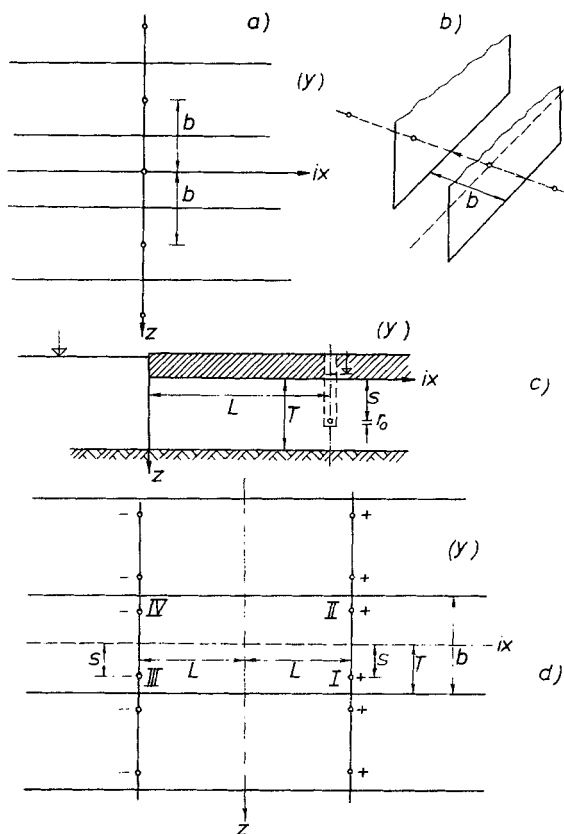


Fig. 4.16. Basic idea of the method of mirror representation.

Accordingly

$$\bar{\omega} = \frac{q}{2\pi} \ln \frac{1}{j\pi} \sin \bar{y} = \frac{q}{\pi} \ln \sin \bar{y} + \frac{q}{2\pi} \ln \frac{1}{j\pi} \quad (4.8.7)$$

The second term on the right-hand side of eqn. (4.8.7) has the meaning of an arbitrary constant; therefore

$$\bar{\omega} = \frac{q}{2\pi} \ln \sin \bar{y} + c \quad (4.8.8)$$

Hence

$$\begin{aligned}\sin \bar{y} &= \text{Exp} \left[\frac{2\pi}{q} (\varphi + i\psi) - c \right] = \text{Exp} \left(\frac{2\pi\varphi}{q} - c \right) \text{Exp} \frac{2\pi i\psi}{q} = \\ &= \text{Exp} \left(\frac{2\pi\varphi}{q} - c \right) \left(\cos \frac{2\pi}{q} \psi + i \sin \frac{2\pi}{q} \psi \right)\end{aligned}$$

$$\sin \bar{y} = \sin \bar{z} \cosh \bar{x} + i \cos \bar{z} \sinh \bar{x}$$

On separating the above into the real and imaginary parts, we obtain for the potential function and for the stream function

$$\begin{aligned}\sin \bar{z} \cosh \bar{x} &= \text{Exp} \left(\frac{2\pi\varphi}{q} - c \right) \cos \frac{2\pi\psi}{q} \\ \cos \bar{z} \sinh \bar{x} &= \text{Exp} \left(\frac{2\pi\varphi}{q} - c \right) \sin \frac{2\pi\psi}{q}\end{aligned}\tag{4.8.9}$$

It follows from these two equations that

$$\sin^2 \bar{z} \cosh^2 \bar{x} + \cos^2 \bar{z} \sinh^2 \bar{x} = \text{Exp} 2 \left(\frac{2\pi\varphi}{q} - c \right)$$

hence

$$\begin{aligned}\varphi &= \frac{q}{4\pi} \ln (\sin^2 \bar{z} \cosh^2 \bar{x} + \cos^2 \bar{z} \sinh^2 \bar{x}) + c = \\ &= \frac{q}{4\pi} \ln (\sin^2 \bar{z} + \sinh^2 \bar{x}) + c = \\ &= \frac{q}{4\pi} \ln \left(\frac{1 - \cos 2\bar{z}}{2} + \frac{\cosh 2\bar{x} - 1}{2} \right) + c \\ \varphi &= \frac{q}{4\pi} \ln \frac{\cosh 2\bar{x} - \cos 2\bar{z}}{2} + c\end{aligned}\tag{4.8.10}$$

Further, from eqn. (4.8.9)

$$\frac{\text{tgh } \bar{x}}{\text{tg } \bar{z}} = \text{tg } \frac{2\pi\psi}{q}$$

and hence

$$\psi = \frac{q}{2\pi} \arctg (\cotg \bar{z} \text{tgh } \bar{x})\tag{4.8.11}$$

Eqn. (4.8.11) indicates the way in which the volume flow is distributed in the flow field. Choose, for example, $\bar{z} = 0$; then $\cotg \bar{z} = \infty$, $\arctg \infty = \pi/2, 3\pi/2, 5\pi/2$, etc., with the corresponding $\psi = q/4, 3q/4$, etc. For $\bar{z} = \pi/2$, $\cotg \bar{z} = 0$, $\arctg 0 = 0, \pi, 2\pi$, etc., with the corresponding $\psi = 0, q/2, q$, etc. The dividing lines (the axes of symmetry) between the sinks are clearly streamlines.

We can form a mental picture of an infinite series of sinks by placing two parallel mirrors at a certain distance, $b/2$, from one of the sinks. Tracing the rays we see that – according to well-known laws of optics – the image of the sink is reflected first from one and then from the other mirror, and so on, until we gain an impression of an infinite series of sinks. The contact between the mirrors and the plane on which they are situated (Fig. 4.16b) is also reflected repeatedly. The optical interpretation corresponds to our problem. The literature often refers to mirror representation which characterizes, in abbreviated form, the application of the principle of superposition in cases when the flow in a field is studied with the help of fictitious images distributed symmetrically about a certain straight line or plane. In the case being considered, there exists, in fact, only one real sink; all the others are regarded as auxiliary, mirror-reflected and hence unreal images of that single real object. The effects of the real sink are superimposed on those of all the reflected sinks.

We shall illustrate the application of the principle of superposition by a simple example of a horizontal drain collecting water percolating from a river (Fig. 4.16c). The overburden is assumed to be of relatively poor permeability. The drain of radius r_0 lies at a depth s under the interface between the overburden and the permeable stratum at a distance L from the river bank.

We shall solve the problem by mirror representation according to Fig. 4.16d. The axes of symmetry in strips of width b are taken for the interfaces between the permeable layer and the overburden. Therefore, we add to the sink *I* representing the drain, a symmetrical fictitious sink *II* in the same strip. The two boundary lines of the strip are considered to be the contact edges of mirrors which form images of an infinite linear series of sinks. The z axis describes the shore line which is simultaneously an equipotential line. Hence we place in it the contact of an additional mirror in which the points *I*, *II* reflect symmetrically. In this way we obtain the systems *III*, *IV*. Since the latter systems are assumed to contain sources, we affix to the volume flow there a negative sign.

In accordance with the previous results, we obtain, by means of simple translation of the coordinate axes:

for system *I*

$$\varphi_I = \frac{q}{4\pi} \ln \frac{\cosh 2(\bar{x} - L) - \cos 2(\bar{z} - \bar{s})}{2} + c_I \quad (4.8.12)$$

$$\psi_I = \frac{q}{2\pi} \arctg [\cotg (\bar{z} - \bar{s}) \operatorname{tgh} (\bar{x} - L)] \quad (4.8.13)$$

for system *II*

$$\varphi_{II} = \frac{q}{4\pi} \ln \frac{\cosh 2(\bar{x} - \bar{L}) - \cos(\bar{z} + \bar{s})}{2} + c_{II} \quad (4.8.14)$$

$$\psi_{II} = \frac{q}{2\pi} \operatorname{arctg} [\cotg(\bar{z} + \bar{s}) \operatorname{tgh}(\bar{x} - \bar{L})] \quad (4.8.15)$$

for system *III*

$$\varphi_{III} = -\frac{q}{4\pi} \ln \frac{\cosh 2(\bar{x} + \bar{L}) - \cos 2(\bar{z} - \bar{s})}{2} + c_{III} \quad (4.8.16)$$

$$\psi_{III} = -\frac{q}{2\pi} \operatorname{arctg} [\cotg(\bar{z} - \bar{s}) \operatorname{tgh}(\bar{x} + \bar{L})] \quad (4.8.17)$$

for system *IV*

$$\varphi_{IV} = -\frac{q}{4\pi} \ln \frac{\cosh 2(\bar{x} + \bar{L}) - \cos 2(\bar{z} + \bar{s})}{2} + c_{IV} \quad (4.8.18)$$

$$\psi_{IV} = -\frac{q}{2\pi} \operatorname{arctg} [\cotg(\bar{z} + \bar{s}) \operatorname{tgh}(\bar{x} + \bar{L})] \quad (4.8.19)$$

where $c_I, c_{II}, c_{III}, c_{IV}$ are constants.

Generally, after superposition of all effects

$$\varphi = \varphi_I + \varphi_{II} + \varphi_{III} + \varphi_{IV} \quad (4.8.20)$$

$$\psi = \psi_I + \psi_{II} + \psi_{III} + \psi_{IV} \quad (4.8.21)$$

Thus, for example, the velocity potential

$$\varphi = \frac{q}{4\pi} \ln \frac{[\cosh 2(\bar{x} - \bar{L}) - \cos 2(\bar{z} - \bar{s})] [\cosh 2(\bar{x} - \bar{L}) - \cos 2(\bar{z} + \bar{s})]}{[\cosh 2(\bar{x} + \bar{L}) - \cos 2(\bar{z} - \bar{s})] [\cosh 2(\bar{x} + \bar{L}) - \cos 2(\bar{z} + \bar{s})]} + c \quad (4.8.22)$$

where c is a constant.

Turning now to the actual flow region: on the assumption that $b = 2T$, eqn. (4.8.2) yields

$$\varphi = \frac{q}{4\pi} \ln \frac{\{\cosh [\pi(x - L)/T] - \cos [\pi(z - s)/T]\} \{\cosh [\pi(x - L)/T] - \cos [\pi(z + s)/T]\}}{\{\cosh [\pi(x + L)/T] - \cos [\pi(z - s)/T]\} \{\cosh [\pi(x + L)/T] - \cos [\pi(z + s)/T]\}} + c \quad (4.8.23)$$

Assume that the potential lines around a sink are nearly circular; hence on the circle of radius r_0 , which represents the outer contour of the drain, $\varphi = \varphi_0$. Assume also

that the length L is greater by far than r_0 , and that the drain lies neither immediately on the surface of the relatively impervious subsoil nor on the interface between the overburden and the permeable stratum. Noting that on the drain perimeter, $r_0 = s - z$, $z + s = 2s - r_0$, we obtain

$$\varphi_0 = \frac{q}{4\pi} \ln \frac{[1 - \cos(\pi r_0/T)] \{1 - \cos[\pi(2s - r_0)/T]\}}{[\cosh(2\pi L/T) - \cos(\pi r_0/T)] \{\cosh(2\pi L/T) - \cos[\pi(2s - r_0)/T]\}} + c \quad (4.8.24)$$

Substituting in eqn. (4.8.23) we obtain

$$\begin{aligned} \varphi - \varphi_0 = & \frac{q}{4\pi} \ln \frac{\{\cosh[\pi(x - L)/T] \cos[\pi(z - s)/T]\} \{\cosh[\pi(x - L)/T] - \\ & - \cos[\pi(z + s)/T]\}}{\{\cosh[\pi(x + L)/T] - \cos[\pi(z - s)/T]\} \{\cosh[\pi(x + L)/T] - \\ & - \cos[\pi(z + s)/T]\}} \times \\ & \times \frac{[\cosh(2\pi L/T) - \cos(\pi r_0/T)] \{\cosh(2\pi L/T) - \cos[\pi(2s - r_0)/T]\}}{[(1 - \cos(\pi r_0/T)) \{1 - \cos[\pi(2s - r_0)/T]\}} \end{aligned} \quad (4.8.25)$$

On the z axis, i.e. at the points $x = 0$, let $\varphi = \varphi_L$. Then

$$q = \frac{4\pi(\varphi_L - \varphi_0)}{\ln \frac{[\cosh(2\pi L/T) - \cos(\pi r_0/T)] \{\cosh(2\pi L/T) - \cos[\pi(2s - r_0)/T]\}}{[1 - \cos(\pi r_0/T)] \{1 - \cos[\pi(2s - r_0)/T]\}}} \quad (4.8.26)$$

4.9. Variational methods

In a sense, the principle of superposition also finds application in solutions of steady-state flow by variational methods. Suppose that in a range S there exists a flow defined by the distribution of lines of equal piezometric heads h .

Consider the given steady flow to be characterized by a certain distribution of the value $h(x, z)$ inside S . According to Chapter 1, a unit volume of permeable medium is loaded by the force γJ (γ is the unit weight of fluid). Take the force acting through the distance h to perform the work A . Set up the integral for the increments of energy work in the directions of the coordinate axes for the whole region S :

$$\begin{aligned} \gamma \int_S \left[\frac{\partial}{\partial x} \left(h \frac{\partial h}{\partial x} \right) + \frac{\partial}{\partial z} \left(h \frac{\partial h}{\partial z} \right) \right] dS &= \int_S \left(\frac{\partial A}{\partial x} + \frac{\partial A}{\partial z} \right) dS \\ \int_S \left(\frac{\partial A}{\partial x} + \frac{\partial A}{\partial z} \right) dS &= \gamma \int_S \left[\left(\frac{\partial h}{\partial x} \right)^2 + \left(\frac{\partial h}{\partial z} \right)^2 + h \left(\frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial z^2} \right) \right] dS \end{aligned} \quad (4.9.1)$$

According to eqn. (1.4.20b), in steady flow, the third term in the square brackets is equal to zero; hence

$$\int_S \left(\frac{\partial A}{\partial x} + \frac{\partial A}{\partial z} \right) dS = \gamma \int_S \left[\left(\frac{\partial h}{\partial x} \right)^2 + \left(\frac{\partial h}{\partial z} \right)^2 \right] dS = \gamma I \quad (4.9.2)$$

Since the flow forms on the condition of minimum expended work, it is necessary, from the energy standpoint, that the integral

$$I = \int_S \left[\left(\frac{\partial h}{\partial x} \right)^2 + \left(\frac{\partial h}{\partial z} \right)^2 \right] dS = c_1 \quad (4.9.3)$$

should have a certain minimal value.

Since simultaneously $h = -\varphi/k$,

$$I = \int_S \left[\left(\frac{\partial \varphi}{\partial x} \right)^2 + \left(\frac{\partial \varphi}{\partial z} \right)^2 \right] dS = c_2 \quad (4.9.4)$$

where c_1, c_2 are constants.

The integral I is a function of φ , or a function of h , and for this reason is called the functional.

Conversely, the problem in which one is required to find the distribution of h in the range S , can evidently be solved by trial and error. A certain distribution $h(x, z)$ is proposed, and we inquire whether the value of the functional is minimal. The problem is solved by establishing a distribution of the values of h such that for it, $I = \min$. The same holds for the integral I defined in terms of the velocity potential φ .

Assume that the region S has a boundary L on which are prescribed the values φ_L . Assume further that in the region S there is known the distribution of the approximate values of $\bar{\varphi}$, the values on the boundary L being $\bar{\varphi}_L = \varphi_L$. Note that inside S there are deviations $\bar{\varphi}$ from the exact value φ , and assume that the differences are not too large:

$$\bar{\varphi}(x, z) = \varphi(x, z) + \varepsilon \eta(x, z) \quad (4.9.5)$$

where the value ε is sufficiently small. The function $\eta(x, z)$ is continuous in the range S , and takes the value zero on the boundary L . The function $\bar{\varphi}(x, z)$ is an allied ("varied") function of the function $\varphi(x, z)$. Consider the functional

$$\bar{I} = \int_S \left[\left(\frac{\partial \bar{\varphi}}{\partial x} \right)^2 + \left(\frac{\partial \bar{\varphi}}{\partial z} \right)^2 \right] dS \quad (4.9.6)$$

and assume that

$$\bar{I} \geq I \quad (4.9.7)$$

Since the value of the functional I is minimal, the functional $\bar{I}$ will also have a value close to the minimum value, provided that $\bar{I} \rightarrow I$, i.e. on the condition that

$$\left. \frac{d\bar{I}}{d\varepsilon} \right|_{\varepsilon \rightarrow 0} = 0 \quad (4.9.8)$$

Actually

$$\begin{aligned} \frac{d\bar{I}}{d\varepsilon} &= \frac{d}{d\varepsilon} \int_S \left[\left(\frac{\partial \varphi}{\partial x} + \varepsilon \frac{\partial \eta}{\partial x} \right)^2 + \left(\frac{\partial \varphi}{\partial z} + \varepsilon \frac{\partial \eta}{\partial z} \right)^2 \right] dS = \\ &= 2 \int_S \left[\left(\frac{\partial \varphi}{\partial x} + \varepsilon \frac{\partial \eta}{\partial x} \right) \frac{\partial \eta}{\partial x} + \left(\frac{\partial \varphi}{\partial z} + \varepsilon \frac{\partial \eta}{\partial z} \right) \frac{\partial \eta}{\partial z} \right] dS \end{aligned}$$

Assume that $\varepsilon \rightarrow 0$, so that

$$\frac{d\bar{I}}{d\varepsilon} = 2 \int_S \left(\frac{\partial \varphi}{\partial x} \frac{\partial \eta}{\partial x} + \frac{\partial \varphi}{\partial z} \frac{\partial \eta}{\partial z} \right) dS$$

Consider that

$$\frac{\partial \varphi}{\partial x} \frac{\partial \eta}{\partial x} - \eta \frac{\partial^2 \varphi}{\partial x^2} + \eta \frac{\partial^2 \varphi}{\partial x^2} = \frac{\partial}{\partial x} \left(\eta \frac{\partial \varphi}{\partial x} \right) - \eta \frac{\partial^2 \varphi}{\partial x^2}$$

and similarly,

$$\frac{\partial \varphi}{\partial z} \frac{\partial \eta}{\partial z} = \frac{\partial}{\partial z} \left(\eta \frac{\partial \varphi}{\partial z} \right) - \eta \frac{\partial^2 \varphi}{\partial z^2}$$

The condition for the existence of an extreme value of the functional is

$$\frac{d\bar{I}}{d\varepsilon} = 2 \int_S \left[\frac{\partial}{\partial x} \left(\eta \frac{\partial \varphi}{\partial x} \right) + \frac{\partial}{\partial z} \left(\eta \frac{\partial \varphi}{\partial z} \right) \right] dS - 2 \int_S \eta \left(\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial z^2} \right) dS = 0$$

By Ostrogradsky's theorem

$$\int_S \left[\frac{\partial}{\partial x} \left(\eta \frac{\partial \varphi}{\partial x} \right) + \frac{\partial}{\partial z} \left(\eta \frac{\partial \varphi}{\partial z} \right) \right] dS = \int_L \eta \frac{\partial \varphi}{\partial n} dL$$

where n is the direction of the normal line pointing towards the interior of S .

As noted earlier, the function $\eta(x, z)$ has the value zero on L . Therefore

$$\frac{d\bar{I}}{d\varepsilon} = -2 \int_S \eta \left(\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial z^2} \right) dS = 0 \quad (4.9.9)$$

This equation is always satisfied because, independently of η , eqn. (1.4.20a) applies.

Evidently, the problem being considered can be converted to the case when we are required to find the minimum of the integral $\bar{I}$ constructed for some approximating

function $\bar{\varphi}(x, z)$ having the values $\bar{\varphi}_L = \varphi_L$ on the boundaries. We can regard this function as the result of superposition of the partial solutions $\bar{\varphi}_j$ ($j = 1, 2, 3, \dots, n$)

$$\bar{\varphi} = \bar{\varphi}_0 + a_1 \bar{\varphi}_1 + a_2 \bar{\varphi}_2 + \dots + a_n \bar{\varphi}_n \quad (4.9.10)$$

where

a_j — constants

$\bar{\varphi}_j$ — partial solutions

We choose the function $\bar{\varphi}_0$ so as to obtain $\bar{\varphi}_0 = \varphi_L$ on the boundary. For the remaining functions, we assume that $(\bar{\varphi}_j)_L = 0$ on the boundaries.

The functional (4.9.6) will have a minimum if the partial solutions satisfy the condition

$$\frac{dJ_j}{da_j} = -2 \int_S \eta_j \left(\frac{\partial^2 \bar{\varphi}_j}{\partial x^2} + \frac{\partial^2 \bar{\varphi}_j}{\partial z^2} \right) dS = 0 \quad (4.9.11)$$

Since the form of the function $\eta(x, z)$ is not prescribed, we can set $\eta(x, z) = a \bar{\varphi}_j(x, z)$. The minimization of the functional will be achieved on the condition that

$$\begin{aligned} \sum_{j=1}^n \int_S a_j \bar{\varphi}_1 \left(\frac{\partial^2 \bar{\varphi}_j}{\partial x^2} + \frac{\partial^2 \bar{\varphi}_j}{\partial z^2} \right) dS &= - \int_S \bar{\varphi}_1 \left(\frac{\partial^2 \bar{\varphi}_0}{\partial x^2} + \frac{\partial^2 \bar{\varphi}_0}{\partial z^2} \right) dS \\ \sum_{j=1}^n \int_S a_j \bar{\varphi}_2 \left(\frac{\partial^2 \bar{\varphi}_j}{\partial x^2} + \frac{\partial^2 \bar{\varphi}_j}{\partial z^2} \right) dS &= - \int_S \bar{\varphi}_2 \left(\frac{\partial^2 \bar{\varphi}_0}{\partial x^2} + \frac{\partial^2 \bar{\varphi}_0}{\partial z^2} \right) dS \\ &\dots\dots\dots \\ \sum_{j=1}^n \int_S a_j \bar{\varphi}_n \left(\frac{\partial^2 \bar{\varphi}_j}{\partial x^2} + \frac{\partial^2 \bar{\varphi}_j}{\partial z^2} \right) dS &= - \int_S \bar{\varphi}_n \left(\frac{\partial^2 \bar{\varphi}_0}{\partial x^2} + \frac{\partial^2 \bar{\varphi}_0}{\partial z^2} \right) dS \end{aligned} \quad (4.9.12)$$

The system of equations (4.9.12) is theoretically satisfied regardless of the choice of the coefficient a_j , provided we have

$$\frac{\partial^2 \bar{\varphi}_j}{\partial x^2} + \frac{\partial^2 \bar{\varphi}_j}{\partial z^2} = 0, \quad j = 0, 1, 2, 3, \dots, n \quad (4.9.13)$$

In practical calculations the form of the functions $\bar{\varphi}_j$ is chosen to be as simple as possible in order to facilitate the evaluation of the integrals. The selected functions need not correspond to the condition (4.9.13). In such cases the minimization of the functional is ensured by a suitable choice of the coefficients a_j , eqn. (4.9.12) being used for their determination. By selecting a large enough n , we can make the solution more exact.

Assume, for example, that the equation of the boundary L is $F(x, z) = 0$. Therefore, we can choose

$$\begin{aligned} \bar{\varphi}_1 &= F(x, z), & \bar{\varphi}_2 &= x F(x, z), & \bar{\varphi}_3 &= z F(x, z) \\ \bar{\varphi}_4 &= x^2 F(x, z), & \bar{\varphi}_5 &= z^2 F(x, z), & \bar{\varphi}_6 &= xz F(x, z) \end{aligned}$$

The procedure stemming from the method elaborated by Galerkin assumes a linear combination of functions satisfying the prescribed boundary condition. Generally, however, the selected functions might not correspond to the fundamental partial differential equation.

In another approach, suggested by Trefftz, the solution proceeds from a linear combination of the functions $\bar{\varphi}_j$ which, on the other hand, correspond to the initial partial differential equation but fail to satisfy the specified boundary conditions. In many cases, Trefftz's method is more advantageous because it leads to the evaluation of line integrals, a task much easier than the evaluation of the surface integrals of system (4.9.12). The various modifications of variational methods, referred to in the literature, are essentially combinations of the basic ideas outlined above.

The aim of this section was to show one of the possible applications of the principle of superposition. The variational methods represent a theoretical point of departure for the numerical method of finite elements.

All numerical methods possess a certain disadvantage in that a particular solution can be used to appraise only a narrow class of practical problems. Analytical methods are more general.

4.10. Solution of some cases of groundwater motion in non-homogeneous flow regions

We shall now demonstrate how the principle of superposition of velocity potentials can be used in solutions of problems involving groundwater flow in regions consisting of two soils of different permeabilities. Consider, for example, a plane (Fig. 4.17a) containing a dividing line representative of the interface between a soil with a coefficient of permeability k_1 , and a soil with a coefficient of permeability k_2 . Point sinks with volume flows q_1, q_2 are present at distances a_1, a_2 from the dividing line. The mathematical description of the conditions obtaining in the plane, presented in Section 3.1.6, will be correct in the case that the piezometric head h at the points of the interface between the layers is the same, regardless of whether those points are approached from the left or from the right. The second condition is the equality of the velocity components in the direction of the normal to the interface.

The solution of the problem is described by Pirverdyan who makes use of fictitious sinks q'_1 and q''_1 in the region I containing the sink q_1 . According to Fig. 4.17b, the principle of superposition applied to the region I , leads to

$$-h_1 = \frac{q_1}{2\pi k_1} \ln r_1 + \frac{q'_1}{2\pi k_1} \ln r'_1 + \frac{q''_1}{2\pi k_1} \ln r''_1 + c_1 \quad (4.10.1)$$

At the points of the interface, we simplify the symbols used as follows:

$$-h_1 = \frac{q_1}{2\pi k_1} \ln r_1 + \frac{q'_1}{2\pi k_1} \ln r_1 + \frac{q''_1}{2\pi k_1} \ln r_2 + c \quad (4.10.2)$$

Having regard to the sense of motion toward the sinks, we obtain the velocity components in the direction of the normal to the interface as

$$v_{1n} = \frac{q_1}{2\pi r_1} \frac{q_1}{r_1} - \frac{q'_1}{2\pi r'_1} \frac{a_1}{r'_1} - \frac{q''_1}{2\pi r''_1} \frac{a_2}{r''_1} \quad (4.10.3)$$

r, r'_1, r''_1 are the distance between the points of the interface and the respective sinks.

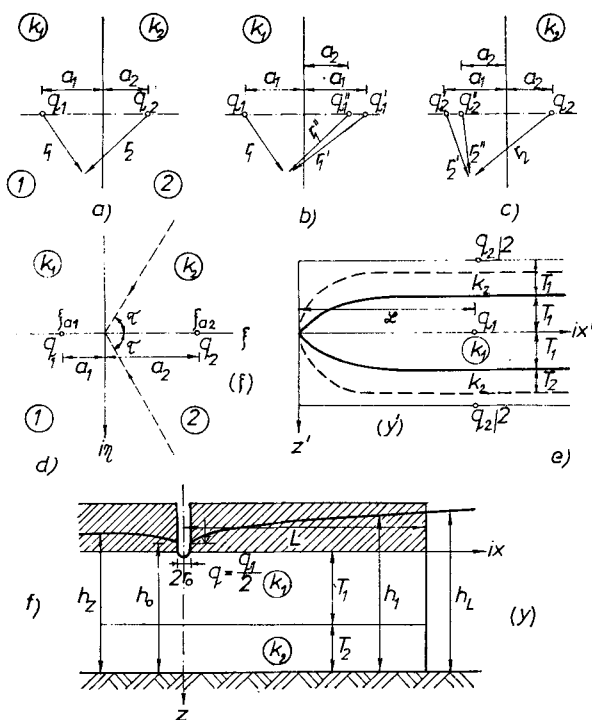


Fig. 4.17. Calculation of the flow field in a non-homogeneous region.

In a simplified description $r'_1 = r_1, r''_1 = r_2$

$$v_{1n} = \frac{q_1 a_1}{2\pi r_1^2} - \frac{q'_1 a_1}{2\pi r_1^2} - \frac{q''_1 a_2}{2\pi r_2^2} \quad (4.10.4)$$

Similarly, we shall use the fictitious sinks q'_2 and q''_2 in the region 2 containing the sink q_2 (Fig. 4.17c). The principle of superposition applied to the region 2, leads to

$$-h_2 = \frac{q_2}{2\pi k_2} \ln r_2 + \frac{q'_2}{2\pi k_2} \ln r'_2 + \frac{q''_2}{2\pi k_2} \ln r''_2 + c_2 \quad (4.10.5)$$

At the points of the interface between the layers, $r'_2 = r_1$, $r''_2 = r_2$ and

$$-h_2 = \frac{q_2}{2\pi k_2} \ln r_2 + \frac{q'_2}{2\pi k_2} \ln r_1 + \frac{q''_2}{2\pi k_2} \ln r_2 + c \quad (4.10.6)$$

Bearing in mind the sense of motion in the direction of the normal, we obtain on the interface

$$v_{2n} = -\frac{q_2}{2\pi r_2} \frac{a_2}{r_2} + \frac{q'_2}{2\pi r'_2} \frac{a_1}{r'_2} + \frac{q''_2}{2\pi r''_2} \frac{a_2}{r''_2} \quad (4.10.7)$$

$$v_{2n} = -\frac{q_2 a_2}{2\pi r_2^2} + \frac{q'_2 a_1}{2\pi r_1^2} + \frac{q''_2 a_2}{2\pi r_2^2} \quad (4.10.8)$$

The conditions applying to the interface lead to the following system of equations

$$\frac{q_1 a_1}{r_1^2} - \frac{q'_1 a_1}{r_1^2} - \frac{q''_1 a_2}{r_2^2} = -\frac{q_2 a_2}{r_2^2} + \frac{q'_2 a_1}{r_1^2} + \frac{q''_2 a_2}{r_2^2} \quad (4.10.9)$$

$$\frac{q_1}{k_1} \ln r_1 + \frac{q'_1}{k_1} \ln r_1 + \frac{q''_1}{k_1} \ln r_2 + c_1 = \frac{q_2}{k_2} \ln r_2 + \frac{q'_2}{k_2} \ln r_1 + \frac{q''_2}{k_2} \ln r_2 + c_2 \quad (4.10.10)$$

Eqn. (4.10.10) can be satisfied in the case that $c_1 = c_2$ and simultaneously,

$$\frac{q_1}{k_1} = \frac{q'_2}{k_2} - \frac{q'_1}{k_1}$$

$$\frac{q_2}{k_2} = \frac{q''_1}{k_1} - \frac{q''_2}{k_2}$$

Eqn. (4.10.9) is satisfied when

$$q_1 = q'_1 + q'_2$$

$$q_2 = q''_1 + q''_2$$

hence

$$q'_1 = \frac{k_1 - k_2}{2k_2} q'_2, \quad q''_2 = \frac{k_2 - k_1}{2k_1} q''_1$$

Given q_1, q_2 we obtain the relations

$$q'_1 = \frac{k_1 - k_2}{k_1 + k_2} q_1, \quad q''_1 = \frac{2k_1 q_2}{k_1 + k_2}$$

$$q'_2 = \frac{2k_2 q_1}{k_1 + k_2}, \quad q''_2 = -\frac{k_1 - k_2}{k_1 + k_2} q_2$$

Substituting in eqn. (4.10.1) we have for the region 1

$$-h_1 = \frac{1}{2\pi k_1} \left(q_1 \ln r_1 + q_1 \frac{k_1 - k_2}{k_1 + k_2} \ln r'_1 + q_2 \frac{2k_1}{k_1 + k_2} \ln r''_1 \right) + c_1 \quad (4.10.11)$$

Similarly, eqn. (4.10.6) yields for the region 2

$$-h_2 = \frac{1}{2\pi k_2} \left(q_2 \ln r_2 - q_2 \frac{k_1 - k_2}{k_1 + k_2} \ln r''_2 + q_1 \frac{2k_2}{k_1 + k_2} \ln r'_2 \right) + c_2 \quad (4.10.12)$$

The physical meaning of the superposition carried out above will become especially clear when considering the simple example of just a single sink in the region 1. Assuming that $q_2 = 0$ we obtain, in place of eqns. (4.10.11), (4.10.12),

$$-h_1 = \frac{1}{2\pi k_1} \left(q_1 \ln r_1 + q_1 \frac{k_1 - k_2}{k_1 - k_2} \ln r'_1 \right) + c_1 \quad (4.10.13)$$

$$-h_2 = \frac{q_1}{\pi(k_1 + k_2)} \ln r'_2 + c_2 \quad (4.10.14)$$

In eqn. (4.10.13) appears the simultaneous effect of the sinks q'_1, q''_1 which we consider to be in a homogeneous region with a coefficient of permeability k_1 . In eqn. (4.10.14) we consider only one sink, q''_2 , in a homogeneous region with a coefficient of permeability k_2 . This fictitious sink is at the same location as the real sink q_1 . This implies that the equipotential lines in the region 1 are circles, with centre at the point of location of the sink q_1 . The sink q'_2 emphasizes the function of the real sink, and essentially describes the effect of non-homogeneity in the whole flow region.

The solution just presented provides an exact result which can be used in approximate analyses of more complicated problems. Consider the original specification including one sink in the region 1 and another sink in the region 2. At the point of intersection of the line connecting the two sinks, and the line of the interface between the layers, $r_1 = a_1, r_2 = a_2$.

Choose the volume flows q_1, q_2 and the distances between the sinks in such a way that no water will flow at that point. Bearing in mind the sense of the flow, eqns. (4.10.3), (4.10.7) give

$$v_{1n} = \frac{q_1}{2\pi a_1} - \frac{q'_1}{2\pi a_2} - \frac{q''_1}{2\pi a_2} = \frac{1}{2\pi} \left(\frac{q_1}{a_1} - \frac{k_1 - k_2}{k_1 + k_2} \frac{q_1}{a_1} - \frac{2k_1}{k_1 + k_2} \frac{q_2}{a_2} \right)$$

$$-v_{2n} = \frac{q_2}{2\pi a_2} - \frac{q'_2}{2\pi a_1} - \frac{q''_2}{2\pi a_2} = \frac{1}{2\pi} \left(\frac{q_2}{a_2} - \frac{2k_2}{k_1 + k_2} \frac{q_1}{a_1} + \frac{k_1 - k_2}{k_1 + k_2} \frac{q_2}{a_2} \right)$$

the condition that $|v_{1n}| = |v_{2n}|$ yields

$$\frac{q_1}{a_1} \left(1 - \frac{k_1 - k_2}{k_1 + k_2} \right) - \frac{2k_1}{k_1 + k_2} \frac{q_2}{a_2} = \frac{q_2}{a_2} \left(1 + \frac{k_1 - k_2}{k_1 + k_2} \right) - \frac{2k_2}{k_1 + k_2} \frac{q_1}{a_1}$$

which leads to

$$\frac{k_2 q_1}{a_1} = \frac{k_1 q_2}{a_2}, \quad \frac{a_2}{a_1} = \frac{k_1 q_2}{k_2 q_1} \quad (4.10.15)$$

Now assume that the stream of water flowing into the sink q_1 is separated from the stream of water flowing into the sink q_2 by linear dividing streamlines (Fig. 4.17d). The water comes from great distances where the equipotential lines are circular. The angle τ between the dividing streamlines and the line connecting the two sinks, can be found from this condition, for we have

$$\frac{q_1}{q_2} = \frac{\pi k_1 + 2[(\pi/2) - \tau] k_2}{2k_2 \tau} = \frac{\pi(k_1 + k_2)}{2k_2 \tau} - 1 \quad (4.10.16)$$

and simultaneously

$$\tau = \frac{\pi q_2 (k_1 + k_2)}{2k_2 (q_1 + q_2)} \quad (4.10.18)$$

Substituting eqn. (4.10.16) in eqn. (4.10.15) we have

$$\frac{a_2}{a_1} = \frac{k_1}{k_2} \left/ \left[\frac{\pi(k_1 + k_2)}{2k_2 \tau} - 1 \right] \right. \quad (4.10.18)$$

Suppose now that the flow being discussed is represented in the complex plane (ζ), with the sinks lying on the real axis ξ . The abscissae of the points of the sinks are ξ_{a1} , ξ_{a2} . We shall map the (ζ) plane conformally onto the (y') plane.

We start by first mapping the (ζ) plane onto the auxiliary plane (ζ')

$$\zeta' = \sqrt{\xi}$$

The (ζ) plane transforms into the half-plane (ζ'). The negative part of the real axis ξ transforms into the imaginary axis $i\eta'$. All the rays in the (ζ) plane issuing from the origin and making an angle τ with the ξ axis, transform into corresponding rays issuing from the origin in the (ζ') half-plane. However, after the transformation, the angle between the rays and the real axis ζ' will be halved. From the above exposition follows the new position of the dividing streamlines. The sink q_2 will split into halves, each of which will transform into the real axis of the (ζ') plane.

In the next step we make use of the transformation function

$$\zeta' = \sin \frac{\pi y'}{4T_1}$$

which we have already analyzed several times. This function maps the (ζ') plane onto the (y') plane. Of special interest is a strip of width $4T_1$ lying parallel to the ix' axis. We evidently have inside this strip

$$\zeta = \sin^2 \frac{\pi y'}{4T_1} \quad (4.10.19)$$

According to eqn. (4.10.11), to the left of the imaginary axis we have

$$\begin{aligned} -h_1 = \frac{1}{2\pi k_1} & \left[q_1 \ln |\zeta - \zeta_{a1}| + q_1 \frac{k_1 - k_2}{k_1 + k_2} \ln |\zeta + \zeta_{a1}| + \right. \\ & \left. + q_2 \frac{2k_1}{k_1 + k_2} \ln |\zeta + \zeta_{a2}| \right] + c_1 \end{aligned} \quad (4.10.20)$$

Similarly, according to (4.10.12)

$$\begin{aligned} -h_2 = \frac{1}{2\pi k_2} & \left[q_2 \ln |\zeta + \zeta_{a2}| - q_2 \frac{k_1 - k_2}{k_1 + k_2} \ln |\zeta - \zeta_{a2}| + \right. \\ & \left. + q_1 \frac{2k_2}{k_1 + k_2} \ln |\zeta - \zeta_{a1}| \right] + c_2 \end{aligned} \quad (4.10.21)$$

Considering Fig. 4.17e we see that the interface between the layers of soil as well as the dividing streamlines will map into curves in the (y') plane. At greater distances from the origin the curves will change to straight lines parallel to the imaginary axis ix' , with T_2 always smaller than T_1 . The ratio between T_2 and T_1 depends on the value of the angle τ in the (ζ) plane:

$$\tau = \frac{\pi(T_1 - T_2)}{2T_1}$$

Substituting in eqn. (4.10.16) we obtain

$$\frac{q_2}{q_1} = \frac{2k_2\tau}{\pi(k_1 + k_2) - 2k_2\tau} = \frac{k_2(T_1 - T_2)}{T_1(k_1 + k_2) - k_2(T_1 - T_2)} = \frac{k_2(T_1 - T_2)}{k_1T_1 + k_2T_2} \quad (4.10.22)$$

and on substituting in eqn. (4.10.15) we have

$$\frac{\zeta_{a2}}{\zeta_{a1}} = \frac{k_1(T_1 - T_2)}{k_1T_1 + k_2T_2} \quad (4.10.23)$$

Eqns. (4.10.22) and (4.10.23) enable us to express q_2/ξ_{a2} as linear functions of q_1, ξ_{a1} . On substituting the result in eqn. (4.10.20) we obtain

$$\begin{aligned} -h_1 = & \frac{q_1}{2\pi k_1} \left[\ln |\zeta - \zeta_{a1}| + \frac{k_1 - k_2}{k_1 + k_2} \ln |\zeta + \zeta_{a1}| + \right. \\ & \left. + \frac{k_2(T_1 - T_2)}{k_1 T_1 + k_2 T_2} \frac{2k_1}{k_1 + k_2} \ln \left| \zeta + \frac{k_1(T_1 - T_2)}{k_1 T_1 + k_2 T_2} \zeta_{a1} \right| \right] + c_1 \quad (4.10.24) \end{aligned}$$

Together with eqn. (4.10.19), eqn. (4.10.24) provides a description of the situation obtaining in that part of the (y') plane which is characterized by the coefficient of permeability k_1 . If we assume the sink to be very remote from the origin, the pattern of flow will resemble that shown in Fig. 4.17f. The river water enters a permeable stratum consisting of two layers of different permeabilities. The thickness of the lower layer is less than that of the upper one. At a sufficiently great distance from the river is a canal cut through an overburden of relatively smaller permeability of the upper permeable layer. The canal bottom is of approximately circular perimeter (radius r_0). The dimensions of the canal bottom are small compared with the thickness of the upper layer. The surface of the relatively impervious subsoil was formed with the help of the sink q_2 in which we are no longer interested.

The above idea indicates the subsequent procedure of the solution in which we assume that the origin of the mapping plane has been moved directly to the sink. In the new (y) plane

$$y' = \mathcal{L} + y$$

where $\mathcal{L}$ is a very large value.

On the imaginary axis of the (y') plane

$$\eta' = \sinh \frac{\pi y'}{4T_1} = \sinh \frac{\pi(\mathcal{L} + y)}{4T_1}$$

Taking note of the magnitude of $\mathcal{L}$, we may write, according to eqn. (4.10.19),

$$\zeta = \frac{1}{4} \text{Exp} \frac{\pi(\mathcal{L} + y)}{2T_1}$$

Using the notation

$$\begin{aligned} A &= \frac{2k_1 k_2 (T_1 - T_2)}{(k_1 + k_2)(T_1 k_1 + T_2 k_2)} = \frac{2k_2 B}{k_1 + k_2} \\ B &= \frac{k_1(T_1 - T_2)}{T_1 k_1 + T_2 k_2} \end{aligned}$$

eqn. (4.10.24) gives along the imaginary axis in the (y') plane

$$\begin{aligned} -h_1 = & \frac{q}{\pi k_1} \left\{ \ln \left[\frac{\text{Exp}(\pi \mathcal{L}/2T_1)}{4} \left| \text{Exp} \frac{\pi x}{2T_1} - 1 \right| \right] + \right. \\ & + \frac{k_1 - k_2}{k_1 + k_2} \ln \left[\frac{\text{Exp}(\pi \mathcal{L}/2T_1)}{4} \left| \text{Exp} \frac{\pi x}{2T_1} + 1 \right| \right] + \\ & \left. + A \ln \left[\frac{\text{Exp}(\pi \mathcal{L}/2T_1)}{4} \left| \text{Exp} \frac{\pi x}{2T_1} + B^2 \right| \right] \right\} + c_1 \end{aligned} \quad (4.10.25)$$

where q is the volume flow through the canal bottom.

At the point $x = r_0$, $h_1 = h_0$, at the point $x = L$, $h_1 = h_L$. From these conditions we determine the constant c_1 . On substituting it in eqn. (4.10.25) we have

$$\begin{aligned} h_L - h_1 = & -\frac{q}{\pi k_1} \left[\ln \frac{\text{Exp}(\pi L/2T_1) - 1}{\text{Exp}(\pi r_0/2T_1) - 1} + \frac{k_1 - k_2}{k_1 + k_2} \ln \frac{\text{Exp}(\pi L/2T_1) + 1}{\text{Exp}(\pi r_0/2T_1) + 1} + \right. \\ & \left. + A \ln \frac{\text{Exp}(\pi L/2T_1) + B^2}{\text{Exp}(\pi r_0/2T_1) + B^2} \right] \end{aligned} \quad (4.10.26)$$

$$q = \frac{-\pi k_1 (h_L - h_0)}{\ln \frac{\text{Exp}(\pi L/2T_1) - 1}{\text{Exp}(\pi r_0/2T_1) - 1} + \frac{k_1 - k_2}{k_1 + k_2} \ln \frac{\text{Exp}(\pi L/2T_1) + 1}{\text{Exp}(\pi r_0/2T_1) + 1} + A \ln \frac{\text{Exp}(\pi L/2T_1) + B^2}{\text{Exp}(\pi r_0/2T_1) + B^2}} \quad (4.10.27)$$

4.11. Flow in media with continuously varying permeability

We have outlined the solution to cases when a given region contains permeable zones, each with definite properties. The zones with different properties link up with one another in a clearly defined way. In practice, one also comes across configurations of permeable media in which the coefficient of permeability varies gradually, continuously. At each point, a coefficient of permeability k is given, which is a function of position. The initial equation applying to a two-dimensional flow defined in the x, z system, is

$$\frac{\partial}{\partial x} \left(k \frac{\partial h}{\partial x} \right) + \frac{\partial}{\partial z} \left(k \frac{\partial h}{\partial z} \right) = 0 \quad (4.11.1)$$

where

h — piezometric head at a point of the x, z plane.

The equation can also be written as

$$k \left(\frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial z^2} \right) + \frac{\partial k}{\partial x} \frac{\partial h}{\partial x} + \frac{\partial k}{\partial z} \frac{\partial h}{\partial z} = 0 \quad (4.11.2)$$

The second and third term on the left-hand side of eqn. (4.11.2) can be neglected, provided that

$$\frac{\partial k}{\partial x} \frac{\partial h}{\partial x} \rightarrow 0, \quad \frac{\partial k}{\partial z} \frac{\partial h}{\partial z} \rightarrow 0 \quad (4.11.3)$$

In this case we can use eqn. (3.1.6). The invalidity of eqn. (4.11.3) compels us to solve more complicated relationships. In a number of problems, the task is simplified by the introduction of a substitution based on the assumption that we know the functional dependence of the coefficient of permeability on the position of a point in the x, z plane

$$k = k_0 f(x, z) \quad (4.11.4)$$

where

- $f(x, z)$ – a continuous function,
- k_0 – a fixed, chosen value of the coefficient of permeability,
- k – value of the coefficient of permeability at a given point of the x, z plane.

Consider now the function $F(x, z, k_0 h)$

$$F = k_0 h \sqrt{f} \quad (4.11.5)$$

where

h – piezometric head at a given point of the x, z plane.

The partial derivatives of eqn. (4.11.5) in the directions of the coordinate axes are

$$\frac{\partial F}{\partial x} = k_0 \left(\sqrt{f} \frac{\partial h}{\partial x} + \frac{h}{2\sqrt{f}} \frac{\partial f}{\partial x} \right), \quad \frac{\partial F}{\partial z} = k_0 \left(\sqrt{f} \frac{\partial h}{\partial z} + \frac{h}{2\sqrt{f}} \frac{\partial f}{\partial z} \right)$$

By differentiating again and rearrangement we obtain the relations

$$\begin{aligned} \frac{\partial^2 F}{\partial x^2} &= k_0 \left[\sqrt{f} \frac{\partial^2 h}{\partial x^2} + \frac{1}{\sqrt{f}} \frac{\partial h}{\partial x} \frac{\partial f}{\partial x} + h \frac{\partial^2(\sqrt{f})}{\partial x^2} \right] \\ \frac{\partial^2 F}{\partial z^2} &= k_0 \left[\sqrt{f} \frac{\partial^2 h}{\partial z^2} + \frac{1}{\sqrt{f}} \frac{\partial h}{\partial z} \frac{\partial f}{\partial z} + h \frac{\partial^2(\sqrt{f})}{\partial z^2} \right] \end{aligned}$$

Hence

$$\begin{aligned} \sqrt{f} \left(\frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial z^2} \right) &= k_0 \left(f \frac{\partial^2 h}{\partial x^2} + \frac{\partial h}{\partial x} \frac{\partial f}{\partial x} + f \frac{\partial^2 h}{\partial z^2} + \frac{\partial h}{\partial z} \frac{\partial f}{\partial z} \right) + \\ &+ k_0 h \sqrt{f} \left[\frac{\partial^2(\sqrt{f})}{\partial x^2} + \frac{\partial^2(\sqrt{f})}{\partial z^2} \right] = \frac{\partial}{\partial x} \left(k_0 f \frac{\partial h}{\partial x} \right) + \frac{\partial}{\partial z} \left(k_0 f \frac{\partial h}{\partial z} \right) + \\ &+ F \left[\frac{\partial^2(\sqrt{f})}{\partial x^2} + \frac{\partial^2(\sqrt{f})}{\partial z^2} \right] \end{aligned}$$

is k_T , and it is assumed that this value can be used approximately in the range of the outer contour of the lining. It is also assumed that through the lining with a coefficient of permeability k_b , the water flows axi-symmetrically. The resistance of the lining is expressed by the resistance of an equivalent ring of material of natural rock. In other words, we assume that the water enters a smaller tunnel without lining, having a radius r_0 .

We are thus comparing the hydraulic resistances of two rings formed of two different materials. The piezometric head is h_T on the outside, and h_D on the inside of the ring. According to eqns. (4.8.1) and (1.4.16) we have for an axi-symmetrical flow

$$k_b h_D = \frac{q}{2\pi} \ln r_D + c$$

$$k_b h_T = \frac{q}{2\pi} \ln r_T + c$$

so that

$$k_b(h_D - h_T) = \frac{q}{2\pi} \ln \frac{r_D}{r_T}$$

where

- k_b — coefficient of permeability of the lining,
- q — quantity of seepage percolating through the whole lining,
- r_D — radius of outer contour of the lining,
- r_T — radius of inner contour of the lining,
- h_D, h_T — piezometric heads on the inside and outside of the lining.

The piezometric heads are measured from a reference plane placed at the level of the free surface. Hence $h = z - p/\gamma$.

The equivalent ring of natural rock is bounded by a circle of radius r_D on the outside, and by a circle of radius r_0 on the inside. Since the corresponding piezometric heads are h_D and h_T ,

$$k_T(h_D - h_T) = \frac{q}{2\pi} \ln \frac{r_D}{r_0}$$

where

- k_T — coefficient of permeability at a depth corresponding to the position of the tunnel axis.

Since the quantity q flows through both rings, we have

$$\frac{k_T}{\ln(r_D/r_T)} = \frac{k_b}{\ln(r_D/r_0)}$$

and from this equation we obtain the radius of the substitute tunnel

$$r_0 = r_D \text{Exp} \left(- \frac{k_T}{k_b} \ln \frac{r_D}{r_T} \right) \quad (4.11.8)$$

It is, of course, assumed that $k_b < k_T$. Therefore $r_0 < r_D$. The difference between the radii of the outer and inner contour of the substitute ring of natural rock, is the substitute length with the aid of which we describe the effect of the lining. Moreover, we have simplified the problem by examining the flow in the x, z region in which the permeability is assumed to vary continuously.

Hencky's hypothesis can be the point of departure for deriving the effect of the weight of the overburden on the seepage properties of the rock. At a given point, the dependence of strain on the mean of principal stresses is assumed to be logarithmic. The linear law which we shall employ in the subsequent discussion, is acceptable within the range of comparatively small strains. The value of the coefficients of permeability decreases with depth and we have

$$k = k_0 \text{Exp} (-Cz) \quad (4.11.9)$$

Hence

$$k_T = k_0 \text{Exp} (-Cz_T) \quad (4.11.10)$$

where

C — is a constant with the dimension of $[m^{-1}]$,

z_T — the ordinate of the tunnel axis.

The free surface runs in the region with a comparatively low overburden, where the material is relatively more permeable. Since we suppose that the seepage into the tunnel will be small, we can accept the assumption of a horizontal course for the free surface (Fig. 4.18). We place the x axis at the level of the free surface and choose the coefficient of permeability on this level $k = k_0$.

The z axis points downward; hence the continuous variation of permeability is described by the function

$$f = \text{Exp} (-Cz) \quad (4.11.11)$$

On the right-hand side of eqn. (4.11.6) we have

$$\frac{F}{\sqrt{f}} \left[\frac{\partial^2(\sqrt{f})}{\partial x^2} + \frac{\partial^2(\sqrt{f})}{\partial z^2} \right] = \frac{C^2 F}{4} \quad (4.11.12)$$

In place of eqn. (4.11.6) we obtain the equation

$$\frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial z^2} - \frac{C^2}{4} F = 0 \quad (4.11.13)$$

The pressure on the free surface is assumed to be zero. Since there $z = 0$, we have $h = 0$ and also $F = 0$ on the x axis. The discharge near the tunnel is assumed to be axi-symmetrical. These conditions are satisfied by the solution

$$F = \alpha \left[K_0 \left\{ \frac{C}{2} \sqrt{x^2 + (z - z_T)^2} \right\} - K_0 \left\{ \frac{C}{2} \sqrt{x^2 + (z + z_T)^2} \right\} \right] \quad (4.11.14)$$

where $K_0(\xi)$ is the modified zero-order Bessel function of the second kind (the McDonald function), and α the constant of integration.

On the circumference of the substitute tunnel, $z = z_T$, $x = r_0$. Since simultaneously $z_T \gg r_0$,

$$F_T = \alpha \left[K_0 \left(\frac{Cr_0}{2} \right) - K_0(Cz_T) \right] \quad (4.11.15)$$

Suppose that

$$F_T = k_0 h_T \text{Exp} \left(-\frac{Cz_T}{2} \right)$$

so that

$$\alpha = \frac{k_0 h_T \text{Exp} \left(-\frac{1}{2} Cz_T \right)}{K_0\left(\frac{1}{2} Cr_0\right) - K_0(Cz_T)} \quad (4.11.16)$$

After calculating the constant of integration α and substituting it in eqn. (4.11.14) we have

$$h = h_T \frac{K_0\left\{\frac{1}{2} C \sqrt{x^2 + (z - z_T)^2}\right\} - K_0\left\{\frac{1}{2} C \sqrt{x^2 + (z + z_T)^2}\right\}}{K_0\left(\frac{1}{2} Cr_0\right) - K_0(Cz_T)} \text{Exp} \left[\frac{C(z - z_T)}{2} \right] \quad (4.11.17)$$

On the outside of the lining, where $z = z_T$, $x = r_D$, $z_T \gg r_D$, we have, approximately,

$$h_D = h_T \frac{K_0\left(\frac{1}{2} Cr_D\right) - K_0(Cz_T)}{K_0\left(\frac{1}{2} Cr_0\right) - K_0(Cz_T)} \quad (4.11.18)$$

A flow of water takes place in the permeable medium. The pressure manifests itself through a neutral and an effective stress. The first is co-determined by the pressure head

$$\frac{p}{\gamma} = z - h \quad (4.11.19)$$

where p is the pressure, and γ the unit weight of water. The effective stress induced by the flow, is co-determined by the gradient h .

Eqn. (4.11.17) readily yields the components of the gradient in the directions of the coordinate axes, viz.

$$\frac{\partial h}{\partial x} = \frac{-h_T C x \text{Exp} \left[\frac{1}{2} C (z - z_T) \right]}{2[K_0(\frac{1}{2} C r_0) - K_0(C z_T)]} \times \\ \times \left[\frac{K_1\{\frac{1}{2} C \sqrt{[x^2 + (z - z_T)^2]}\}}{\sqrt{[x^2 + (z - z_T)^2]}} - \frac{K_1\{\frac{1}{2} C \sqrt{[x^2 + (z + z_T)^2]}\}}{\sqrt{[x^2 + (z + z_T)^2]}} \right] \quad (4.11.20)$$

$$\frac{\partial h}{\partial z} = \frac{C}{2} \left\{ h - \frac{h_T \text{Exp} \left[\frac{1}{2} C (z - z_T) \right]}{K_0(\frac{1}{2} C r_0) - K_0(C z_T)} \times \right. \\ \times \left[\frac{z - z_T}{\sqrt{[x^2 + (z - z_T)^2]}} K_1\{\frac{1}{2} C \sqrt{[x^2 + (z - z_T)^2]}\} - \frac{z + z_T}{\sqrt{[x^2 + (z + z_T)^2]}} \times \right. \\ \left. \left. \times K_1\{\frac{1}{2} C \sqrt{[x^2 + (z + z_T)^2]}\} \right] \right\} \quad (4.11.21)$$

where $K_1(\zeta)$ is the modified first-order Bessel function of the first kind.

Since the discharge into the substitute tunnel is assumed to be symmetrical, we have

$$q = -2\pi r_0 \frac{\partial h}{\partial x} \Big|_{x=r_0, z=z_T} = \frac{\pi C r_0 h_T k_T}{K_0(\frac{1}{2} C r_0) - K_0(C z_T)} \left[K_1\left(\frac{C r_0}{2}\right) - \frac{r_0}{2 z_T} K_1(C z_T) \right] \quad (4.11.22)$$

and, neglecting the second term in the square brackets,

$$q = \frac{\pi C r_0 h_T k_T K_1(\frac{1}{2} C r_0)}{K_0(\frac{1}{2} C r_0) - K_0(C z_T)} \quad (4.11.23)$$

The approximate nature of the solution presented above is implicit in the assumptions. As eqn. (1.11.17) implies, it is not exactly true that h is constant on the inner contour of the substitute ring. The equipotential lines are almost circular, however.

We should perhaps mention the fact that the pressures and the gradients of the axi-symmetrical flow inside the lining are related in a simple way to the pressures and the gradients in the substitute ring. The relevant calculation is omitted, as it would yield nothing new methodologically.

The solution ignores the possible effect of the pressure of the flowing water on the rock permeability, as well as the effect of the state of stress in the neighbourhood of the tunnel on the seepage properties of the material. Compared to the action of the weight of the rock which co-determines the value of the coefficient of permeability, these effects are assumed to be relatively small from the hydraulic standpoint. Obviously, such an assumption is not always acceptable, and a solution of practical problems should be preceded by an analysis of the geological structure of the terrain,

properties of the permeable medium and static conditions. The approximate solution outlined in this section is applicable to cases of unyielding lining, capable of withstanding the effects of all applied forces.

4.12. Problem of steady-state transfer of moisture

The basic ideas of the methods of approximate solutions of various problems have also been found useful in studies concerned with fields defined by linear functions or by functions amenable to linearization. As an example, consider the problem of transfer of moisture in the mathematical description of which we have employed the concept of the moisture velocity potential $W(x, z)$. This is a function whose derivatives in the directions of the coordinate axes give the velocity components along those axes. The unsaturated zone is regarded as a continuum.

Since in heavy soils the effect of gravitation is not predominant, the steady-state transfer of moisture is defined by eqn. (1.6.69), rearranged as follows:

$$\frac{\partial^2 W}{\partial x^2} + \frac{\partial^2 W}{\partial z^2} = 0 \quad (4.12.1)$$

The value of the function $W(x, z)$ depends on moisture and has a maximum in completely saturated soils. Eqn. (4.12.1) is amenable to solution under prescribed boundary conditions.

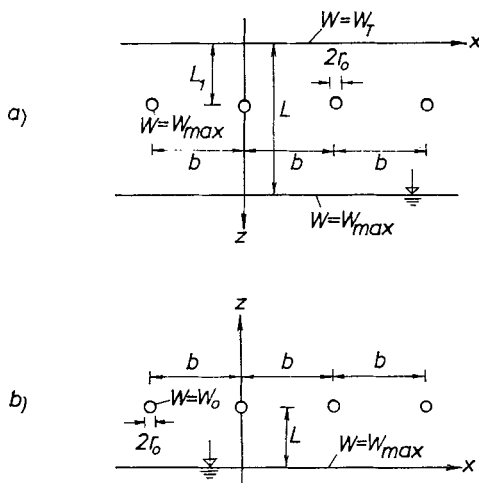


Fig. 4.19. Irrigation drainage.

Consider the case shown schematically in Fig. 4.19a. Above the free surface of gravitational water, there exists an unsaturated zone. To the moisture maintained on the surface of the terrain, corresponds the value W_T . At a depth L_1 is placed

a group of drains filled with water. Similarly as on the free surface, $W = W_{\max}$ on the perimeter of the drains. It is assumed that

$$\frac{L_1}{b} > 0.5, \quad \frac{L - L_1}{b} > 0.5, \quad \frac{r_0}{b} \ll 1 \quad (4.12.2)$$

Methodically, the solution of eqn. (4.12.1) will be based on the principle of superposition of the moisture velocity potential and the superposition of velocities. Since the procedure is similar to that used earlier, we merely write down the resulting equation

$$W = \frac{q}{4\pi} \ln \frac{\cosh[2\pi(z + L_1)/b] - \cos(2\pi x/b)}{\cosh[2\pi(z - L_1)/b] - \cos(2\pi x/b)} + \frac{q_0 z}{b} + c \quad (4.12.3)$$

where

q — quantity of moisture emanating from a drain,

c — a constant,

q_0 — quantity of moisture emanating from the free surface toward the surface of terrain in a strip of width b ,

b — spacing of drains.

On the straight line $z = 0$, $W = W_T$, at the point $z = L_1$, $x = r_0$, $W = W_{\max}$. Similarly, on the level $z = L$, we assume that $W \approx W_{\max}$. Hence $c = W_T$, and

$$\begin{aligned} & \frac{W_{\max} - W}{W_{\max} - W_T} = \\ & \frac{\ln \frac{\cosh[2\pi(z + L_1)/b] - \cos(2\pi x/b)}{\cosh[2\pi(z - L_1)/b] - \cos(2\pi x/b)} - \frac{z}{L_1} \ln \frac{\cosh(4\pi L_1/b) - \cos(2\pi r_0/b)}{1 - \cos(2\pi r_0/b)}}{\ln \frac{\cosh[2\pi(L + L_1)/b] - 1}{\cosh[2\pi(L - L_1)/b] - 1} - \frac{L}{L_1} \ln \frac{\cosh[4\pi L_1/b] - \cos(2\pi r_0/b)}{1 - \cos(2\pi r_0/b)}} \end{aligned} \quad (4.12.4)$$

Moisture at a point is calculated from eqn. (1.6.56) rearranged as follows:

$$w = w_0 + {}^{n-1}\sqrt{\left(\frac{n-1}{C} W\right)} \quad (4.12.5)$$

where

n — exponent in Averyanov's formula,

C — coefficient of moisture conductivity,

w_0 — threshold moisture, not necessarily identical with the moisture w_T on the terrain surface, in terms of which we have defined W_T .

We have, further,

$$q = \frac{4\pi(W_T - W_{\max})}{\ln \frac{\cosh [2\pi(L + L_1)/b] - 1}{\cosh [2\pi(L - L_1)/b] - 1} - \frac{L}{L_1} \ln \frac{\cosh (4\pi L_1/b) - \cos (2\pi r_0/b)}{1 - \cos (2\pi r_0/b)}} \quad (4.12.6)$$

$$q_0 = - \frac{1}{L_1} \frac{q}{4\pi} \ln \frac{\cosh (4\pi L_1/b) - \cos (2\pi r_0/b)}{1 - \cos (2\pi r_0/b)} \quad (4.12.7)$$

Theoretically, the inflow to the terrain surface should be equal to the evapotranspiration characterized by the velocity v_0 ,

$$v_0 = \int_{-b/2}^{+b/2} \frac{dW}{dz} dx = \frac{1}{b} (q_0 + q) \quad (4.12.8)$$

The theory becomes invalid once the sum of the inflows to the terrain surface becomes smaller than evaporation and the biological consumption of plants. The prescribed moisture W_T would be hard to maintain without irrigation.

The problem can be specified in another way. We are interested in knowing the resulting moisture effect on the terrain surface when a system of drains spaced b apart is situated at a depth L_1 . The radius of the collecting drains is r_0 . The calculation is carried out using eqns. (4.12.6), (4.12.7). The quantities q , q_0 are obtained for different W_T chosen successively. The value of W_T for which eqn. (4.12.8) is satisfied, will be considered acceptable. If the first trial is not successful, the position and the spacing of the drains are changed, and the procedure is repeated.

Construction of irrigation drains poses a number of additional problems. From the point of view of hydraulics, the most significant by far is the effect of the technology of the construction of the various elements. When the elements are placed in open ditches, the medium becomes non-homogeneous because the back fill of the ditches is usually more permeable than the natural soil. It is recommended that due regard be paid to practical experience and applicable rules which are not based on theory alone.

If applied on its own, the diffusion theory also has some shortcomings. By way of illustration, we consider the example shown schematically in Fig. 4.19b. The brickwork of a building extends below the free surface of groundwater. Water rises, and moisture is prevented from propagating by a system of drain pipes. The effect of evaporation through the face and back of the wall is ignored.

The problem is defined with the help of eqn. (4.12.1), with $W = W_{\max}$ in $z = 0$, and the minimum value W_0 corresponding to w_0 assumed to be at $v = r_0$, $z = L$. It is readily established that

$$\frac{W - W_0}{W_{\max} - W_0} = 1 - \frac{\ln \frac{\cosh [2\pi(z + L)/b] - \cos (2\pi x/b)}{\cosh [2\pi(z - L)/b] - \cos (2\pi x/b)}}{\ln \frac{\cosh (4\pi L/b) - \cos (2\pi r_0/b)}{1 - \cos (2\pi r_0/b)}} \quad (4.12.9)$$

For points with a large ordinate z , the above simplifies to

$$\frac{W_z - W_0}{W_{\max} - W_0} = 1 - \frac{4\pi L}{b \ln \frac{\cosh(4\pi L/b) - \cos(2\pi r_0/b)}{1 - \cos(2\pi r_0/b)}} \quad (4.12.10)$$

where W_z is the value of the moisture velocity potential at points $z \rightarrow \infty$, i.e. at great heights above the axis of the drains.

If the spacing of the drain pipes is chosen to be quite large, i.e. on the condition that

$$\frac{L}{b} \rightarrow 0, \quad \frac{r_0}{b} \rightarrow 0 \quad (4.12.11)$$

we obtain

$$\frac{W_z - W_0}{W_{\max} - W_0} = 1, \quad W_z = W_{\max} \quad (4.12.12)$$

As eqns. (4.12.12) suggest, the wall — regardless of its height — will be completely saturated in most cases. This conclusion of our solution is at variance with theory which sets the maximum possible height of capillary elevation at 10 m.

Bearing in mind the unsteady moisture transfer, the advocates of the diffusion theory claim that its applicability is restricted to dynamic phenomena. What they fail to specify is where the boundary between stationarity and non-stationarity should be placed.

We can accept the hypothesis that the diffusion theory will be of use in cases involving moisture transfer, i.e. on the assumption of water motion in an unsaturated zone. We must, of course, determine the limit velocity at which the motion will cease. Practical solutions call for corrections paying heed to any available experience.

4.13. Superposition of pressures

Another function which, according to Section 1.4, can be used to describe a flow in permeable media, is the dependence of the pressure p on the position of a point in space. Under the prevailing effect of gravity, we examine the consequences of force action and determine the components of acceleration with the help of the force potential. The gravitational force potential is a continuous function which is defined in both the saturated and the unsaturated zones.

A steady-state flow in a saturated medium, under the predominant effect of gravity, is described by the Laplace equation (1.4.20c) which can be solved for given boundary conditions. One of these conditions can be the free surface of gravitational water. The pressure along this line is assumed to be atmospheric, regarded as relatively zero ($p = 0$). There is no water in the pores above the free surface. However, this part

of the space can be regarded as a medium filled with water; the force which acts there, is in a direction opposite to that acting below the free surface. Fictitious water is virtually weightless but within the set of forces being considered we must also account for factors which have "annulled" the effect of gravity.

This approach makes it possible to continualize the whole space irrespective of whether or not a free surface exists there. We can use the example shown in Fig. 4.20a to illustrate our remarks. Inside a massif situated between two reservoirs, is a tunnel from which percolating water is drawn off. The tunnel duct is filled with air at atmospheric pressure ($p = 0$). The air pressure also acts on the levels of water in the reservoirs. The pressure below the level there increases linearly with depth. Above the levels in the reservoirs, on the other hand, the water pressure is assumed to diminish linearly. The pressures have negative values.

From below, the massif is bounded by the surface of a deep-situated relatively impervious layer. Hence we know all the boundary conditions for constructing a square grid. The lines $p = \text{const.}$ are isobars, and the isobar which belongs to the free surface on which $p = 0$, is also the depression curve. In the case being considered the free surface terminates on the tunnel contour so that from a portion of the tunnel duct perimeter water falls in drops into air. For a different configuration of the boundary conditions we would find the depression curve to be continuous and passing above the tunnel. In that case free seepage of water would be observed on the whole perimeter of the tunnel duct.

The family of isobars enables us to calculate the velocity potential φ because

$$\varphi = -k \left(\frac{p}{\gamma} + z \right)$$

where

k — coefficient of permeability,

γ — unit weight of water,

z — vertical ordinate of a point.

We can easily construct a flow net and discover that the effect of the relatively impervious subsoil can be found by means of the mirror representation, or in other words, by the application of the principle of superposition (Section 4.8). This shows that the use of the mirror representation is also correct in the case under consideration.

The superposition of pressures on which the mirror representation method is based, facilitates numerical solution of some practical cases. Consider, for example, the earth massif shown schematically in Fig. 4.20b. Inside the massif is a pipe with a substitute diameter (see Sect. 4.11). The pressure in the pipe is assumed to be p_0 . Since the height of the pipe centreline above the surface of the impervious subsoil is comparatively large, we can assume the pressure on the x axis to be $p = \gamma H$. On the boundaries, the pressure increases linearly with the depth below the water level in the reservoirs.

The boundary condition is satisfied by the superposition of two pressure fields. Consider first the action of a sink at the point $x = a$, $z = b$, with zero pressure on the boundaries. The field is defined by the pressure p' , and inside the flow region

$$p' = c_1 \ln \frac{\{\cosh [\pi(z - b)/L] - \cos [\pi(x - a)/L]\} \{\cosh [\pi(z + b)/L] - \cos [\pi(x + a)/L]\}}{\{\cosh [\pi(z - b)/L] - \cos [\pi(x + a)/L]\} \{\cosh [\pi(z + b)/L] - \cos [\pi(x - a)/L]\}} \quad (4.13.1)$$

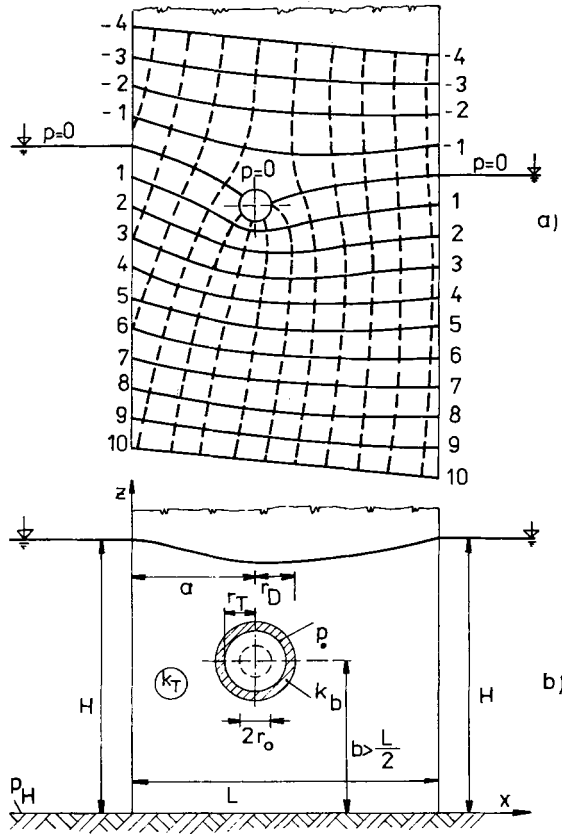


Fig. 4.20. Superposition of pressures.

Consider simultaneously another pressure field defined by the function p'' : we assume that

$$p'' = c_2 z \quad (4.13.2)$$

inside the massif as well as on its boundaries.

Superposing the two partial effects we obtain the description of the resulting field

$$p = p' + p'' + c_3 \quad (4.13.3)$$

where c_1, c_2, c_3 are constants.

The function (4.13.3) satisfies the requirement of a linear growth of pressure on the boundaries, as it depends on the depth below the level in the reservoirs. The values of the constants of eqn. (4.13.4) are obtained from the following conditions:

$$\begin{aligned} p &= 0 & \text{at the point } x = 0, \quad z = H \\ p &= p_H & \text{at the point } z = 0 \\ p &= p_0 & \text{at the point } x = a + r_0, \quad z = b \end{aligned} \quad (4.13.4)$$

where p_0 is the average pressure inside of the tunnel.

As the last of conditions (4.13.4) suggests, the effect of pressure variations is neglected within the range of the duct contour, and the isobars in the neighbourhood of the sink are assumed to be circular. In this sense, the solution is approximate.

The equation obtained after calculating the constants is:

$$\begin{aligned} p = & \ln \frac{\{\cosh [\pi(z - b)/L] - \cos [\pi(x - a)/L]\} \cdot \{\cosh [\pi(z + b)/L] - \cos [\pi(x + a)/L]\}}{\{\cosh [\pi(z - b)/L] - \cos [\pi(x + a)/L]\} \cdot \{\cosh [\pi(z + b)/L] - \cos [\pi(x - a)/L]\}} + \\ = & \left[p_0 - p_H \left(1 - \frac{b}{H} \right) \right] \ln \frac{[1 - \cos (\pi r_0/L)] \{\cosh (2\pi b/L) - \cos [\pi(2a + r_0)/L]\}}{\{1 - \cos [\pi(2a + r_0)/L]\} [\cosh (2\pi b/L) - \cos (\pi r_0/L)]} + \\ & + p_H \left(1 - \frac{z}{H} \right) \end{aligned} \quad (4.13.5)$$

Since $p = 0$ on the free surface, we can use eqn. (4.13.5) to determine the positions of the points of the depression curve. For a chosen x , the corresponding z is obtained by successive approximations.

The method of superposition of pressures is not intended exclusively for problems with the free surface as a boundary. It has general application and is sometimes helpful in studies of problems not clearly defined. Thus, for example, one might not know whether or not a free surface exists at all. Engineering interpretation does not even require that all the conditions should be known in detail.

As an example consider the problem of an underpass underneath a river effected by the method of protected roof (Fig. 4.21a). The figure shows in dashed lines a road tunnel which is to be built immediately below the river bottom. This is made feasible by means of an impervious slab which forms the tunnel roof and assists in overcoming the effect of uplift on the finished structure. In the preparatory phase, before beginning to construct the tunnel, two galleries with permeable timbering are provided at a suf-

ficient depth. Frontal hermetical closures will enclose in them cylindrical spaces from which air can be evacuated. Evacuation produces the sub-atmospheric pressure p_0 . Its effect is to give rise at the point 2 to a pressure $p = 0$ which corresponds to atmospheric. The driving of the tunnel can then go ahead in favourable conditions under the protection of the slab. The quantity of interest is the value of the necessary sub-atmospheric pressure.

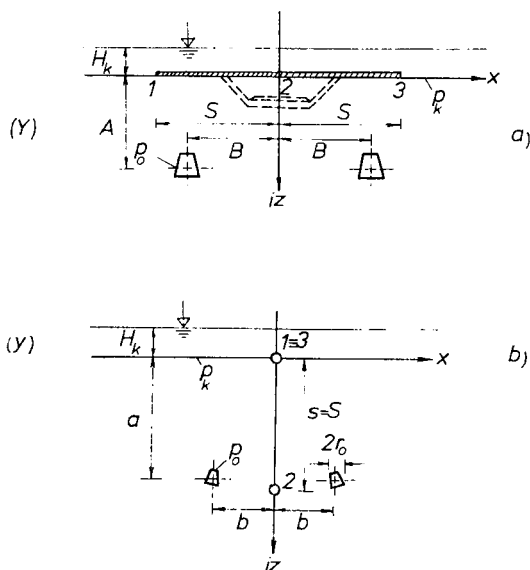


Fig. 4.21. Method of protected roof.

The structure is shown schematically in Fig. 4.21a. The Gauss plane (Y) has its real axis on the level of the river bottom. The (Y) plane is transformed into the (y) plane whose configuration is shown in Fig. 4.21b. In terms of the present symbols, the transformation function (used already in Section 4.1) is

$$y = \sqrt{(S^2 + Y^2)}, \quad y = x + iz, \quad Y = X + iZ \quad (4.13.6)$$

The function (4.13.6) implies the relations between the coordinates of points of the two planes:

$$\begin{aligned} x^2 &= \frac{1}{2} \left[\sqrt{[(S^2 + X^2 - Z^2)^2 + 4X^2Z^2]} + S^2 + X^2 - Z^2 \right] \\ z^2 &= \frac{1}{2} \left[\sqrt{[(S^2 + X^2 - Z^2)^2 + 4X^2Z^2]} - S^2 - X^2 + Z^2 \right] \end{aligned} \quad (4.13.7)$$

Eqn. (4.13.7) enables us to determine the line segments a, b corresponding to the line segments A, B in Fig. 4.21a. The shape of the contour of the transformed galleries is obtained in a similar way. The curvilinear trapezoids in the (y) plane are replaced by circles of radius r_0 , with sinks at their centres.

The pressure on the river bottom is $p_k = \gamma H_k$. At very distant points of the (Y) plane and at the corresponding points of the (y) plane, the pressure grows linearly with the depth below the river level. The solution is obtained by superposing the results of an analysis of two partial problems. One defines the distribution of pressure p' assuming sources to be active, and $p' = 0$ on the level of the river bottom. The corresponding pressure is given by

$$p' = c_1 \ln \frac{[(x+b)^2 + (z-a)^2][(x-b)^2 + (z-a)^2]}{[(x+b)^2 + (z+a)^2][(x-b)^2 + (z+a)^2]} + c_3 \quad (4.13.8)$$

The other partial solution defines the linear increase of pressure in the whole (Y) region ($p'' = c_2 Z$). Using eqn. (4.13.3) the partial results are superposed, and the constants c_1, c_2, c_3 are calculated from the following conditions:

$$p = p_k \quad \text{on the line } z = 0,$$

$$p = p_k + \gamma y \quad \text{at the points } x \rightarrow \infty, \quad z = a \pm A$$

$$p = 0 \quad \text{at the point } x = 0, \quad z = s, \quad \text{i.e. at the point 2 of the (y) plane.}$$

After calculating the constants we obtain the equation

$$p = p_k + \gamma Z - \frac{p_k}{2 \ln [b^2(s-a)^2/b^2(s+a)^2]} \times \\ \times \ln \frac{[(x+b)^2 + (z-a)^2][(x-b)^2 + (z-a)^2]}{[(x+b)^2 + (z+a)^2][(x-b)^2 + (z+a)^2]} \quad (4.13.9)$$

On the contour of the gallery where $x = b + r_0, z = a, p = p_0$

$$p_0 = p_k \left(1 - \frac{\ln \sqrt{\{[(2b+r_0)^2 + 4a^2][r_0^2 + 4a^2]\}}}{\ln \{[b^2 + (s+a)^2]/[b^2 + (s-a)^2]\}} \right) + \gamma A \quad (4.13.10)$$

For engineering reasons the pressure p_0 cannot drop below a certain limit. The design is not feasible so long as $p_0 < -7 \cdot 10^4$ Pa. As the solution suggests, the configuration of the slab dimensions, and the position and dimensions of the evacuation elements may be chosen at will.

The initial partial differential equation describes a flow in homogeneous or anisotropic permeable media. In non-homogeneous spaces the solution of practical problems is more difficult.

4.14. Two-dimensional flow under the combined law of flow

The combined law of flow assumes the concurrent existence of a linear, pre-linear and post-linear regime. The situation is described by the equation

$$\frac{\partial}{\partial x} \left(k_{cx} \frac{\partial h}{\partial x} \left| \frac{\partial h}{\partial x} \right|^{-1/2} \right) + \frac{\partial}{\partial z} \left(k_{cz} \frac{\partial h}{\partial z} \left| \frac{\partial h}{\partial z} \right|^{-1/2} \right) = 0 \quad (4.14.1)$$

which stems from the assumption that the velocity vector is collinear with the gradient vector. Although this assumption simplifies the very essence of the problem, analytical solutions of eqn. (4.14.1) are extremely difficult and feasible only in very simple cases.

For example, an analysis of the axi-symmetrical flow in the neighbourhood of a pressure tunnel without lining may be of practical importance. The radius of the tunnel is denoted by r_0 ; the pressures acting on the tunnel contour are assumed to correspond to the piezometric head h_0 . The origin of the coordinate system, from which the radius vectors r are measured, is assumed to lie on the tunnel axis. A circular equipotential line $h = h_R = \text{const.}$ is assumed to be situated at a distance $r = R$.

For the axi-symmetrical flow in a homogeneous medium, eqn. (4.14.1) takes the form

$$\frac{\partial}{\partial r} (2\pi r k_c) \sqrt{\frac{\partial h}{\partial r}} = 0 \quad (4.14.2)$$

$$k_c = \sqrt{\frac{vk}{1 + (vk/k_t)}} \quad (4.14.3)$$

where

- k_c — coefficient of permeability in the combined regime of flow,
- k — coefficient of permeability under the linear law,
- k_t — coefficient of permeability in fully developed turbulence,
- v — flow velocity.

It is evident from eqn. (4.14.3) that the effect of pre-linearity is not assumed to predominate, and attention is concentrated on the region of the linear law and on the post-linear regime.

Eqn. (4.14.3) is substituted in eqn. (4.14.2) and because of the axial symmetry, it is assumed that

$$v = \frac{Q}{2\pi r}$$

The resulting equation is integrated, and after calculating the constants of integration we obtain

$$h_R - h = \frac{Q}{2\pi k} \left(\ln \frac{R}{r} + \frac{kQ}{2\pi k_t^2} \frac{R-r}{rR} \right) \quad (4.14.4)$$

The discharge Q into the tunnel is calculated from the equation

$$Q = -\frac{\pi k_t^2 r_0 R}{k(R - r_0)} \left\{ \ln \frac{R}{r_0} - \sqrt{\left[\left(\ln \frac{R}{r_0} \right)^2 + \frac{4k^2(h_R - h_0)(R - r_0)}{k_t^2 r_0 R} \right]} \right\} \quad (4.14.5)$$

In the flow field, the non-linearity of the law of flow manifests itself to various degrees. The force action of the flowing water depends on the flow velocity at any particular point of the field. The gradient J required in the calculation, is obtained by differentiating eqn. (4.14.4) with respect to r :

$$J = \frac{\partial h}{\partial r} = \frac{Q}{2\pi k r} \left(1 + \frac{kQ}{2\pi k_t^2 r} \right) \quad (4.14.6)$$

The force acting on unit volume of the permeable medium is calculated from the equation

$$F = \gamma J = \frac{\gamma Q}{2\pi k r} \left(1 + \frac{kQ}{2\pi k_t^2 r} \right) \quad (4.14.7)$$

The second term on the right-hand side of eqn. (4.14.6) describes the effect of the turbulent component of the combined law. If it is neglected, the equation becomes

$$F = \frac{\gamma Q}{2\pi k r} \quad (4.14.8)$$

Of interest to practising engineers may be the solution carried out on the assumption of fully developed turbulence in the entire annular region. The relevant basic equation is obtained by neglecting the effect of $\ln(R/r)$ in eqn. (4.14.4). Then

$$h_R - h = \frac{Q^2}{4\pi^2 k_t^2} \frac{R - r}{rR} \quad (4.14.9)$$

$$Q = 2\pi k_t \sqrt{\left[\frac{Rr_0}{R - r_0} (h_R - h_0) \right]} \quad (4.14.10)$$

$$J = \left(\frac{Q}{2\pi k_t r} \right)^2 \quad (4.14.11)$$

$$F = \gamma \left(\frac{Q}{2\pi k_t r} \right)^2 \quad (4.14.12)$$

The force action of a turbulent flow is clearly greater than that of water flowing according to the linear law.

CHAPTER 5

PLANE STEADY FLOW OF GROUNDWATER

Plane flow is essentially a special case of three-dimensional flow. We assume that the equipotential surfaces have vertical directrices and that the principles underlying the hydraulic theory of groundwater motion apply (Section 1.4). A simplified treatment can be used in a number of practical problems involving flows in large areas and free surfaces with small inclinations. The advantages of the method will become apparent when studying flows through non-homogeneous stratified soils where we may apply the theory of the G potential explained in Section 1.4.7.

5.1. Mapping the plane flow in the complex plane

Assume that we have a stratified soil lying on the horizontal surface of a relatively impervious subsoil. Think of the base of the permeable stratum as lying in the complex plane $z = x + iy$. The flow of water is characterized by the distribution of the complex potential $\Theta(x, y)$

$$\Theta(z) = G(x, y) + i\Psi(x, y) \quad (5.1.1)$$

In eqn. (5.1.1), $G(x, y)$ denotes the value of the Girinsky potential, the derivatives of which in the directions of the coordinate axes give uniquely the components of the volume flow vector in the directions of those axes. The same values would be obtained by taking the negative derivatives of the function $\Psi(x, y)$ in the directions of opposite axes.

A basic idea of the meaning of the complex potential $\Theta(z)$ is obtained from an analysis of the parallel flow in the (z) plane, shown in Fig. 5.1a, which takes place in the positive direction of the real axis. Denoting by q_0 the volume flow per unit width of flow we may write, in keeping with the definition of the potential G ,

$$G = \int q_0 dx = q_0 x + c_1 \quad (5.1.2)$$

Similarly, following the definition of the function Ψ , we write

$$\Psi = \int q_0 dy = q_0 y + c_2 \quad (5.1.3)$$

where c_1, c_2 are constants.

On substituting eqns. (5.1.2), (5.1.3) in eqn. (5.1.1) we have

$$\Theta(x, y) = q_0 x + i q_0 y + c = q_0 z + c \quad (5.1.4)$$

Eqn. (5.1.4) describes a parallel flow. Lines parallel to the x axis are assumed to be the plan view of the stream surfaces, while lines parallel to the iy axis are assigned the meaning of the lines of equal values of the potential G . The distance h between the points of the free surface (which moves in the $(n + 1)$ th layer of the stratum at a given point) and the surface of the relatively impervious subsoil is obtained with the help of eqn. (1.7.7)

$$h(x, y) = -\frac{1}{k_{n+1}} \sum_{j=1}^n T_j (k_j - k_{j+1}) + \sqrt{\left\{ \left[\frac{\sum_{j=1}^n T_j (k_j - k_{j+1})}{k_{n+1}} \right]^2 - \frac{2}{k_{n+1}} \left[G(x, y) - \frac{1}{2} \sum_{j=1}^n T_j^2 (k_j - k_{j+1}) \right] \right\}} \quad (5.1.5)$$

As eqn. (5.1.5) implies, the lines of equal values of h (hydroisohypsos) are not identical with the lines of equal values of G . However, given the distribution of the values of G in a field, we can readily determine the distribution of the values of h (i.e. the distribution of the ordinates of the points of the free surface).

From the mathematical point of view, the description of plane seepage does not differ from the methods which we have used in investigations relating to two-dimensional flow. In place of the complex velocity we consider here the complex specific discharge $U(x, y)$ (i.e. the discharge per unit width of flow)

$$U(x, y) = q_x - i q_y$$

Another example of plane flow is the flow into (from) a point sink (source) which passes through the whole saturated stratum. In the complex plane (z) this flow is described by the function

$$\Theta(z) = \frac{Q}{2\pi} \ln z + c = \frac{Q}{2\pi} \ln r e^{i\delta} + c \quad (5.1.6)$$

where

Q — is the discharge,

r, δ — are the polar coordinates of a point in the (z) plane.

It follows from eqn. (5.1.6) that

$$G(x, y) + i\Psi(x, y) = \frac{Q}{2\pi} \ln r + i \frac{Q}{2\pi} \delta + c$$

$$G(x, y) = \frac{Q}{2\pi} \ln r + c \quad (5.1.7)$$

$$\Psi(x, y) = \frac{Q}{2\pi} \delta \quad (5.1.8)$$

The lines of $G = \text{const.}$ are circles of radius r , the lines of $\Psi(x, y) = \text{const.}$ are straight lines passing through the origin and making an angle δ with the positive direction of the x axis (Fig. 5.1b).

In the case being considered, the complex discharge is

$$U(z) = \frac{\partial \Theta}{\partial z} = \frac{Q}{2\pi z} = \frac{Q}{2\pi r} e^{-i\delta} = q_x - iq_y \quad (5.1.9)$$

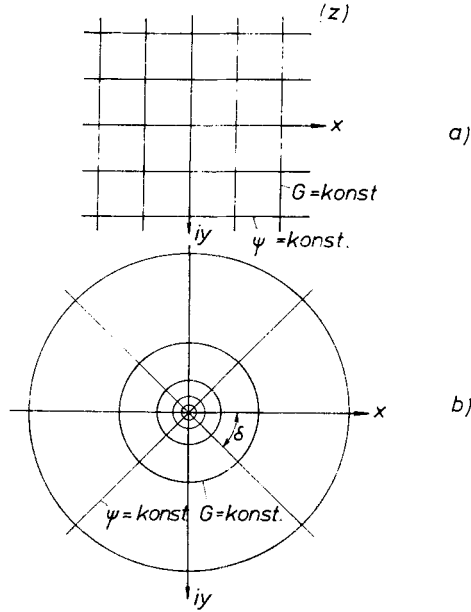


Fig. 5.1. Example of conformal mapping of plane flow.

From eqn. (5.1.9) we obtain

$$q_x - iq_y = \frac{Q}{2\pi r} (\cos \delta - i \sin \delta)$$

$$q_x = \frac{Q}{2\pi r} \cos \delta \quad (5.1.10)$$

$$q_y = \frac{Q}{2\pi r} \sin \delta \quad (5.1.11)$$

The actual specific volume flow q is calculated from the equation

$$q = \sqrt{(q_x^2 + q_y^2)} = \frac{Q}{2\pi r} \quad (5.1.12)$$

The expression (5.1.12) corresponds to the definition of the specific volume flow, i.e. in the case being considered, to the flow per unit length of a circle of radius r .

An essentially different type of flow in the (z) plane is described by the function

$$\Theta(z) = \frac{iI}{2\pi} \ln z + c \quad (5.1.13)$$

In an analysis of this function we shall not be satisfied merely with the principal value of the logarithm. More generally, it holds on separation of the real and imaginary parts that

$$G(x, y) = -\frac{I}{2\pi} (\delta + 2n\pi) + c, \quad n = 0, 1, 2, \dots \quad (5.1.14)$$

$$\Psi(x, y) = \frac{I}{2\pi} \ln r \quad (5.1.15)$$

The values of $G = \text{const.}$ lie on straight lines passing through the origin, the values of $\Psi = \text{const.}$ on circles of radius r . The complex discharge

$$\begin{aligned} U(z) &= \frac{iI}{2\pi z} = \frac{iI}{2\pi r} e^{-i(\delta + 2n\pi)} = \\ &= \frac{iI}{2\pi r} [\cos(\delta + 2n\pi) - i \sin(\delta + 2n\pi)] \end{aligned} \quad (5.1.16)$$

Hence

$$q_x = \frac{I}{2\pi r} \sin(\delta + 2n\pi) \quad (5.1.17)$$

$$q_y = \frac{I}{2\pi r} \cos(\delta + 2n\pi) \quad (5.1.18)$$

$$q = \sqrt{(q_x^2 + q_y^2)} = \frac{I}{2\pi r} \quad (5.1.19)$$

Imagine the lines $G = \text{const.}$ and $\Psi = \text{const.}$ as lying on a so-called Riemann surface (for easier interpretation think of the surface as being a helicoid with an infinitely small lead). This essentially leads to an analogy with the potential vortex known from theoretical hydraulics. The value

$$I = 2\pi r q \quad (5.1.20)$$

is the volume flow which characterizes the properties (intensity) acquired by the vortex in its coming into being.

5.2. Conformal mapping of plane flow

In Section 5.1 we showed the possibility of considering a plane flow in the complex plane. Hence we can again apply the principles of conformal mapping of the (ζ) plane onto the (z) plane. The following relations hold good in this process:

$$\frac{dU}{dz} = \frac{dU}{d\zeta} \left| \frac{d\zeta}{dz} \right| \quad (5.2.1)$$

$$q_z = q_\zeta \left| \frac{d\zeta}{dz} \right| \quad (5.2.2)$$

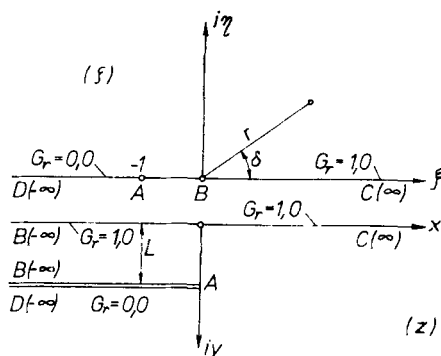


Fig. 5.2. Solution of the discharge into a cut near a river.

As an example of the practical application of conformal mapping, consider the transformation of the (ζ) plane (Fig. 5.2) into the (z) plane effected by the function

$$z = \frac{L}{\pi} (\zeta + 1 + \ln \zeta), \quad \zeta = r e^{i\delta} \quad (5.2.3)$$

From the above we obtain for the principal value of the logarithm of the complex number that

$$x + iy = \frac{L}{\pi} (r \cos \delta + i r \sin \delta + 1 + \ln r + i\delta)$$

$$x = \frac{L}{\pi} (r \cos \delta + \ln r + 1) \quad (5.2.4)$$

$$y = \frac{L}{\pi} (r \sin \delta + \delta) \quad (5.2.5)$$

We find that the points on the negative branch of the ξ axis have been transformed into the points of the slit in the (z) plane. The points on the positive branch of the ξ axis have been transformed into the x axis in the (z) plane.

The increments in the values of x , y in the (z) plane correspond to the increments in the values of $d\delta$ in the (ξ) plane

$$\begin{aligned} dx &= -\frac{L}{\pi} r \sin \delta d\delta \\ dy &= \frac{L}{\pi} (r \cos \delta + 1) d\delta \end{aligned} \quad (5.2.6)$$

$$ds = \sqrt{[(dx)^2 + (dy)^2]} = \frac{L d\delta}{\pi} \sqrt{(r^2 + 2r \cos \delta + 1)} \quad (5.2.7)$$

Consider now the particular case of a potential vortex at the origin of the (ξ) plane. On the positive branch of the ξ axis, let $G = G_{\max}$, on the negative branch, $G = G_{\min}$. The reduced value of the potential, G_r , is defined by

$$G_r = \frac{G - G_{\min}}{G_{\max} - G_{\min}} \quad (5.2.8)$$

According to eqn. (5.1.14) we obtain for the principal value of the logarithm

$$G_r = \frac{-I\delta/2\pi}{-I\pi/2\pi} = \frac{\delta}{\pi}$$

The reduced value of the specific discharge can be considered to be

$$q_r = \frac{2q}{I} = \frac{2}{2\pi r} = \frac{1}{\pi r} \quad (5.2.9)$$

On the positive branch of the ξ axis, we will clearly have $G_r = 1$, $q_r = 1/\pi\xi$, while on the negative branch of the ξ axis, $G_r = 0$, $q_r = 1/\pi|\xi|$. Hence in the (z) plane, the x axis is the line of equal values of the potential G , with $G_r = 1.0$. We can think of the x axis as corresponding to the course of the river shore line. The perimeter of the slit in the (z) plane is also the line of equal values of the potential G , with $G_r = 0$. The perimeter of the slit corresponds to the walls of a very narrow cut which passes through the whole saturated stratum. The water percolates from the river into the cut which, though very long, ends on the imaginary axis in the (z) plane. The distribution of the volume flow along the boundaries of the flow region is calculated from the relation

$$q_r = \frac{dG_r}{ds} = \frac{1}{L} \sqrt{\frac{1}{r^2 + 2r \cos \delta + 1}} \quad (5.2.10)$$

For $\delta = \pi$, i.e. for the perimeter of the cut, the quantity

$$q_r = \left| \frac{1}{L(r-1)} \right| \quad (5.2.11)$$

Letting $r \rightarrow 1$, we obtain $q_r = \infty$. For $r \rightarrow 0$, $q_r = 1/L$. The flow concentrates at the end of the cut. At the points of the cut which are sufficiently removed from the end, we obtain

$$q_r \approx \frac{1}{L} \quad (5.2.12)$$

The above is an expression which we know from the theory of one-dimensional steady flow of groundwater (Chapter 2).

The same results may also be obtained in another way, Along the ξ axis in the (ζ) plane

$$(q_r)_\zeta = -i(q_r)_\eta = -i \frac{1}{\pi|\xi|} \quad (5.2.13)$$

The transformation function (5.2.3) implies that

$$\frac{dz}{d\zeta} = \frac{L}{\pi} \left| \frac{1+\zeta}{\zeta} \right|$$

At the points on the real axis of the (ζ) plane

$$\frac{dz}{d\zeta} = \frac{L}{\pi} \left| \frac{1+\xi}{\xi} \right|$$

Substituting in eqn. (5.2.2), we obtain along the x axis in the (z) plane

$$(q_r)_z = - \frac{i}{\pi|\xi|} \left| \frac{\xi}{1+\xi} \right| \frac{\pi}{L} = -i \frac{1}{|\xi+1|} \frac{1}{L} \quad (5.2.14)$$

Along the image of the shore line, the specific volume flow $(q_r)_y$ clearly varies within the limits

$$0 \leq (q_r)_y \leq \frac{1}{L}$$

Along the cut, on the side facing the river, the specific volume flow varies within the limits

$$\frac{1}{L} \leq (q_r)_y \leq \infty ,$$

on the side away from the river (where $-\infty \leq \xi \leq -1$), within the limits

$$0 \leq (q_r)_y \leq \infty$$

In places which are at a sufficient distance from the end of the cut, the water merely percolates to one side, directly from the river, while the seepage which comes from the hinterland, is very small.

5.3. Circular inversion

Consider a circle of radius R in the $(\bar{\zeta})$ plane. By means of the transformation function

$$\zeta = \frac{R^2}{\bar{\zeta}} \quad (5.3.1)$$

we shall map the $(\bar{\zeta})$ plane onto the (ζ) plane. We clearly have

$$rc^{i\delta} = \frac{R^2}{\bar{r}c^{i\delta}}$$

$$r(\cos \delta + i \sin \delta) = \frac{R^2}{\bar{r}}(\cos \delta - i \sin \delta)$$

Equating the real and imaginary parts we obtain

$$r \cos \delta = \frac{R^2}{\bar{r}} \cos \delta$$

$$r \sin \delta = \frac{R^2}{\bar{r}} \sin (-\delta) = \frac{R^2}{\bar{r}} \sin (2\pi - \delta)$$

According to Fig. 5.3, points exterior to a circle in the $(\bar{\zeta})$ plane, transform into points interior to a circle in the (ζ) plane. The radius vector $\bar{r}$ changes to the radius vector $r = R^2/\bar{r}$. The transformation also results in reflection about the real axes of the plane. The length effect of the transformation may be calculated from the relation

$$\frac{d\zeta}{d\bar{\zeta}} = - \frac{R^2}{(\bar{\zeta})^2} \quad (5.3.2)$$

Assume that the reflection merely has a formal meaning, and think of the original plane as being coincident with the transformed plane. Regard the corresponding points as lying on the same ray issuing from the origin. Eqn. (5.3.1) plainly implies that

$$\frac{r}{R} = \frac{R}{\bar{r}} \quad (5.3.3)$$

$$r = \frac{R^2}{\bar{r}}, \quad \bar{r} = \frac{R^2}{r} \quad (5.3.4)$$

$$r\bar{r} = R^2 \quad (5.3.5)$$

Eqn. (5.3.5) expresses the first of Euclid's axioms. Accordingly, the point A corresponding to the point $\bar{A}$, lies at the point of intersection of the ray and the polar drawn for the point $\bar{A}$ relative to the circle.

To each point exterior to the circle corresponds a certain point interior to the circle, and we speak of the inversion of conjugate points (circular inversion).

Assume, following Fig. 5.3c, that a sink originated at the point A . At the points of the circle of radius R we note a definite distribution of the values of the potential G .

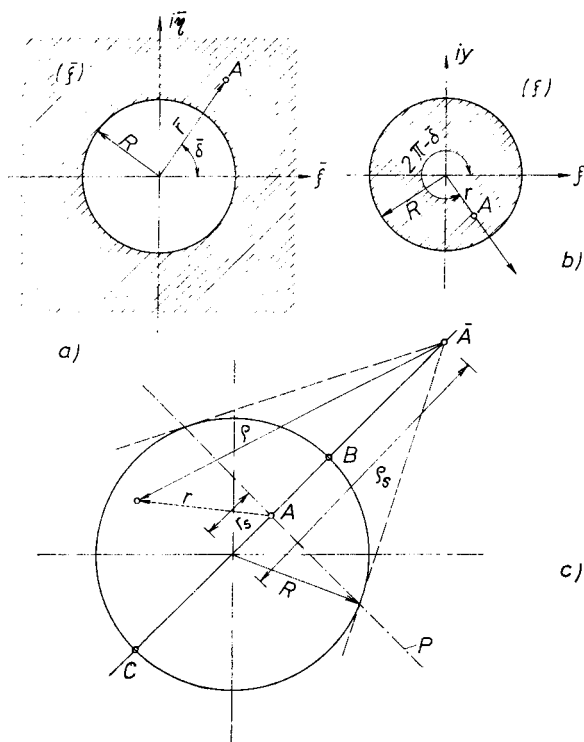


Fig. 5.3. Circular inversion.

If a sink of equal strength is assumed to exist at the point $\bar{A}$, the distribution of the potential G on the circle will be the same as in the former case, for the sink at the point A can be thought of as having come into being only after conformal mapping of the sink at the point $\bar{A}$ had taken place.

In plane flow, the principle of superposition applies just as it does in two-dimensional flow. If a sink of strength Q is assumed to exist at the point A , and a source with a volume flow $-Q$ at the point $\bar{A}$, the potential on the circle $G = \text{const.}$ according to eqn. (5.1.7), is, after superposition,

$$G = \frac{Q}{2\pi} \ln \frac{r}{\varrho} + c \quad (5.3.6)$$

the ratio $r/\varrho = \text{const.}$ on the circle. The validity of this can easily be demonstrated

at points B, C . Thus, for example, at the point B

$$\frac{r}{\varrho} = \frac{R - r_s}{\varrho_s - R} = \frac{R - r_s}{(R^2/r_s) - R} = \frac{r_s}{R}$$

and at the point C

$$\frac{r}{\varrho} = \frac{R + r_s}{\varrho_s + R} = \frac{R + r_s}{(R^2/r_s) + R} = \frac{r_s}{R} \quad (5.3.7)$$

Eqn. (5.3.7) will be used to calculate the constant c of eqn. (5.3.6). Consider the following value of G_r on the circle of radius R :

$$G_R = \frac{Q}{2\pi} \ln \frac{r_s}{R} + c$$

$$c = G_R - \frac{Q}{2\pi} \ln \frac{r_s}{R}$$

On substituting in eqn. (5.3.6) we have

$$G - G_R = \frac{Q}{2\pi} \ln \frac{rR}{\varrho r_s} \quad (5.3.8)$$

Assume that the equipotential lines in the vicinity of the sink are nearly circular. Choose one of them, with radius r_0 , and consider the potential on this line to be $G = G_0$. On the circle circumference

$$r = r_0, \quad \varrho \approx \varrho_s - r_s = \frac{R^2}{r_s} - r_s = \frac{R^2 - r_s^2}{r_s}$$

On substituting in eqn. (5.3.8)

$$G_0 - G_R = \frac{Q}{2\pi} \ln \frac{Rr_0}{R^2 - r_s^2}$$

Hence

$$Q = \frac{2\pi(G_R - G_0)}{\ln [(R^2 - r_s^2)/Rr_0]} \quad (5.3.9)$$

Assuming that the sink is situated centrally, $r_s = 0$, and

$$Q = \frac{2\pi(G_R - G_0)}{\ln (R/r_0)} \quad (5.3.10)$$

We shall call eqn. (5.3.10) Dupuit's formula. It is also implied directly by eqn. (5.1.7).

In the simple case being considered we obtain for $r = R$

$$\begin{aligned}
 G_R &= \frac{Q}{2\pi} \ln R + c \\
 c &= G_R - \frac{Q}{2\pi} \ln R \\
 G - G_R &= \frac{Q}{2\pi} \ln \frac{r}{R} = -\frac{Q}{2\pi} \ln \frac{R}{r}
 \end{aligned} \tag{5.3.11}$$

and on substituting $r = r_0$, $G = G_0$, we have eqn. (5.3.10).

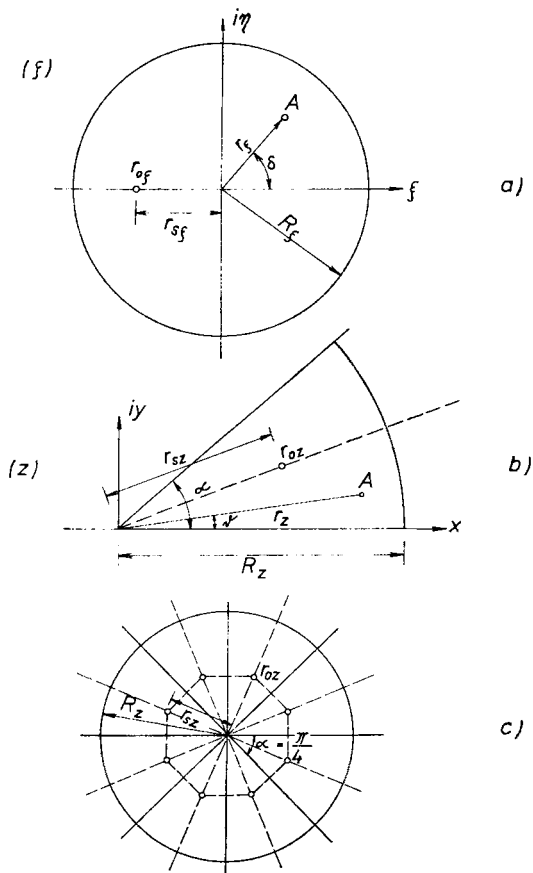


Fig. 5.4. Battery of wells.

The solution of a well placed centrally within a circle, is of considerable theoretical significance. The case may further be elaborated by the method of conformal mapping. Referring to Fig. 5.4, we transform the (ζ) plane onto the (z) plane by

means of the function

$$\zeta = z^m, \quad m = \frac{2\pi}{\alpha} \quad (5.3.12)$$

According to de Moivre's theorem

$$\zeta + i\eta = r_z^m (\cos m\vartheta + i \sin m\vartheta)$$

Hence

$$\zeta = r_z^m \cos m\vartheta \quad (5.3.13)$$

$$\eta = r_z^m \sin m\vartheta \quad (5.3.14)$$

Eqns. (5.3.12) may also be written as follows:

$$r_\zeta (\cos \delta + i \sin \delta) = r_z^m (\cos m\vartheta + i \sin m\vartheta) \quad (5.3.15)$$

Hence

$$r_\zeta \cos \delta = r_z^m \cos m\vartheta$$

$$r_\zeta \sin \delta = r_z^m \sin m\vartheta$$

$$r_\zeta = r_z^m, \quad \delta = m\vartheta \quad (5.3.16)$$

Points lying on a circle in the (ζ) plane again transform into a circle. Straight lines with polar angle δ become straight lines with a polar angle ϑ . Consider in the (ζ) plane a well of a small radius $r_{0\zeta}$ and let $r_{s\zeta}$ be its distance from the origin. The corresponding well in the (z) plane is of radius r_{0z} and lies at a distance r_{sz} from the origin. From the transformation function we have

$$\frac{d\zeta}{dz} = mz^{m-1} \quad (5.3.17)$$

and, therefore,

$$r_{0\zeta} = r_{0z} \left| \frac{d\zeta}{dz} \right| = r_{0z} m r_{sz}^{m-1} \quad (5.3.18)$$

Substituting the results in eqn. (5.3.9) we have

$$Q = \frac{2\pi(G_R - G_0)}{\ln [(R_z^{2m} - r_{sz}^{2m})/r_{0z} m r_{sz}^{m-1} R_z^m]} = \frac{2\pi(G_R - G_0)}{\ln \left[R_z^m \left(1 - \frac{r_{sz}^{2m}}{R_z^{2m}} \right) / m r_{0z} r_{sz}^{m-1} \right]} \quad (5.3.19)$$

where

G_R — the value of the potential on a circle of radius R_z ,

G_0 — the value of the potential on the contour of a well of radius r_{0z} .

The function (5.3.12) maps the (ζ) plane into the interior of a sector in the (z) plane. In this way we obtain, for integer values of m , the pattern of flow toward one of a whole battery of wells (shown in Fig. 5.4 for $\alpha = \pi/4$). Eqn. (5.3.19) defines the discharge of one well. The output of the entire battery is obtained by multiplying the result of eqn. (5.3.19) by the number of wells, i.e. by m .

Of practical interest is the value of G at the centre of the battery. It is calculated directly from eqn. (5.3.8) following the substitution

$$\begin{aligned} r_s &= r_{s\zeta} = r_{sz}^m \\ R &= R_\zeta = R_z^m \\ \varrho &= \frac{R_\zeta^2}{r_{s\zeta}} = \frac{R_z^{2m}}{r_{sz}^m} \end{aligned}$$

We obtain

$$G - G_R = \frac{Q}{2\pi} \ln \frac{r_{sz}^m R_z^m}{R_z^{2m} r_{sz}^m / r_{sz}^m} = m \frac{Q}{2\pi} \ln \frac{r_{sz}}{R_z}$$

and after a simple rearrangement

$$G = G_R - \frac{mQ}{2\pi} \ln \frac{R_z}{r_{sz}} \quad (5.3.20)$$

In conclusion, note again that the discharge Q can be taken as either negative or positive depending on whether we are dealing with seepage of water into a well, or with infiltration of water from a well.

5.4. Interference among wells of small radii (Superposition of the G potential)

Consider first the case which we have solved by the application of circular inversion, i.e. a sink at point A placed eccentrically inside a circle of radius R , with a corresponding source at the point $\bar{A}$. Suppose now that the eccentricity has become so large as to change the circle into a straight line. The eccentricity $r_s \approx R$. On this assumption, eqn. (5.3.8) yields

$$G - G_R = \frac{Q}{2\pi} \ln \frac{r}{\varrho} \quad (5.4.1)$$

In accordance with the principle of superposition, we could consider m sinks of strength Q inside the circle, and the same number of conjugate sources outside the circle. If $R \rightarrow \infty$, the conjugate points are distributed symmetrically about a straight line (Fig. 5.5a). Denote by r_i the distance between a chosen point and the i -th sink,

and by ϱ_i the distance between that point and the conjugate source. Then

$$G - G_R = \frac{1}{2\pi} \sum_{i=1}^m Q_i \ln \frac{r_i}{\varrho_i} \quad (5.4.2)$$

For a system of sinks which are of equal strength Q we obtain

$$G - G_R = \frac{Q}{2\pi} \sum_{i=1}^m \ln \frac{r_i}{\varrho_i} \quad (5.4.3)$$

Eqn. (5.4.3) can also be written as follows:

$$G - G_R = \frac{Q}{2\pi} \ln \prod_{i=1}^m \frac{r_i}{\varrho_i} \quad (5.4.4)$$

Along the straight line $G_R = \text{const.}$, and we may, therefore, regard it as the image of the shore line of a river from which the water infiltrates into the wells.

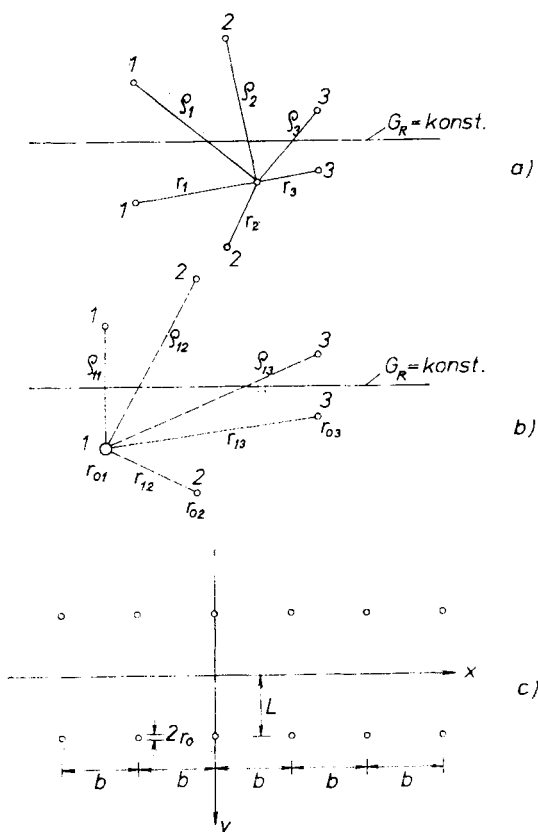


Fig. 5.5. Superposition of effects of wells.

Although the method of mirror representation would provide further results of practical interest, an analysis made using it would hardly bring anything new methodologically. We therefore continue our exposition by considering a well which receives water from a curvilinear, e.g. parabolic, boundary.

In the discussion we shall fall back upon the results obtained in Section 4.8, especially upon the solution of a system of sources and sinks positioned at equal mutual distances (Fig. 4.16a).

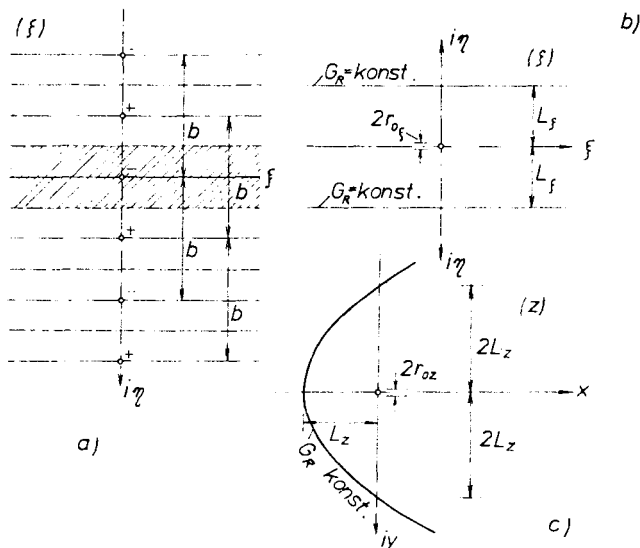


Fig. 5.6. Infiltration from a curved boundary of the region of flow.

Consider two systems in the (ζ) plane, one containing a series of sinks, the other a series of sources. Place the objects in accordance with the scheme shown in Fig. 5.6, and consider the real and the imaginary axis to be opposite to their respective counterparts in Fig. 4.16a. For the series of sinks we obtain, in place of eqn. (4.8.8), the equation

$$\Theta_- = \frac{Q}{2\pi} \ln \sinh \frac{\pi\zeta}{b} + c_-$$

Since the sources are displaced by $b/2$ relative to the sinks, their effect may be expressed by the function

$$\Theta_+ = \frac{Q}{2\pi} \ln \cosh \frac{\pi\zeta}{b} + c_+$$

After superposition, the complex potential is defined by the relation

$$\Theta = \frac{Q}{2\pi} \ln \tanh \frac{\pi\zeta}{b} + c \quad (5.4.10)$$

where Q is the strength of a sink.

Of interest in the (ζ) plane is a single strip (Fig. 5.6b), or actually, a half of that strip, which lies below the ξ axis. Essentially, the discharge Q relates to a half of a sink. The water which it receives comes from a linear shore line. Following an obvious change of symbols (Fig. 5.6b) we have

$$\Theta(\zeta) = \frac{Q}{\pi} \ln \tanh \frac{\pi\zeta}{4L_\zeta} + c \quad (5.4.11)$$

The potential function G is obtained by comparing the real parts on the two sides of eqn. (5.4.11)

$$G = \frac{Q}{\pi} \ln \left| \tanh \frac{\pi\zeta}{4L_\zeta} \right| + c = \frac{Q}{\pi} \ln \frac{\sqrt{[\sinh^2(\pi\xi/2L_\zeta) + \sin^2(\pi\eta/2L_\zeta)]}}{\cosh(\pi\xi/2L_\zeta) + \cos(\pi\eta/2L_\zeta)} + c$$

Hence

$$G = \frac{Q}{2\pi} \ln \frac{\sinh^2(\pi\xi/2L_\zeta) + \sin^2(\pi\eta/2L_\zeta)}{[\cosh(\pi\xi/2L_\zeta) + \cos(\pi\eta/2L_\zeta)]^2} + c \quad (5.4.12)$$

Suppose that $G = G_R$ on the image of the shore line, i.e. at the points $\eta = L_\zeta$. Eqn. (5.4.12) gives $c = G_R$. Thus the final form of the equation is

$$G - G_R = \frac{Q}{2\pi} \ln \frac{\sinh^2(\pi\xi/2L_\zeta) + \sin^2(\pi\eta/2L_\zeta)}{[\cosh(\pi\xi/2L_\zeta) + \cos(\pi\eta/2L_\zeta)]^2} \quad (5.4.13)$$

The function mapping the (ζ) half-plane onto the (z) plane is

$$z = \zeta^2 \quad (5.4.14)$$

Separating it into the real and imaginary parts we have

$$x = \xi^2 - \eta^2 \quad (5.4.15)$$

$$y = 2\xi\eta \quad (5.4.16)$$

$$\eta = \pm \sqrt{\frac{-x + \sqrt{(x^2 + y^2)}}{2}} \quad (5.4.17)$$

$$\xi = \pm \frac{y\sqrt{2}}{2\sqrt{[-x + \sqrt{(x^2 + y^2)}]}} = \frac{y}{\sqrt{2[-x + \sqrt{(x^2 + y^2)}]}} \quad (5.4.18)$$

According to Fig. 5.6c, the image of the straight lines $\eta = \text{const.}$ in the (z) plane will be a family of confocal parabolas with the focus at the origin. The transformed shore line is clearly of a parabolic shape. The abscissa of the parabola vertex is

$$L_z = L_\zeta^2 \quad (5.4.19)$$

and the radius of curvature at the vertex is $2L_z$.

Assume that in the (ζ) plane the equipotential lines around the sink were circular, and that one of them — that of radius $r_{0\zeta}$ corresponded to the contour of an hydraulically perfect well. In the (z) plane a well of radius r_{0z} corresponds to it, where

$$r_{0z} = r_{0\zeta}^2 \quad (5.4.20)$$

On substituting the respective expressions in eqn. (5.4.13) we obtain

$$G - G_R = \frac{Q}{2\pi} \ln \frac{\sinh^2 \frac{\pi y}{2\sqrt{\{2L_z[-x + \sqrt{(x^2 + y^2)}]\}}} + \sin^2 \frac{\pi}{2} \sqrt{\frac{-x + \sqrt{(x^2 + y^2)}}{2L_z}}}{\left(\cosh \frac{\pi y}{2\sqrt{\{2L_z[-x + \sqrt{(x^2 + y^2)}]\}}} + \cos \frac{\pi}{2} \sqrt{\frac{-x + \sqrt{(x^2 + y^2)}}{2L_z}} \right)^2} \quad (5.4.21)$$

Thus, for example, at the point $x = 0$, $y = 2L_z$

$$G - G_R = \frac{Q}{2\pi} \ln \frac{\sinh^2 \frac{1}{2}\pi + 1}{\cosh^2 \frac{1}{2}\pi} = 0$$

On the well contour, say, at the point $x = 0$, $y = r_{0z}$, we assume that $G = G_0$; therefore

$$G_0 - G_R = \frac{Q}{2\pi} \ln \frac{\sinh^2 \frac{1}{2}\pi \sqrt{(r_0/2L_z)} + \sin^2 \frac{1}{2}\pi \sqrt{(r_0/2L_z)}}{[\cosh \frac{1}{2}\pi \sqrt{(r_0/2L_z)} + \cos \frac{1}{2}\pi \sqrt{(r_0/2L_z)}]^2}$$

Hence

$$Q = \frac{2\pi(G_R - G_0)}{\ln \frac{[\cosh \frac{1}{2}\pi \sqrt{(r_0/2L_z)} + \cos \frac{1}{2}\pi \sqrt{(r_0/2L_z)}]^2}{\sinh^2 \frac{1}{2}\pi \sqrt{(r_0/2L_z)} + \sin^2 \frac{1}{2}\pi \sqrt{(r_0/2L_z)}}} \quad (5.4.22)$$

As the case of shore-line infiltration from a parabolic boundary shows, even fairly complicated problems are amenable to analytical treatment. Conformal mapping which transforms a specified region onto a circle or another suitable closed area, makes it possible to convert the problem in question to one which has been, or can be, solved analytically.

5.5. Theoretically favourable location of small-diameter wells

From practical considerations, the best way of locating wells along a given line, is that which results in equal strength of all wells, while satisfying the condition that the shore line (or any other specified boundary) should be loaded uniformly.

We shall outline the principle of the appropriate theoretical solution by considering seepage into a linear array of wells located on the axis of symmetry of a parabolic shore line.

We start from a linear system of hydraulically perfect wells of equal strengths, spaced at equal distances in the (ζ) plane. Water flows into the wells unrestricted from both sides of the region. Evidently, the linear system shown in Fig. 5.7a satisfies the specified requirements. Because the flow is symmetrical, only a half of the flow field

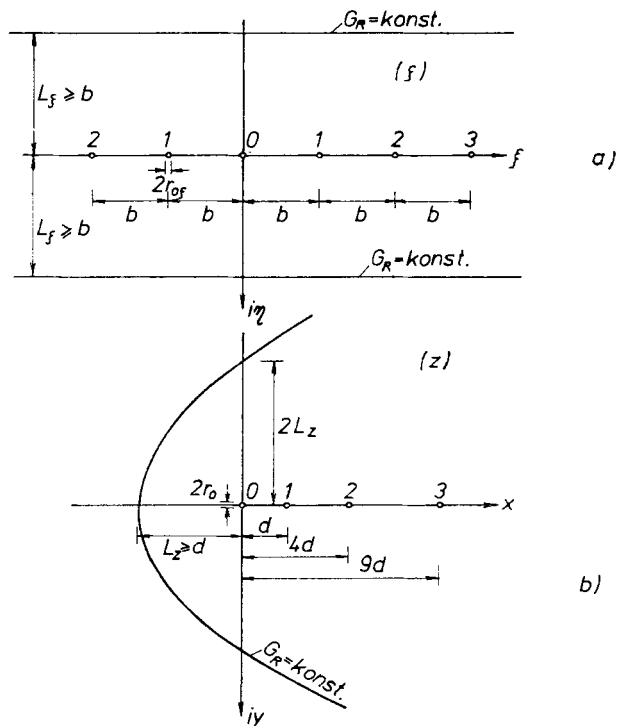


Fig. 5.7. Distribution of wells of nearly equal strengths.

need be examined. Based on the results obtained in Section 4.8, the complex potential in the (ζ) plane can be taken to be

$$\Theta = \frac{Q}{\pi} \ln \sinh \frac{\pi \zeta}{b} + c \quad (5.5.1)$$

where Q is the discharge into a half of a well.

Separating the above into the real and imaginary parts yields

$$G = \frac{Q}{\pi} \ln \left| \sinh \frac{\pi \zeta}{b} \right| + c = \frac{Q}{\pi} \ln \sqrt{\left(\sinh^2 \frac{\pi \xi}{b} \cos^2 \frac{\pi \eta}{b} + \cosh^2 \frac{\pi \xi}{b} \sin^2 \frac{\pi \eta}{b} \right)} + c,$$

$$G = \frac{Q}{2\pi} \ln \left(\cosh \frac{2\pi \xi}{b} - \cos \frac{2\pi \eta}{b} \right) + c \quad (5.5.2)$$

where c is a constant.

It is of interest to note that for

$$\xi \geq b \quad (5.5.3)$$

the first term in the parentheses of eqn. (5.5.2) predominates over the remaining ones. As a result of this, the value of G turns out to be nearly constant for a certain, chosen value of η . This means that the parallels to the ξ axis will be, approximately, also the lines of equal values of G . If this is so, then the specific discharge along those lines must be nearly constant. Supposing now, with condition (5.5.3) applying, that one of the parallels to the ξ axis is the image of the shore line, we may state that the requirement of uniform load is satisfied on that line. We choose the image of the shore line to be at a distance $\eta = L_\zeta$ from the ξ axis, and assume that there, $G = G_R$:

$$G_R = \frac{Q}{2\pi} \ln \left(\cosh \frac{2\pi\xi}{b} - \cos \frac{2\pi L_\zeta}{b} \right) + c$$

Substitution in eqn. (5.5.2) gives

$$G - G_R = \frac{Q}{2\pi} \ln \frac{\cosh (2\pi\xi/b) - \cos (2\pi\eta/b)}{\cosh (2\pi\xi/b) - \cos (2\pi L_\zeta/b)} \quad (5.5.4)$$

Number the wells in keeping with Fig. 5.7a, and use the function (5.4.14) to map the (ζ) half-plane conformally onto the (z) plane. The zero-th well will map onto the focus of the parabolic image of the shore line, with $r_0 = r_{0\zeta}^2$ after the transformation. The remaining wells will map onto the real axis of the (z) plane. The abscissae of their centres will be

$$x_n = n^2 b^2 \quad (5.5.5)$$

$n = 1, 2, \dots, \infty$.

Denote by the symbol d the distance between the zero-th well and the first well. Evidently $d = b^2$; hence we may write, in place of eqn. (5.5.5)

$$x_n = n^2 d \quad (5.5.6)$$

The distance between the n -th well and the $(n - 1)$ th well is equal to $(2n - 1) d$. The radii of the wells were, of course, changed by the transformation, generally according to the relation

$$r_{0n} = r_{0\zeta} \left| \frac{dz}{d\zeta} \right| = 2r_{0\zeta} n^2 d \quad (5.5.7)$$

Since different well radii are hardly convenient in practice, consider the objects to be of the same radius and neglect the effect of the transformation in that respect. Carry out the calculation for the zero-th well, the position and diameter of which in the (z) plane has already been determined.

The distribution of the potential function is obtained from eqn. (5.5.4) on substituting eqns. (5.4.15) to (5.4.18), viz.

$$G - G_R = \frac{Q}{2\pi} \ln \frac{\cosh \pi y \sqrt{\frac{2}{d[-x + \sqrt{(x^2 + y^2)]}} - \cos 2\pi \sqrt{\frac{[-x + \sqrt{(x^2 + y^2)]}{2d}}}}{\cosh \pi y \sqrt{\frac{2}{d[-x + \sqrt{(x^2 + y^2)]}} - \cos 2\pi \sqrt{\frac{L_z}{d}}}} \quad (5.5.8)$$

where Q is the strength of the zero-th well.

At the point $x = 0$, $y = r_0$

$$G_0 - G_R = \frac{Q}{2\pi} \ln \frac{\cosh \pi \sqrt{(2r_0/d)} - \cos 2\pi \sqrt{(r_0/2d)}}{\cosh \pi \sqrt{(2r_0/d)} - \cos 2\pi \sqrt{(L_z/d)}} \quad (5.5.9)$$

from which

$$Q = \frac{2\pi(G_R - G_0)}{\ln \frac{\cosh \pi \sqrt{(2r_0/d)} - \cos 2\pi \sqrt{(L_z/d)}}{\cosh \pi \sqrt{(2r_0/d)} - \cos 2\pi \sqrt{(r_0/2d)}}} \quad (5.5.10)$$

Assume further that there are sinks in the (ζ) plane spaced so close to one another along the real axis as to produce a parallel flow in the $i\eta$ direction, and assume that $G = G_0$ on the line connecting the sinks. Evidently, the specific discharge

$$q_\zeta = \frac{G_R - G_0}{L_\zeta} \quad (5.5.11)$$

Again using the function (5.4.14), transform the (ζ) half-plane onto the (z) plane; this will give the image of the flow from a parabolic shore line into an hydraulically perfect cut terminating at the origin of the (z) plane. Along the cut

$$q_z = q_\zeta \left| \frac{d\zeta}{dz} \right| = q_\zeta \frac{1}{2\xi} = \frac{G_R - G_0}{2\sqrt{(xL_z)}} \quad (5.5.12)$$

In the segment $x_1 \leq x \leq x_2$, the quantity

$$Q = \frac{G_R - G_0}{\sqrt{L_z}} \int_{x_1}^{x_2} \frac{dx}{\sqrt{x}} = \frac{G_R - G_0}{2\sqrt{L_z}} (\sqrt{x_2} - \sqrt{x_1}) \quad (5.5.13)$$

will flow into the cut from both sides.

Now divide the line of the cut into segments with equal total volume flows. The first segment will be of length d . The abscissae of the ends of the remaining segments are x_n ($n = 1, 2, \dots, \infty$).

It follows from eqn. (5.5.13) that

$$\sqrt{x_n - (n-1)d} = \sqrt{d}$$

Hence

$$x_n = n^2 d \quad (5.5.14)$$

Eqn. (5.5.14) is the same as eqn. (5.5.6). This fact implies the possibility of using the analysis of flow into a cut for an approximate determination of the positions of hydraulically perfect wells of equal strengths. The line of the linear array of wells is replaced by a fictitious cut and the distribution of the specific quantity of flow q along the array is determined. The line is then divided into segments with equal total volume flows, and wells are placed at the points of division. Finally, a check is made to ensure that the division into segments has been correctly done.

5.6. Shore line infiltration into large-diameter wells

It was assumed in Sections 5.4 and 5.5 that in a given flow field there exists cooperation and interference among wells of small radii. The exposition proceeded from the theory of a point source around which the equipotential lines are nearly circular. One of the small circles was taken for the outer contour of a well. In practical cases this assumption is not always acceptable, however.

As an example, we mention the discharge into excavation pits or gravel pits which may sometimes be regarded as large-diameter wells. The objects are supplied by shore infiltration.

We shall explain the problem by making use of the laws of circular inversion. Consider a sink at the point A and a conjugate source at the point $\bar{A}$. The circle relative to which the inversion was made, is known to be of radius R . According to Section 5.3 and Fig. 5.8a it satisfies

$$r\varrho = R^2 \quad (5.6.1)$$

At the centre of the circle

$$\varrho = \varrho_s = 2a + r_s, \quad r = r_s$$

To the given pair of points, there clearly corresponds a whole family of circles of radius R with centres on the line connecting the sink and the source. Each circle, including that with $R = \infty$, is an equipotential surface. If, on the other hand, we know the radius $R = r_0$ of the well on the contour of which $G = G_0 = \text{const.}$, and the distance $r_s + a = R + L$ of its centre from the image of the shore line, we can calculate the position of the fictitious sink as well as that of the conjugate fictitious source which, theoretically, replace the effect of the discharge from a large well.

In fact we have

$$a = L + r_0 - r_s \quad (5.6.2)$$

and eqn. (5.6.1) yields directly

$$r_s = (L + r_0) \left\{ 1 - \sqrt{1 - \left(\frac{r_0}{L + r_0} \right)^2} \right\} \quad (5.6.3)$$

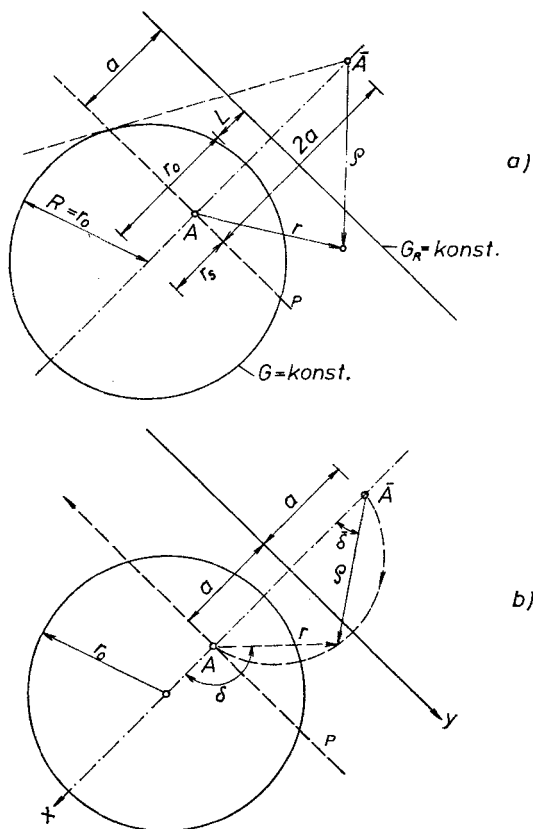


Fig. 5.8. Discharge into a large-diameter well.

On the image of the shore line, $G = G_R$; hence according to eqn. (5.3.6)

$$G - G_R = \frac{Q}{2\pi} \ln \frac{r}{\varrho} \quad (5.6.4)$$

On the well contour, on the axis of symmetry, $r = r_0 - r_s$, $\varrho = 2a - (r_0 - r_s) = 2L + r_0 - r_s$. Therefore,

$$\begin{aligned} G_0 - G_R &= \frac{Q}{2\pi} \ln \frac{r_0 - r_s}{2a - (r_0 - r_s)} \\ Q &= \frac{2\pi(G_R - G_0)}{\ln(2L/(r_0 - r_s) + 1)} \end{aligned} \quad (5.6.5)$$

The shape of the streamlines is obtained with the help of the stream function

$$\Psi = \frac{Q}{2\pi} (\delta - \bar{\delta})$$

$$(\delta - \bar{\delta}) = \operatorname{arctg} [\operatorname{tg} (\delta - \bar{\delta})] = \operatorname{arctg} \frac{\operatorname{tg} \delta - \operatorname{tg} \bar{\delta}}{1 + \operatorname{tg} \delta \operatorname{tg} \bar{\delta}} \quad (5.6.6)$$

According to Fig. 5.8b

$$\operatorname{tg} \delta = \frac{y}{x - a}, \quad \operatorname{tg} \bar{\delta} = -\frac{y}{x + a}$$

On substituting in eqn. (5.6.6) we obtain

$$\Psi = \frac{Q}{2\pi} \operatorname{arctg} \frac{2ay}{x^2 + y^2 - a^2} \quad (5.6.7)$$

The lines of equal values of Ψ are clearly circles centred on the y axis. One of these circles has an infinitely large radius and coincides with the x axis.

5.7. Notes on the time variations of plane flow

A permeable stratum consists of soils having different coefficients of permeability k and different dynamically effective porosities μ_e . The velocity of motion depends on k , μ_e . Water particles move fastest through a soil whose characteristic ratio k/μ_e is maximal.

As the theory assumes, the free surface generally intersects the interfaces between layers so that water need not flow through a certain layer throughout the flow region. A theoretical analysis of what this fact implies for studies of the velocities of particle motion through a flow field, is very difficult and nearly impossible if we consider the theory of plane seepage to be adequate for the purpose. Nevertheless, even informative data obtained by its application, will help us in understanding the process involved.

As a simple example, consider the flow into an hydraulically perfect cut, from a parallel river edge. The shore line lies at a distance L from the cut. Assume first that the soil is homogeneous, and follow the motion of a water particle near the surface of a relatively impervious subsoil.

The time t required for a point to travel distance L on a straight path is

$$t = \mu_e \int_0^L \frac{dx}{v_x} = \mu_e \int_0^L \frac{h}{q} dx = \frac{\mu_e}{q} \int_0^L h dx \quad (5.7.1)$$

where q is the specific volume flow, v_x — the seepage velocity of flow along the path

considered. According to Chapter 2, in a parallel flow

$$G = G_R - \frac{x}{L}(G_R - G_0) \quad (5.7.2)$$

G — the value of the potential at the point x ,

G_R, G_0 — the value of the potential along the shore line and the cut, respectively.

In a homogeneous soil we also have

$$G = -k \frac{h^2}{2}, \quad h = \sqrt{\left(-\frac{2G}{k}\right)} \quad (5.7.3)$$

and on substituting eqn. (5.7.3) in eqn. (5.7.2) we obtain

$$h = \sqrt{\left\{ \frac{2}{k} \left[\frac{x}{L}(G_R - G_0) - G_R \right] \right\}} \quad (5.7.4)$$

The specific volume becomes

$$q = \frac{\partial G}{\partial x} = -\frac{G_R - G_0}{L} \quad (5.7.5)$$

Hence, eqn. (5.7.1) yields

$$t = -\frac{\mu_e L \sqrt{2}}{(G_R - G_0) \sqrt{k}} \int_0^L \sqrt{\left[\frac{x}{L}(G_R - G_0) - G_R \right]} dx$$

and on integration

$$t = \frac{2 \sqrt{2} \mu_e L^2}{3 \sqrt{k} (G_R - G_0)^2} [(-G_R)^{3/2} - (-G_0)^{3/2}] \quad (5.7.6)$$

Consider now the example of a soil profile containing two layers. The lower layer has a thickness T , a coefficient of permeability k , and a dynamically effective porosity μ_e . The upper layer is relatively impervious. Accordingly, we shall examine a water particle which proceeds in a straight line through the pressure zone in the lower layer. Therefore, substituting $h = T$ in eqn. (5.7.1) we have

$$t = \frac{\mu_e T}{q} \int_0^L dx = \frac{\mu_e T L}{q} \quad (5.7.7)$$

Substitution of eqn. (5.7.5) gives

$$t = -\frac{\mu_e T L^2}{G_R - G_0} = \frac{\mu_e L^2}{k(h_R - h_0)} \quad (5.7.8)$$

The assumption of a pressure regime of flow leads to a simpler formula. It seems that because of the non-homogeneity of natural soils, calculations of the time of particle travel are approximate to such a degree that even in problems concerned with plane seepage defined in a rather general manner, we may assume, informally at least, the existence of pressure flow.

As an example, consider the travel of a water particle from a river into a large well (Section 5.6). Consider the shortest path which, according to Fig. 5.8b, lies on the x axis. Because of axial symmetry, we obtain for an independently working sink

$$q_- = \frac{Q}{2\pi r}$$

An independently working source produces

$$q_+ = \frac{Q}{2\pi\varrho} = \frac{Q}{2\pi(2a - r)}$$

The resulting specific flow on the line connecting the source and the sink is

$$q = -\frac{aQ}{\pi} \frac{1}{r(2a - r)}$$

The average velocity v in the pressure horizon of thickness T is given by

$$v = -\frac{aQ}{T\pi} \frac{1}{r(2a - r)} \quad (5.7.9)$$

and substituting in eqn. (5.7.1) we obtain

$$t = \mu_e \int_{r=a}^{r=r_0} \frac{dr}{v} = -\frac{\mu_e T\pi}{aQ} \int_a^{r_0} r(2a - r) dr$$

and after integration

$$t = -\frac{\mu_e T\pi}{Q} \left[\frac{2}{3} a^2 - r_0^2 \left(1 - \frac{r_0}{3a} \right) \right] \quad (5.7.10)$$

Paying heed to eqn. (5.6.6) in eqn. (5.7.10) and to the sense of flow, we have

$$\begin{aligned} t &= -\frac{\mu_e T \ln [2L/(r_0 - r_s) + 1]}{2(G_R - G_0)} \left[\frac{2}{3} a^2 - r_0^2 \left(1 - \frac{r_0}{3a} \right) \right] \\ t &= \frac{\mu_e \ln [2L/(r_0 - r_s) + 1]}{k(h_R - h_0)} \left[\frac{a^2}{3} - \frac{r_0^2}{2} \left(1 - \frac{r_0}{3a} \right) \right] \end{aligned} \quad (5.7.11)$$

For a small well, eqn. (5.7.11) simplifies to

$$t = \frac{\mu_e a^2 \ln \frac{2L}{r_0}}{3k(h_R - h_0)} \quad (5.7.12)$$

In Section 4.7 we discussed the calculation of the time of travel of a water particle in two-dimensional flow. By introducing the assumption of plane pressure seepage, we have actually converted the problem of plane flow to that of two-dimensional flow. All that is necessary is to imagine a vertical two-dimensional flow in the horizontal plane. A case of practical importance is, for example, that of seepage from a river into a linear array of wells spaced at equal distances. We shall not present here any details of the solution, for this would only entail a modification of the procedure outlined above.

5.8. Composition of plane flows

In Chapter 2 we discussed the problem of one-dimensional non-uniform groundwater motion along an inclined plane and arrived at the practical conclusion that solutions obtained by superposition of partial results are highly acceptable in many cases.

The method is equally applicable to problems of plane flow, provided that the assumptions of the theory are satisfied, especially that of vertical equipotential lines. The motion along an inclined plane representing the surface of a relatively impervious subsoil can be described by the suggested method whenever the angle of inclination of the level and the angle of inclination of the plane along which the motion occurs, are both small.

As an example, consider the discharge into an hydraulically perfect well working in parallel flow. The uniform flow is assumed to be parallel to the x axis. The sink representing the well is placed at the origin of the (z) plane. The streamlines of the two elementary flows and the streamlines of the resulting flow are shown in light dash lines in Fig. 5.9a, and the construction can be plainly seen in that figure.

To simplify, assume that the free surface runs within the range of a layer with a low coefficient of permeability, underlain by a water-bearing stratum of thickness T . The coefficient of permeability of the stratum is k . In this case, the level in the ground plane may be regarded as a two-dimensional flow, and the strategy can be based on the definition of the complex potential. The flow takes place parallel to the inclined subsoil plane; it is, therefore, sufficient to examine the conditions in a strip of unit thickness:

$$\begin{aligned}\omega(z) &= \varphi + i\psi, \quad z = x + iy \\ \omega(z) &= zv + \frac{\bar{Q}}{2\pi} \ln z + c\end{aligned}\tag{5.8.1}$$

where v — is the velocity of flow,

$\bar{Q}$ — the strength of the sink,

c — a constant.

Separating the above into the real and imaginary parts we have

$$\varphi = xv + \frac{\bar{Q}}{2\pi} \ln r + c = xv + \frac{\bar{Q}}{2\pi} \ln \sqrt{(x^2 + y^2)} + c \quad (5.8.2)$$

$$\psi = yv + \frac{\bar{Q}}{2\pi} \delta = yv + \frac{\bar{Q}}{2\pi} \operatorname{arctg} \frac{y}{x} \quad (5.8.3)$$

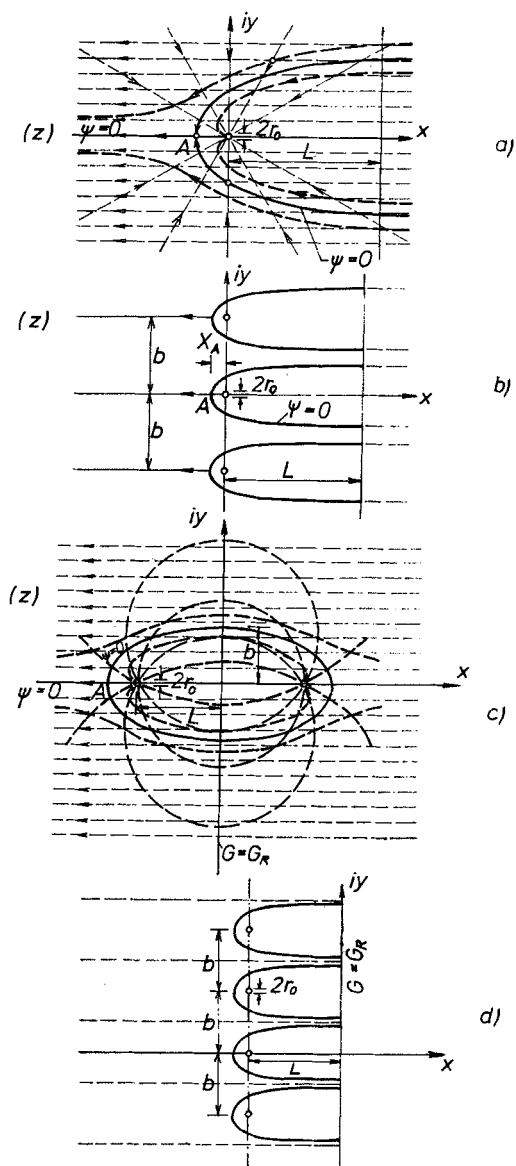


Fig. 5.9. Composition of plane flows.

The equations were obtained by superposition of plane flows. Of special interest in the resulting pattern is the streamline $\psi = 0$ the equation of which is implied directly by eqn. (5.8.3)

$$yv + \frac{\bar{Q}}{2\pi} \operatorname{arctg} \frac{y}{x} = 0 \quad (5.8.4)$$

For $x = \infty$, this leads to

$$y = \pm \frac{\bar{Q}}{2v} \quad (5.8.5)$$

Eqn. (5.8.5) defines the asymptote to the curve $\psi = 0$, whose equation is

$$x = - \frac{y}{\operatorname{tg} (2\pi yv/\bar{Q})} \quad (5.8.6)$$

The point of intersection of the curve and the y axis is obtained directly from (5.8.4) on setting $x = 0$:

$$y = \pm \frac{\bar{Q}}{4v} \quad (5.8.7)$$

Differentiating eqn. (5.8.2) gives the velocity components in the directions of the coordinate axes

$$v_x = v + \frac{\bar{Q}}{2\pi} \frac{x}{x^2 + y^2} \quad (5.8.8)$$

$$v_y = \frac{\bar{Q}}{2\pi} \frac{y}{x^2 + y^2} \quad (5.8.9)$$

On the x axis, the velocity $v_x = 0$ at the point

$$x = - \frac{\bar{Q}}{2\pi v} \quad (5.8.10)$$

At this point the x axis intersects the curve $\psi = 0$. To the curve also belongs that part of the x axis which lies in the interval

$$-\infty < x \leq - \frac{\bar{Q}}{2\pi v}$$

Transition to the values of h which characterize the positions of the level, viz.

$$h = - \frac{xv}{k} - \frac{\bar{Q}}{2\pi k} \ln \sqrt{(x^2 + y^2)} + c \quad (5.8.11)$$

indicates that the velocity v_x is regarded as positive when water flows in the direction of the x axis. The strength of the sink, $\bar{Q}$, is negative if water percolates into it from the underground.

Assume that in the immediate vicinity of the sink, the equipotential lines are circular. Take an equipotential line of radius r_0 for the outer contour of the well. For $x = 0$, $y = r_0$, $h = h_0$, we have

$$h - h_0 = -\frac{xv}{k} - \frac{\bar{Q}}{2\pi k} \ln \frac{\sqrt{(x^2 + y^2)}}{r_0} \quad (5.8.12)$$

It is easy to see in Fig. 5.9a that at greater distances from the sink, the shape of the equipotential lines tends to that of straight lines perpendicular to the x axis. We choose such a line that intersects the x axis at the point $x = L$. The corresponding piezometric head is $h = h_R$. Suppose further, that the strength of the well is $Q = T\bar{Q}$ and that in uniform flow there flows $q = vT$ per unit width of flow

$$Q = -\frac{2\pi[hT(h_R - h_0) + Lq]}{\ln L/r} \quad (5.8.13)$$

$$h = h_0 - \frac{1}{kT} \left[xq + \frac{Q}{2\pi} \ln \frac{\sqrt{(x^2 + y^2)}}{r_0} \right] \quad (5.8.14)$$

As Fig. 5.9a plainly shows, the well receives a natural flow of water of width Q/q . The range of its effect is restricted, and the reduction is theoretically limited only by the geological structure of the terrain. This conclusion becomes, of course, invalid once we start to consider the function of a whole system of an infinite number of wells which, according to Fig. 5.9b, receive water from a flow running at right angles to the well system. The possible strength of an object is co-determined by the inequality

$$|Q| \leq b|q|$$

where b is the spacing of the wells.

In low meadows near rivers we often come across sedimentary strata lying on the nearly horizontal surface of an impervious medium. The groundwater from the river infiltrates, or, more frequently, flows in the direction of the slopes towards the river. We, therefore note a special case of plane non-uniform flow described in the (z) plane by the complex potential

$$\Theta(z) = zq + \frac{Q}{2\pi} \ln \frac{L+z}{L-z} + c \quad (5.8.15)$$

On separating the above into the real and imaginary parts we have

$$G = xq + \frac{Q}{4\pi} \ln \frac{(L+x)^2 + y^2}{(L-x)^2 + y^2} + c \quad (5.8.16)$$

where G is the Girinsky potential (cf. Chapter 1)

$$\Psi = yq + \frac{Q}{2\pi} \operatorname{arctg} \frac{2Ly}{x^2 + y^2 - L^2} \quad (5.8.17)$$

Clearly, we are superposing the flow in the system sink-source onto the parallel flow in the direction of the x axis. Q is the strength of the sink, q — the volume flow per unit width of the parallel flow.

Fig. 5.9c shows the resulting image of the flow defined by the function $G(x, y)$. The sink in the left-hand half of the (z) plane is regarded as the image of the well; hence just as in the preceding case we write

$$G = G_0, \quad x = -(L - r_0), \quad y = 0, \quad L \gg r_0$$

Eqn. (5.8.16) gives

$$G - G_0 = q(L - x) + \frac{Q}{4\pi} \ln \frac{4L^2[(L + x)^2 + y^2]}{r_0^2[(L - x)^2 + y^2]} \quad (5.8.18)$$

On the y axis, $x = 0$, $G = G_R = \text{const}$. Hence the axis may be regarded as the image of the shore line:

$$Q = \frac{2\pi(G_R - G_0 - qL)}{\ln(2L/r_0)} \quad (5.8.19)$$

By differentiating eqn. (5.8.16) with respect to x we obtain

$$q_x = q + \frac{Q}{2\pi} \left[\frac{L + x}{(L + x)^2 + y^2} + \frac{L - x}{(L - x)^2 + y^2} \right] \quad (5.8.20)$$

on the x axis, $y = 0$; therefore, on this axis

$$q_x = q + \frac{Q}{\pi} \frac{L}{L^2 - x^2} \quad (5.8.21)$$

According to Fig. 5.9c, the value at the point A should be $q_x = 0$; hence

$$x_A = \pm \sqrt{L \left(L + \frac{Q}{\pi q} \right)} \quad (5.8.22)$$

At the point A , the streamline $\psi = 0$ defined by eqn. (5.8.17)

$$-\frac{2Ly}{x^2 + y^2 - L^2} = \text{tg} \frac{2\pi bq}{Q} \quad (5.8.23)$$

branches off.

On substituting $x = 0$ in eqn. (5.8.23) we obtain the following equation for calculating the minor semiaxis $y = b$ of the curve $\psi = 0$:

$$\frac{2Lb}{L^2 - b^2} = \text{tg} \frac{2\pi bq}{Q}$$

In the case of shore line infiltration considered, we assumed Q as well as q to be negative. Consider now the case when the water flows from the slopes to the river (from the hinterland), and accordingly, choose a positive q . Assume at the same time that the well only receives water from the hinterland. In the limiting case, i.e. at the origin of the (z) plane, $q_x = 0$. Eqn. (5.8.21) then yields the condition

$$Q \geq -q\pi L$$

and eqn. (5.8.19) the condition

$$G_0 \leq G_R + qL \left(\ln \sqrt{\frac{2L}{r_0}} - 1 \right)$$

These criteria make it possible to select the draw-off and the position of the level in a well in the case that the river contains impure water whose quality cannot be improved by passing through soil.

Collection of natural flows coming from the hinterland into a river, is more successful when the values of L are suitably large. If the space in the shore zone is extensive enough, conditions created in such cases are conducive to the drawing-off of particles which would otherwise reach the river. However, this conclusion does not apply in full measure if the collection is effected by a linear system of wells spaced at equal distances b . As seen in Fig. 5.9d, the axis of the system is parallel to the linear shape of the shore line.

According to eqn. (4.8), after superposition on a flow with a specific volume flow q , which takes place in the direction of the x axis, we obtain along the x axis

$$G = qx + \frac{Q}{4\pi} \ln \frac{\cosh [2\pi(x+L)/b] - \cos(2\pi y/b)}{\cosh [2\pi(x-L)/b] - \cos(2\pi y/b)} + c \quad (5.8.24)$$

The flow under examination is essentially a flow into a linear row of sinks. Assume that in the vicinity of the sinks the equipotential lines are circular, and that a circle of radius r_0 is simultaneously the image of the contour of an hydraulically perfect well. Setting $G = G_0$, $x = -L$, $y = r_0$, $L \gg r_0$ on the contour, we obtain

$$G_0 = -qL + \frac{Q}{4\pi} \ln \frac{1 - \cos(2\pi r_0/b)}{\cosh(4\pi L/b) - \cos(2\pi r_0/b)} + c$$

On substituting in eqn. (5.8.24) we have

$$G - G_0 = q(L+x) + \frac{Q}{4\pi} \ln \frac{\{\cosh [2\pi(x+L)/b] - \cos(2\pi y/b)\} [\cosh(4\pi L/b) - \cos(2\pi r_0/b)]}{\{\cosh [2\pi(x-L)/b] - \cos(2\pi y/b)\} [1 - \cos(2\pi r_0/b)]}$$

On the y axis where $x = 0$, we choose $G = G_R$

$$G_R - G_0 = qL + \frac{Q}{4\pi} \ln \frac{\cosh(4\pi L/b) - 1}{1 - \cos(2\pi r_0/b)}, \quad r_0 \ll b$$

Hence

$$Q = \frac{4\pi(G_R - G_0 - qL)}{\ln \frac{\cosh(4\pi L/b) - 1}{1 - \cos(2\pi r_0/b)}} \quad (5.8.25)$$

For $L \gg b$, $r_0 \ll b$, eqn. (5.8.25) simplifies to

$$\cosh \frac{4\pi L}{b} = \frac{1}{2} \operatorname{Esp} \left(\frac{4\pi L}{b} \right)$$

$$1 - \cos \frac{2\pi r_0}{b} = \frac{1}{2} \left(\frac{2\pi r_0}{b} \right)^2$$

Then

$$Q = \frac{b(G_R - G_0 - qL)}{L + (b/2\pi) \ln(b/2\pi r_0)} \quad (5.8.26)$$

Differentiating eqn. (5.8.24) with respect to x gives

$$q_x = q + \frac{Q}{2b} \left\{ \frac{\sinh[2\pi(x+L)/b]}{\cosh[2\pi(x+L)/b] - \cos(2\pi y/b)} - \frac{\sinh[2\pi(x-L)/b]}{\cosh[2\pi(x-L)/b] - \cos(2\pi y/b)} \right\}$$

At the origin of the (z) plane, we evidently obtain for $x = 0$, $y = 0$, the relation

$$q_x = q + \frac{Q}{b} \frac{\sinh(2\pi L/b)}{\cosh(2\pi L/b) - 1} = q + \frac{Q}{b \operatorname{tgh}(\pi L/b)}$$

If regard is paid to the signs of the quantities q and Q , the same expression applies to the case of natural flow of water from the hinterland towards the river.

To stop the flow on the shore line at the point $x = y = 0$, or to direct the flow of water toward the river, we must make

$$Q \geq -bq \operatorname{tgh} \frac{\pi L}{b}$$

If a system of wells is situated at a distance $L \geq 0.5b$, and all the flows coming from the slopes are assumed to be collected, the discharge per well will be

$$|Q| = b$$

subject, of course, to the condition that, approximately,

$$G_0 \leq G_R + \frac{b}{2\pi} \ln \frac{b}{2\pi r_0}$$

5.9. Seepage around lateral cut-off walls

Backwater produced by an hydraulic structure gives rise not only to seepage under the object but also to lateral escape of water. Seepage around lateral cut-off walls is quantitatively important. Moreover, the water exerts static action on the structure, retaining walls, etc.

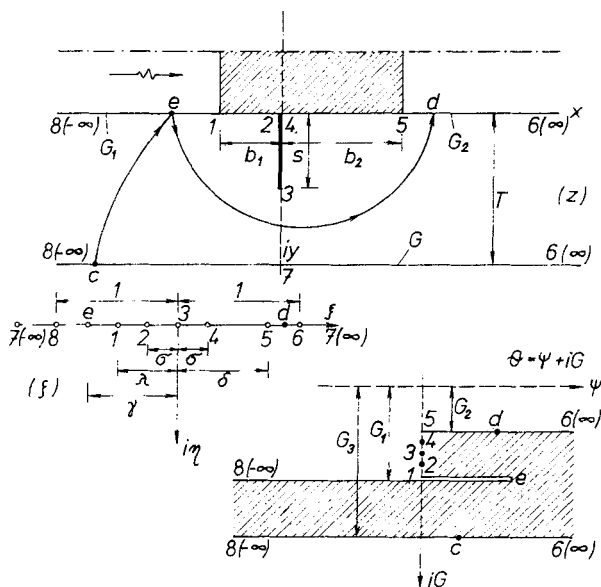


Fig. 5.10. Seepage around lateral cut-off walls.

The theory of plane seepage enables us to solve a number of problems of practical value. To demonstrate its principles, consider a structure founded on a relatively impervious subsoil, provided with a relatively impervious lateral cut-off wall of length s (Fig. 5.10). The structure was built on dry ground, and after the work was finished, there was still a diverted river bed at distance T from it. We consider the Girinsky potential to be G_1 along the shore line of the upstream reservoir, G_2 along the shore line of the downstream reservoir, and G_3 along the shore line of the diverted river bed, and represent the ground plan schematically in the complex plane $z = x + iy$.

The theory of seepage around lateral cut-off walls was elaborated by Nedriga [28]. Except for a few minor modifications and additions, we shall base the discussion that follows on the original version of the theory.

We use the method of conformal mapping (Chapter 3) which enables us to transform a semi-infinite strip with a slit image of the cut-off wall in the (z) plane, onto the (ζ) half-plane without a slit. Further, we shall set up the region of the complex potential in the (θ) plane. When dealing with plane flows, we operate with the func-

tions G , Ψ , and in the construction of the pattern, consider the distribution of the boundary conditions. The situation depends on the character of the flow in the neighbourhood of the structure. If $G_3 < G_1$, water from the diverted river bed flows into both the upstream and the downstream reservoirs. If $G_2 < G_3 < G_1$, several combinations are likely to arise. In the region adjacent to the upstream reservoir, some water flows into the river bed, some by-passes the cut-off wall and proceeds towards the downstream reservoir. Into there also percolates the water which entered the underground from the section of the diverted river bed, lying in the region of the downstream reservoir. The diverted river bed acts as a receiver which collects water percolating from the upstream reservoir, distributes it and enables it to seep back into the underground and in turn, into the downstream reservoir.

Further analysis would reveal the existence of a total of five possible combinations, each representing a definite flow regime. In the (θ) plane, this fact is reflected in the position of the slit.

The situation considered in our solution, is that shown in Fig. 5.10. The mapping of the (θ) plane onto the (ζ) half-plane is effected by the Schwarz-Christoffel integral (Chapter 3)

$$\begin{aligned}
 \Theta &= -iA \int_0^{\zeta} \frac{(\gamma + \zeta) d\zeta}{(1 - \zeta^2) \sqrt{[(\lambda + \zeta)(\delta - \zeta)]}} + B = \\
 &= -iA \left\{ (1 + \gamma) \int_0^{\zeta} \frac{d\zeta}{(1 - \zeta) \sqrt{[(\lambda + \zeta)(\delta - \zeta)]}} - \right. \\
 &\quad \left. - (1 - \gamma) \int_0^{\zeta} \frac{d\zeta}{(1 + \zeta) \sqrt{[(\lambda + \zeta)(\delta - \zeta)]}} \right\} + B \\
 &\quad \int_0^{\zeta} \frac{d\zeta}{(1 - \zeta) \sqrt{[(\lambda + \zeta)(\delta - \zeta)]}} = \\
 &= \frac{1}{\sqrt{[(\lambda + 1)(1 - \delta)]}} \arcsin \frac{\lambda - \delta - 2\lambda\delta + (2 + \lambda - \delta)\zeta}{(\lambda + \delta)(1 - \zeta)} - \\
 &\quad - \frac{1}{\sqrt{[(\lambda + 1)(1 - \delta)]}} \arcsin \frac{\lambda - 2\lambda\delta - \delta}{\lambda + \delta} \\
 &\quad \int_0^{\zeta} \frac{d\zeta}{(1 + \zeta) \sqrt{[(\lambda + \zeta)(\delta - \zeta)]}} = \\
 &= - \frac{1}{\sqrt{[(\delta + 1)(1 - \lambda)]}} \arcsin \frac{\delta - \lambda - 2\lambda\delta - (2 + \delta - \lambda)\zeta}{(\lambda + \delta)(1 + \zeta)} + \\
 &\quad + \frac{1}{\sqrt{[(\delta + 1)(1 - \lambda)]}} \arcsin \frac{\delta - 2\lambda\delta - \lambda}{\lambda + \delta}
 \end{aligned}$$

Accordingly,

$$\begin{aligned}
 \Theta = & -\frac{iA(1+\gamma)}{2\sqrt{[(1+\lambda)(1-\delta)]}} \arcsin \frac{\lambda - \delta - 2\lambda\delta + (2+\lambda-\delta)\zeta}{(\lambda+\delta)(1-\zeta)} - \\
 & -\frac{iA(1-\gamma)}{2\sqrt{[(1+\delta)(1-\lambda)]}} \arcsin \frac{\delta - \lambda - 2\lambda\delta - (2+\delta-\lambda)\zeta}{(\lambda+\delta)(1+\zeta)} + \\
 & +\frac{iA(1+\gamma)}{\sqrt{[(1+\lambda)(1-\delta)]}} \arcsin \frac{\lambda - 2\lambda\delta - \delta}{\lambda + \delta} + \\
 & +\frac{iA(1-\gamma)}{\sqrt{[(1+\delta)(1-\lambda)]}} \arcsin \frac{\delta - 2\lambda\delta - \lambda}{\lambda + \delta} + B
 \end{aligned} \quad (5.9.1)$$

The constants A, B are determined using the known values of Θ at the points 1, 5. At the point 1, $\Theta = iG_1$, $\zeta = -\lambda$, and

$$iG_1 = \frac{iA\pi}{4} \left\{ \frac{1+\gamma}{\sqrt{[(1+\lambda)(1-\delta)]}} - \frac{1-\gamma}{\sqrt{[(1-\lambda)(1+\delta)]}} \right\} + B + C \quad (5.9.2)$$

where

$$\begin{aligned}
 C = & -\frac{iA(1+\gamma)}{\sqrt{[(1+\lambda)(1-\delta)]}} \arcsin \frac{\lambda - 2\lambda\delta - \delta}{\lambda + \delta} + \\
 & +\frac{iA(1-\gamma)}{\sqrt{[(1+\delta)(1-\lambda)]}} \arcsin \frac{\delta - 2\lambda\delta - \lambda}{\lambda + \delta}
 \end{aligned} \quad (5.9.3)$$

At the point 5, $\Theta = iG_2$, $\zeta_5 = \sigma$, and

$$iG_2 = -\frac{iA\pi}{4} \left\{ \frac{1+\gamma}{\sqrt{[(1+\lambda)(1-\delta)]}} - \frac{1-\gamma}{\sqrt{[(1+\delta)(1-\lambda)]}} \right\} + B + C \quad (5.9.4)$$

It follows from eqns. (5.9.2) and (5.9.4) that

$$A = \frac{G_1 - G_2}{\pi} \frac{2}{(1+\gamma)/\sqrt{[(1+\lambda)(1-\delta)]} - (1-\gamma)/\sqrt{[(1+\delta)(1-\lambda)]}} \quad (5.9.5)$$

$$B + C = i \frac{G_1 + G_2}{2} \quad (5.9.6)$$

Another equation needed for the calculation, is obtained by analyzing the situation at the point 6. In the (Θ) plane, this point lies at infinity. If we by-pass it, we must record a change in G . The increment of G will be $(G_3 - G_2)$. In view of the small

radius of the circle used for by-passing the point δ ,

$$i(G_3 - G_2) = -iA \int \frac{(\gamma + \zeta) d\zeta}{(1 - \zeta^2) \sqrt{[(\lambda + \zeta)(\zeta - \delta)]}} = -iA \frac{\pi(\gamma + 1)}{2 \sqrt{[(\lambda + 1)(1 - \delta)]}} \quad (5.9.7)$$

Substituting for the constant A and some rearrangement gives

$$\frac{G_3 - G_1}{G_3 - G_2} = \frac{1 - \gamma}{1 + \gamma} \sqrt{\frac{(\lambda + 1)(1 - \delta)}{(\delta + 1)(1 - \lambda)}} \quad (5.9.8)$$

Hence

$$\gamma = \frac{(G_3 - G_2)\beta - (G_3 - G_1)}{(G_3 - G_2)\beta + (G_3 - G_1)} \quad (5.9.9)$$

where

$$\beta = \sqrt{\frac{(1 + \lambda)(1 - \delta)}{(1 - \lambda)(1 + \delta)}} \quad (5.9.10)$$

substituting eqns. (5.9.9), (5.9.5), (5.9.6) in eqn. (5.9.1) we obtain

$$\Theta = -i \left(\frac{G_3 - G_2}{\pi} \arcsin \frac{\alpha_1 + \alpha_2 \zeta}{1 - \zeta} + \frac{G_3 - G_1}{\pi} \arcsin \frac{\alpha_3 - \alpha_4 \zeta}{1 + \zeta} - \frac{G_1 + G_2}{2} \right) \quad (5.9.11)$$

where

$$\alpha_1 = \frac{\lambda - 2\lambda\delta - \delta}{\lambda + \delta}, \quad \alpha_2 = \frac{2 + \lambda - \delta}{\lambda + \delta}$$

$$\alpha_3 = \frac{\delta - 2\lambda\delta - \lambda}{\lambda + \delta}, \quad \alpha_4 = \frac{2 + \delta - \lambda}{\lambda + \delta} \quad (5.9.12)$$

Points in the (ζ) plane correspond to points in the (z) plane. The transformation function is again found with the help of the Schwarz-Christoffel integral. In the case of a semi-infinite strip with a slit in the (z) plane we shall use the well-known Pavlovsky's solution:

$$\zeta = \pm \sqrt{\left[1 - \frac{\cos^2(\pi s/2T)}{\cosh^2(\pi z/2T)} \right]} \quad (5.9.13)$$

Along the cut-off wall, i.e. along the segment 2, 3, 4, $z = iy$, so that

$$\zeta = \pm \sqrt{\left[1 - \frac{\cos^2(\pi s/2T)}{\cos^2(\pi y/2T)} \right]} \quad (5.9.14)$$

Along the upper edge of the semi-infinite strip, i.e. along the segment 2, 1, 8 or 4, 5, 6, $z = \pm x$, so that

$$\zeta = \pm \sqrt{\left[1 - \frac{\cos^2(\pi s/2T)}{\cosh^2(\pi x/2T)} \right]} \quad (5.9.15)$$

Along the image of the shore line of the diverted river bed, i.e. along the segment 6, 7, 8, $z = \pm x + iT$, so that

$$\zeta = \sqrt{\frac{\cosh^2(\pi x/2T) - \sin^2(\pi s/2T)}{\sinh(\pi x/2T)}} \quad (6.9.16)$$

Accordingly,

$$\lambda = -\sqrt{\left[1 - \frac{\cos^2(\pi s/2T)}{\cosh^2(\pi b_1/2T)}\right]} \quad (5.9.17)$$

$$\delta = \sqrt{\left[1 - \frac{\cos^2(\pi s/2T)}{\cosh^2(\pi b_2/2T)}\right]} \quad (5.9.18)$$

We now have all the formulae required in practical calculations. Thus, e.g. on substituting

$$\Theta = iG, \quad \zeta = \pm \xi$$

along the contour of the hydraulic structure, we obtain from eqn. (5.9.11) the relation

$$G = -\frac{G_3 - G_2}{\pi} \arcsin \frac{\alpha_1 + \alpha_2 \zeta}{1 - \zeta} - \frac{G_3 - G_1}{\pi} \arcsin \frac{\alpha_3 - \alpha_4 \zeta}{1 + \zeta} + \frac{G_1 + G_2}{2} \quad (5.9.19)$$

Along the upstream reservoir

$$\Theta = \Psi + iG_1, \quad \zeta = \xi$$

According to the definition of the stream function in plane flow, we may set $\Psi = Q$, so that the total volume flow Q_e for the segment 1 to 8 becomes

$$Q_e = \frac{G_3 - G_2}{\pi} \operatorname{argcosh} \frac{\alpha_1 + \alpha_2 \zeta}{1 - \zeta} - \frac{G_3 - G_1}{\pi} \operatorname{argcosh} \frac{\alpha_3 - \alpha_4 \zeta}{1 + \zeta} \quad (5.9.20)$$

The specific volume flow along the upstream reservoir is

$$q = \frac{\partial Q}{\partial x} = \frac{\partial Q}{\partial \zeta} \frac{d\zeta}{dx} \quad (5.9.21)$$

The principal characteristics of the flow along the downstream reservoir and the diverted river bed can be calculated in a similar manner. A bifurcation point occurs in places where $q = 0$: at such a point, the flows separate. Thus, e.g. for $G_3 < G_1$, $G_1 < G_2$, the water will percolate from the diverted river bed into the upstream reservoir along the segment $(-\infty)$ to the point e , where $q = 0$. Along the segment from the point e to the point 1, the water will seep from the upstream reservoir around the structure to the downstream reservoir. From the diverted river bed, the water will percolate towards the upstream reservoir along the segment $(-\infty)$ to the point c , and into the downstream reservoir along the segment from the point c to $(+\infty)$.

As noted earlier, there exist several other combinations of the boundary levels. Although a different shape of the complex potential region needs to be considered in each particular case, the solution as outlined satisfies all five combinations. In every case (including the one discussed) we obtain interior angles equal to $\pi/2$ at the points 1, 5, and interior angles equal to zero at the points 6, 8. Hence the general form of the transformation function according to the theory of conformal mapping turns out to be the same in all cases.

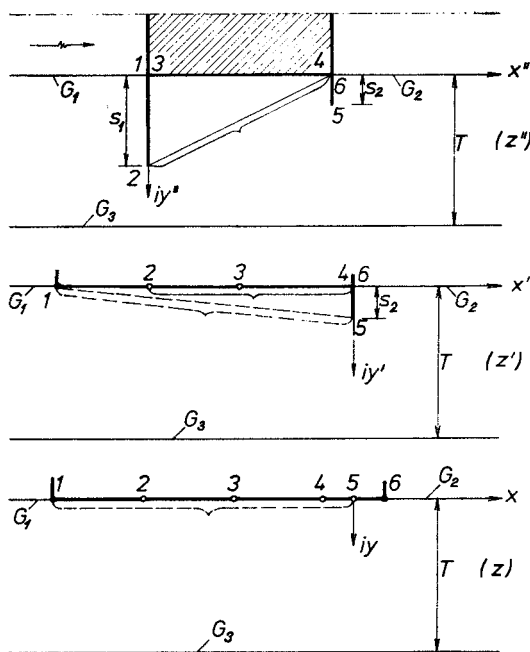


Fig. 5.11. Successive conformal mappings of plane flow.

As the results indicate, a number of singular points are present in the scheme shown in Fig. 5.10. From the point of view of structural safety, the singularities at the points 1, 5 are worthy of special notice. The situation can be improved by sinking the object into the earth mass. As an example, consider the project with two cut-off walls shown in Fig. 5.11.

The specified flow region in the (z'') plane will be processed by the method of successive conformal mappings (Section 4.1). The transformation function

$$z' = \sqrt{[(z'')^2 + s_1^2]}$$

has a simple geometrical interpretation. Following the Pythagorean theorem, measure the distance between the points of the structure contour and the end of the upstream

cut-off wall by means of a pair of compasses, and transfer it as indicated onto the (z') plane. Shift the cut-off wall on the downstream side of the structure. Follow this by graphical transformation of the points of the (z') plane onto the (z) plane. Noting that

$$z = \sqrt{[(z')^2 + s_2^2]}$$

use the Pythagorean theorem again, measure the distance between the points on the structure contour and the end of the remaining cut-off wall, and transfer it as indicated onto the (z) plane. The contour thus obtained in the (z) plane has no cut-off walls. The case may be solved by the application of the theory outlined above (as a somewhat simpler case for $s = 0$).

We find that in practice the shore line of the diverted river bed is hardly ever linear, and that the length of the bed is limited by levees closing its inlet and outlet. Such complicated projects are very difficult to solve analytically. In the simplified example discussed, on the other hand, when no water is pumped from, or artificially brought into, the diverted river bed, it is plain that

$$G_3 = \frac{G_1 - G_2}{2} + G_2 \quad (5.9.22)$$

We further note that for $b/T < 0.5$, the river bed exerts no substantial effect on the variations of the function G along the transformed contour of the structure. Hence, no gross error will be made if along the contour G is calculated using the solution which neglects the effect of the river bed. Upon modifying the relationships derived in Section 3.2.1, we obtain for the case being considered the relation

$$G = \frac{G_1 - G_2}{\pi} \arccos \frac{x}{b} + G_2 \quad (5.9.23)$$

and on substitution we obtain the relations

$$G_r = \frac{G - G_2}{G_1 - G_2}, \quad G = (G_1 - G_2) G_r + G_2 \quad (5.9.24)$$

$$G_r = \frac{1}{\pi} \arccos \frac{x}{b}, \quad 0 \leq G_r \leq 1 \quad (5.9.25)$$

The last formula may also be written as follows:

$$\frac{x}{b} = \cos \pi G_r \quad (5.9.26)$$

Fig. 5.12 shows a simple graphical interpretation of eqn. (5.9.26). The semi-circle of

radius b drawn over the transformed contour of the structure, contains 20 divisions, each corresponding to 5% of the maximum value of G_r . The scale enables us to read off the value of G_r corresponding to any point of the base and hence also to the transformed ends of the cut-off walls. To the value thus ascertained correspond a definite value of G and a definite value of h . We can, therefore, calculate the mean gradients of flow past the cut-off wall on the downstream side of the structure.

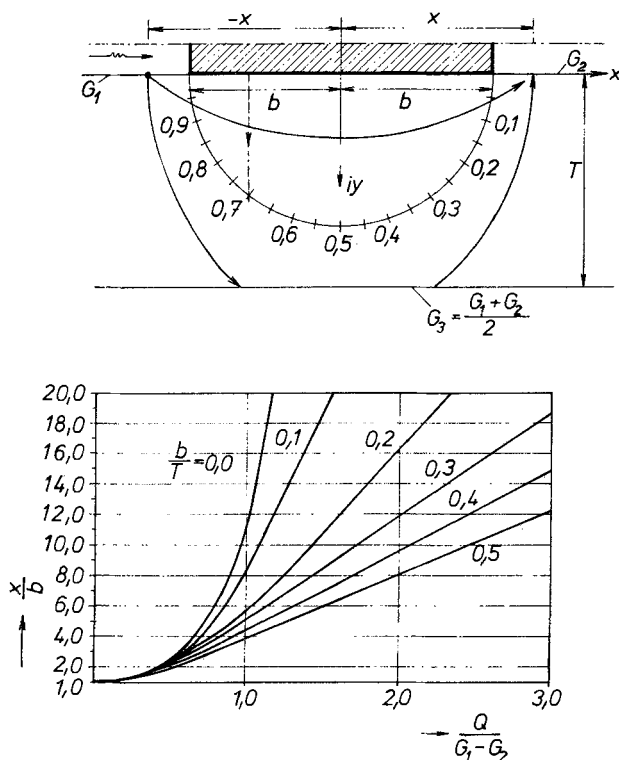


Fig. 5.12. Graphical solution of seepage around a transformed structure contour.

Since the flow pattern in the field between the structure and the river bed is symmetrical, the disposition of the bifurcation points on the upstream and downstream sides is symmetrical as well. The abscissae $x = X$ of the bifurcation points are determined from the simple relation

$$\pm \frac{X}{T} = \frac{2}{\pi} \quad (5.9.27)$$

The volume flow along the shore line is also symmetrical. We can read off its rate for various x/b in the diagram shown in Fig. 5.12 which plots the values of $Q/(G_1 - G_2)$ against x/b . The symbol Q denotes the total volume flow in the segment from the end

of the structure contour to the point with abscissa b . It is clear that in the segment

$$1 \leq \frac{x}{b} \leq \frac{X}{b}$$

Q by-passes the structure directly. In the segment

$$\frac{X}{b} \leq \frac{x}{b} \leq \infty$$

seepage occurs intermediately.

5.10. Quasi-plane flow

One of the assumptions of the theory of plane seepage flow is that the equipotential lines in stratified soils are vertical. If several horizons with different flow regimes exist in a stratum, this condition might not be satisfied. We know, for example, that continuous layers of poor permeability tend to encourage the division of the flow field and enable the water from the neighbouring positions with different flow regimes, to flow over. Since this flow occurs mainly in the vertical direction, we find that in a saturated stratum a vertical flow exists as well as a horizontal flow. Such a flow is called quasi-plane.

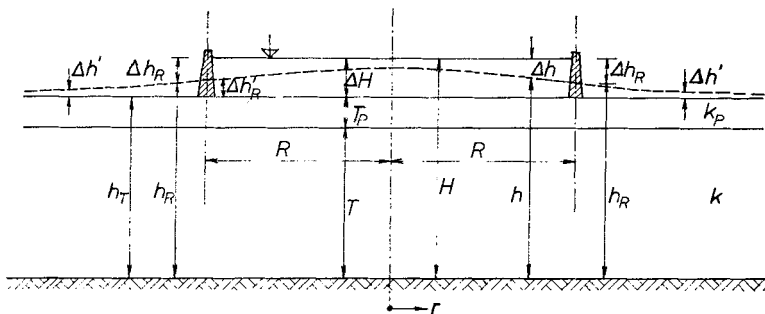


Fig. 5.13. Seepage through the circular bottom of a pond.

Because of the widely differing combinations of permeable and less permeable layers which an engineer is likely to meet with in practice, no general solution of the quasi-plane flow can be offered. Using an example from the field of land reclamation, we shall set out some of the basic principles from which a solution of quasi-plane flow can proceed.

Fig. 5.13 shows schematically a pond embankment of circular ground plan with radius R . The dimensions of the pond being large compared to the width of the embankment, we neglect the horizontal dimension of the latter.

Both the embankment and the pond lie on an overburden of low permeability, whose thickness is T_p and whose coefficient of permeability is k_p . The overburden is underlain by a much thicker and more permeable stratum of thickness T and coefficient of permeability k .

Since the water percolates from the pond symmetrically to all sides, the pressure line has a maximum on the axis of symmetry. Beyond the embankment, the pressure line is elevated above the terrain so that the water which has reached the subsoil through the pond moving downwards, seeps upwards in the hinterland. It is assumed that the water which has thus percolated up, will flow away along the surface of the terrain.

We first take note of the situation in the region of the pond. Assuming that $\partial h / \partial t = 0$, we can generally use eqn. (1.4.67) giving

$$\frac{\partial q_x}{\partial x} + \frac{\partial q_y}{\partial y} + v_0 = 0 \quad (5.10.1)$$

The term v_0 includes the effect of vertical flow downwards through the overburden. According to Darcy's law we obtain in this particular case

$$v_0 = -\frac{k_p}{T_p} (H - h)$$

$$q_x = -kT \frac{\partial h}{\partial x}, \quad q_y = -kT \frac{\partial h}{\partial y}$$

Substituting in eqn. (5.10.1) we have

$$-\frac{\partial^2 h}{\partial x^2} - \frac{\partial^2 h}{\partial y^2} - \frac{k_p}{kTT_p} (H - h) = 0$$

Hence in the notation

$$\beta = \sqrt{\frac{k_p}{kTT_p}} \quad (5.10.2)$$

$$\Delta h = H - h$$

$$\frac{\partial h}{\partial x} = -\frac{\partial(\Delta h)}{\partial x}, \quad \frac{\partial h}{\partial y} = -\frac{\partial(\Delta h)}{\partial y} \quad (5.10.3)$$

we have

$$\frac{\partial^2(\Delta h)}{\partial x^2} + \frac{\partial^2(\Delta h)}{\partial y^2} - \beta^2 \Delta h = 0 \quad (5.10.4)$$

Eqn. (5.10.4) expressed in terms of cylindrical coordinates is

$$\frac{\partial^2(\Delta h)}{\partial r^2} + \frac{1}{r} \frac{\partial(\Delta h)}{\partial r} - \beta^2 \Delta h = 0 \quad (5.10.5)$$

In essence, eqn. (5.10.5) is Bessel's differential equation the solution of which is

$$\Delta h = C_1 I_0(\beta r) + C_2 K_0(\beta r) \quad (5.10.6)$$

where

I_0 — the modified zero order Bessel function of the first kind,
 K_0 — the modified zero order Bessel function of the second kind,
 C_1, C_2 — constants.

The values of the constants C_1, C_2 are determined from the boundary conditions. Clearly, on the axis of symmetry, Δh must have a finite value. For $r \rightarrow 0$, the function $K_0(\beta r)$ tends to infinity, however, and we must therefore set $C_2 = 0$. Hence

$$\Delta h = C_1 I_0(\beta r) \quad (5.10.7)$$

Moreover, the pressure line has a maximum on the axis of symmetry; this means that

$$\frac{\partial(\Delta h)}{\partial r} = 0$$

there. In the case considered, we obtain for $r = 0$

$$\frac{d[I_0(\beta r)]}{dr} = \beta I_1(\beta r) = 0$$

where

I_1 — the modified first order Bessel function of the first kind.

Thus the solution described by eqn. (5.10.7) is satisfactory at the point $r = 0$. According to Fig. 15.13, at the point $r = R$

$$\Delta h = \Delta h_R = C_1 I_0(\beta R)$$

from which it follows that

$$C_1 = \frac{\Delta h_R}{I_0(\beta R)}$$

$$\Delta h = \Delta h_R \frac{I_0(\beta r)}{I_0(\beta R)} \quad (5.10.8)$$

The flow through a cylinder of radius r is

$$Q = -kT2\pi r \frac{\partial h}{\partial r} = 2\pi kTr \frac{\partial(\Delta h)}{\partial r} = 2\pi kTr\beta \Delta h_R \frac{I_1(\beta r)}{I_0(\beta R)} \quad (5.10.9)$$

Hence on the axis of symmetry of the embankment we obtain

$$Q_R = 2\pi kTR\beta \Delta h_R \frac{I_1(\beta R)}{I_0(\beta R)} \quad (5.10.10)$$

When analyzing the situation in the hinterland of the embankment we make use of the equation

$$\frac{\partial q_x}{\partial x} + \frac{\partial q_y}{\partial y} - v_0 = 0$$

Since water seeps upwards to the surface of the terrain, we choose the substitution

$$h - h_T = \Delta h' \quad (5.10.11)$$

$$v_0 = -\frac{k_p}{T_p} \Delta h', \quad q_x = -kT \frac{\partial(\Delta h')}{\partial x}, \quad q_y = -kT \frac{\partial(\Delta h')}{\partial y} \quad (5.10.12)$$

$$\frac{\partial^2(\Delta h')}{\partial x^2} + \frac{\partial^2(\Delta h')}{\partial y^2} - \beta \Delta h' = 0, \quad \beta = \sqrt{\frac{k_p}{kTT_p}} \quad (5.10.12)$$

Eqn. (5.10.12) is formally identical with eqn. (5.10.4); hence the general solution will be of the same form, viz.

$$\Delta h' = C_1 I_0(\beta r) + C_2 K_0(\beta r) \quad (5.10.13)$$

At the points $r \rightarrow \infty$, we must have $\Delta h' \ll \infty$; therefore, we choose $C_1 = 0$, and obtain

$$\Delta h' = C_2 K_0(\beta r)$$

$$\frac{\partial h}{\partial r} = \frac{\partial(\Delta h')}{\partial r} = -C_2 \beta K_1(\beta r) \quad (5.10.14)$$

K_1 is the modified first order Bessel function of the second kind. Hence the flow through a cylinder of radius r is, generally,

$$Q = -kT2\pi r \frac{\partial(\Delta h')}{\partial r} = kT2\pi r C_2 \beta K_1(\beta r)$$

On the axis of the embankment

$$Q_R = kT2\pi R C_2 \beta K_1(\beta R) \quad (5.10.15)$$

Since the equation of continuity of flow under the embankment simultaneously applies, eqn. (5.10.15) must be the same as eqn. (5.10.10)

$$C_2 K_1(\beta R) = \Delta h_R \frac{I_1(\beta R)}{I_0(\beta R)}$$

$$C_2 = \frac{\Delta h_R I_1(\beta R)}{I_0(\beta R) K_1(\beta R)}$$

$$\Delta h' = \frac{\Delta h_R I_1(\beta R) K_0(\beta r)}{I_0(\beta R) K_1(\beta R)} \quad (5.10.16)$$

The flow through a cylinder of radius $r > R$ is

$$Q = 2\pi k T r \beta \Delta h_R \frac{I_1(\beta R) K_1(\beta r)}{I_0(\beta R) K_1(\beta R)} \quad (5.10.17)$$

As is evident from Fig. 5.13, on the axis of the embankment

$$\Delta H = \Delta h'_R + \Delta h_R$$

and, therefore,

$$\Delta H = \frac{\Delta h_R I_1(\beta R) K_0(\beta R)}{I_0(\beta R) K_1(\beta R)} + \Delta h_R$$

From this it follows that

$$\Delta h_R = \frac{\Delta H}{1 + \{[I_1(\beta R) K_0(\beta R)]/[I_0(\beta R) K_1(\beta R)]\}} \quad (5.10.18)$$

Having determined the loss Δh_R on the axis of the embankment, we may regard the solution as complete, for we have established all the values of the constants required in practical computations.

5.11. Method of superposition of partial results

The example discussed in the preceding section was defined in a comparatively simple manner. Solutions of problems involving more general boundary conditions and a flow region of a fairly complex form, are not so easy to obtain.

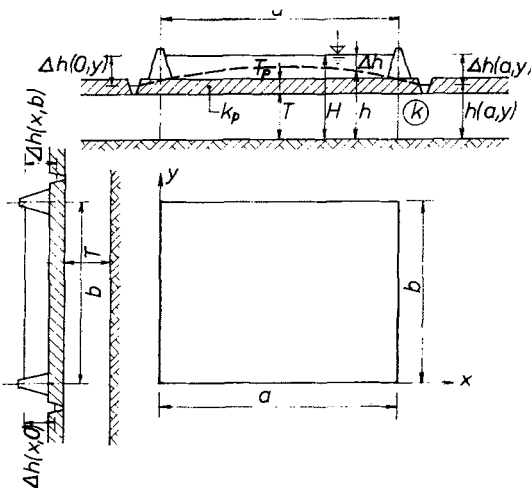


Fig. 5.14. Seepage through the rectangular bottom of a pond.

In many cases the examination is facilitated by decomposition into partial problems. An idea of this procedure will be gained by considering the example shown schematically in Fig. 5.14a.

Drainage elements for collecting and carrying away the seepage, are placed on the perimeter of a rectangular reservoir. Hydraulic resistances of substantial magnitude arise in the seepage of water through the reservoir bottom which is of relatively low permeability. Compared to them, the other losses (produced, for example, by the curvature of the streamlines underneath the embankment and at the entrance to the canal) can be regarded as negligible.

Choose a basic piezometric head (the head corresponding to the water level in the reservoir being the best suited for the purpose) and obtain the difference

$$\Delta h = H - h \quad (5.11.1)$$

We are examining a pseudo-plane flow, and the basic partial differential equation is of a form identical with that of eqn. (5.10.4). Eqn. (5.10.2) also applies, of course. The distribution of Δh on the boundaries may be defined by the following functions:

$$\begin{aligned} \Delta h &= f(0, y) \quad \text{on the line } x = 0 \\ \Delta h &= f(a, y) \quad \text{on the line } x = a \\ \Delta h &= f(x, 0) \quad \text{on the line } y = 0 \\ \Delta h &= f(x, b) \quad \text{on the line } y = b \end{aligned} \quad (5.11.2)$$

One of the partial solutions is obtained by considering only the first of the boundary conditions (5.11.2). In the remaining part of the flow region, the boundary conditions are set equal to zero

$$f(a, y) = f(x, 0) = f(x, b) = 0 \quad (5.11.3)$$

An analysis of this partial problem yields the distribution of the function Δh_{x0} which must satisfy the initial partial differential equation and the stated boundary conditions. We are therefore dealing with a function of x, y as well as of h . For clarity sake we denote it by

$$\Delta h_{x0} = F_{x0}(x, y, h) \quad (5.11.4)$$

In the next partial solution the second of the conditions (5.11.2) is considered, and the remaining ones are set equal to zero. Just as in the former case we consider the function

$$\Delta h_{xa} = F_{xa}(x, y, h) \quad (5.11.5)$$

Similar relations can be set up for the third and fourth partial solutions. The resulting function $\Delta h(x, y)$ is obtained by superposition of the partial results, viz.

$$\Delta h = F_{x0}(x, y, h) + F_{xa}(x, y, h) + F_{y0}(x, y, h) + F_{yb}(x, y, h) \quad (5.11.6)$$

Essentially, the methodical procedure consists of finding functions $F(x, y, h)$ which satisfy eqn. (5.11.6) while meeting the conditions of the partial problems.

Take, for example, the first partial solution and assume it to be in the form of the product of two auxiliary functions $X(x)$, $Y(y)$:

$$\Delta h_{x0} = X(x) Y(y) \quad (5.11.7)$$

Substituting in the equation

$$\frac{\partial^2(\Delta h_{x0})}{\partial x^2} + \frac{\partial^2(\Delta h_{x0})}{\partial y^2} - \beta^2 \Delta h_{x0}, \quad \beta = \sqrt{\frac{k_p}{kTT_p}} \quad (5.11.8)$$

which follows directly from eqn. (5.10.4) we obtain

$$\frac{1}{X(x)} \frac{\partial^2(X)}{\partial x^2} + \frac{1}{Y(y)} \frac{\partial^2(Y)}{\partial y^2} = \beta^2 \quad (5.11.9)$$

We re-write eqn. (5.11.9) in the form

$$\frac{1}{X(x)} \frac{\partial^2 X}{\partial x^2} - \frac{\beta^2}{2} = -\frac{1}{Y(y)} \frac{\partial^2(Y)}{\partial y^2} + \frac{\beta^2}{2} \quad (5.11.10)$$

$$\frac{1}{X(x)} \frac{\partial^2 X}{\partial x^2} - \frac{\beta^2}{2} = \kappa^2 \quad (5.11.11)$$

$$-\frac{1}{Y(y)} \frac{\partial^2(Y)}{\partial y^2} + \frac{\beta^2}{2} = \kappa^2 \quad (5.11.12)$$

where κ^2 is a parameter to which both sides of eqn. (5.11.10) must be equal.

Since equations of the type of eqns. (5.11.11) and (5.11.12) were solved in Chapter 4, we can at once write down the results

$$X = c_1 \cosh \left[x \sqrt{\left(\kappa^2 + \frac{\beta^2}{2} \right)} \right] + c_2 \sinh \left[x \sqrt{\left(\kappa^2 + \frac{\beta^2}{2} \right)} \right]$$

$$Y = C_1 \cos \left[y \sqrt{\left(\kappa^2 - \frac{\beta^2}{2} \right)} \right] + C_2 \sin \left[y \sqrt{\left(\kappa^2 - \frac{\beta^2}{2} \right)} \right]$$

where c_1, c_2, C_1, C_2 are constants.

Since it is required that $\Delta h_{x0} = 0$ for $x = a$, we have

$$c_1 \cosh \left[a \sqrt{\left(\kappa^2 + \frac{\beta^2}{2} \right)} \right] + c_2 \sinh \left[a \sqrt{\left(\kappa^2 + \frac{\beta^2}{2} \right)} \right] = 0$$

from which

$$c_1 = 1$$

$$c_2 = -\frac{1}{\operatorname{tgh} \left[a \sqrt{\left(\kappa^2 + \frac{\beta^2}{2} \right)} \right]}$$

Substituting in eqn. (5.11.7) we obtain

$$\Delta h_{x0} = \frac{\sinh \left[(a-x) \sqrt{\left(\kappa^2 + \frac{\beta^2}{2} \right)} \right]}{\sinh \left[a \sqrt{\left(\kappa^2 + \frac{\beta^2}{2} \right)} \right]} \left\{ C_1 \cos \left[y \sqrt{\left(\kappa^2 - \frac{\beta^2}{2} \right)} \right] + \right. \\ \left. + C_2 \sin \left[y \sqrt{\left(\kappa^2 - \frac{\beta^2}{2} \right)} \right] \right\}$$

At the points of the straight line $y = 0$, we should have $\Delta h_{x0} = 0$; this is possible if $C_1 = 0$. At the points of the straight line $y = b$, we should have $\Delta h_{x0} = 0$. Hence

$$0 = C_2 \sin \left[b \sqrt{\left(\kappa^2 - \frac{\beta^2}{2} \right)} \right]$$

For $C_2 \neq 0$ we obtain

$$b \sqrt{\left(\kappa^2 - \frac{\beta^2}{2} \right)} = n\pi, \quad n = 1, 2, \dots$$

so that

$$\kappa^2 - \frac{\beta^2}{2} = \frac{n^2 \pi^2}{b^2}, \quad \kappa^2 + \frac{\beta^2}{2} = \frac{n^2 \pi^2}{b^2} + \beta^2$$

The parameter κ is multi-valued: a different value is obtained for each n . We showed in Chapter 4 that the resulting pattern of flow can be established by superposing the partial patterns obtained by separate analyses made with different functions on the boundaries of the region. Assume that in the case being considered we can also use the sum

$$\Delta h_{x0} = \sum_{n=1}^{\infty} \frac{\sinh \left\{ (a-x) \sqrt{\left[(n^2 \pi^2 / b^2) + \beta^2 \right]} \right\}}{\sinh \left\{ a \sqrt{\left[(n^2 \pi^2 / b^2) + \beta^2 \right]} \right\}} (C_2)_n \sin \frac{n\pi y}{b} \quad (5.11.13)$$

and eqn. (5.11.13) is to satisfy the boundary condition of the partial solution at $x = 0$.

Assume that in the interval $0 < y < b$, the function $f(0, y)$ is continuous, takes finite values and has not more than a finite number of extreme values. Then we can use the Fourier expansion:

$$\Delta h_{x=0} = f(0, y) = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi y}{b} \quad (5.11.14)$$

The coefficients A_n are given by the integral

$$A_n = \frac{2}{b} \int_0^b f(0, y) \sin \frac{n\pi y}{b} \quad (5.11.15)$$

On the left-hand boundary, $x = 0$; hence we have, in place of eqn. (5.11.13), the equation

$$\Delta h_{x=0} = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi y}{b} = \sum_{n=1}^{\infty} (C_2)_n \sin \frac{n\pi y}{b}$$

which leads to

$$(C_2)_n = A_n$$

The first partial solution can then be formulated as follows:

$$\Delta h_{x0} = \sum_{n=1}^{\infty} A_n \frac{\sinh \{(a-x) \sqrt{[(n^2\pi^2/b^2) + \beta^2]}\}}{\sinh \{a \sqrt{[(n^2\pi^2/b^2) + \beta^2]}\}} \sin \frac{n\pi y}{b} \quad (5.11.16)$$

Since the remaining partial solutions are essentially modifications of the procedure outlined above, we can immediately write the results:

$$\begin{aligned} \Delta h = & \sum_{n=1}^{\infty} \left\{ A_n \frac{\sinh \{(a-x) \sqrt{[(n^2\pi^2/b^2) + \beta^2]}\}}{\sinh \{a \sqrt{[(n^2\pi^2/b^2) + \beta^2]}\}} \sin \frac{n\pi y}{b} + \right. \\ & + B_n \frac{\sinh \{x \sqrt{[(n^2\pi^2/b^2) + \beta^2]}\}}{\sinh \{a \sqrt{[(n^2\pi^2/b^2) + \beta^2]}\}} \sin \frac{n\pi(b-y)}{b} + \\ & + C_n \frac{\sinh \{(b-y) \sqrt{[(n^2\pi^2/b^2) + \beta^2]}\}}{\sinh \{b \sqrt{[(n^2\pi^2/b^2) + \beta^2]}\}} \sin \frac{n\pi x}{b} + \\ & \left. + D_n \frac{\sinh \{y \sqrt{[(n^2\pi^2/b^2) + \beta^2]}\}}{\sinh \{b \sqrt{[(n^2\pi^2/b^2) + \beta^2]}\}} \sin \frac{n\pi(a-x)}{b} \right\} \quad (5.11.17) \end{aligned}$$

$$\begin{aligned} A_n &= \frac{2}{b} \int_0^b f(0, y) \sin \frac{n\pi y}{b} dy, \quad B_n = \frac{2}{b} \int_0^b f(a, b-y) \sin \frac{n\pi(b-y)}{b} dy \\ C_n &= \frac{2}{a} \int_0^a f(x, 0) \sin \frac{n\pi x}{a} dx, \quad D_n = \frac{2}{a} \int_0^a f(a-x, b) \sin \frac{n\pi(a-x)}{a} dx \end{aligned} \quad (5.11.18)$$

The components of the quantity of seepage are calculated using the derivatives:

$$q_x = kT \frac{\partial(\Delta h)}{\partial x}, \quad q_y = kT \frac{\partial(\Delta h)}{\partial y} \quad (5.11.19)$$

The total quantity of seepage along the sides of the rectangle is

$$\begin{aligned} Q \Big|_{x=0} &= kT \int_0^b \frac{\partial(\Delta h)}{\partial x} \Big|_{x=0} dy, \quad Q \Big|_{x=a} = kT \int_0^b \frac{\partial(\Delta h)}{\partial x} \Big|_{x=a} dy \\ Q \Big|_{y=0} &= kT \int_0^a \frac{\partial(\Delta h)}{\partial y} \Big|_{y=0} dx, \quad Q \Big|_{y=b} = kT \int_0^a \frac{\partial(\Delta h)}{\partial y} \Big|_{y=b} dx \end{aligned} \quad (5.11.20)$$

Consider now the case where $\Delta h = \Delta h_0 = \text{const.}$ throughout the perimeter of the reservoir. Then

$$A_n = B_n = C_n = D_n = \frac{2 \Delta h_0}{n\pi} (1 - \cos n\pi)$$

According to eqn. (5.11.17), in this particular case

$$\begin{aligned} \Delta h = \frac{2 \Delta h_0}{\pi} \sum_{n=1}^{\infty} \left\{ \frac{\sinh \{(a-x) \sqrt{[(n\pi/b)^2 + \beta^2]}\} - 1 - \cos n\pi}{n} \frac{\sinh \{x \sqrt{[(n\pi/b)^2 + \beta^2]}\}}{\sinh \{a \sqrt{[(n\pi/b)^2 + \beta^2]}\}} \sin \frac{n\pi y}{b} + \right. \\ \left. + \frac{\sinh \{(b-y) \sqrt{[(n\pi/a)^2 + \beta^2]}\} - 1 - \cos n\pi}{n} \frac{\sinh \{y \sqrt{[(n\pi/a)^2 + \beta^2]}\}}{\sinh \{b \sqrt{[(n\pi/a)^2 + \beta^2]}\}} \sin \frac{n\pi x}{a} \right\} \quad (5.11.21) \end{aligned}$$

This equation satisfies the boundary condition. Consider, for example, the straight line $x = 0$. In the interval $0 \leq y \leq b$ we obtain

$$\begin{aligned} \Delta h = \frac{2 \Delta h_0}{\pi} \sum_{n=1}^{\infty} \frac{1 - \cos n\pi}{n} \sin \frac{n\pi y}{b} = \frac{2 \Delta h_0}{\pi} \left(2 \sin \frac{\pi y}{b} + \frac{2}{3} \sin \frac{3\pi y}{b} + \dots \right) = \\ = \frac{4 \Delta h_0}{\pi} \left(\sin \frac{\pi y}{b} + \frac{1}{3} \sin \frac{3\pi y}{b} + \dots \right) = \Delta h_0 \end{aligned}$$

At the centre of the reservoir, where $x = \frac{1}{2}a$, $y = \frac{1}{2}b$ we have

$$\begin{aligned} \Delta h = \frac{\Delta h_0}{\pi} \sum_{n=1}^{\infty} \frac{(1 - \cos n\pi)^2}{n} \left(\frac{1}{\cosh \{\frac{1}{2}a \sqrt{[(n\pi/b)^2 + \beta^2]}\}} + \right. \\ \left. + \frac{1}{\cosh \{\frac{1}{2}b \sqrt{[(n\pi/a)^2 + \beta^2]}\}} \right) \sin \frac{n\pi}{2} \quad (5.11.22) \end{aligned}$$

So long as the product $a\beta$ and also the product $b\beta$ are sufficiently large, the pressure loss at that location tends to zero. Seepage occurs mainly near the embankments, from the interior of the reservoir.

Since the functional relationships in eqn. (5.11.21) are quite simple, we do not deem it necessary to explain in detail the procedure for calculating eqns. (5.11.19) and (5.11.20). Suffice it to note that the method of superposition of partial results extends in a way the principles outlined in Chapter 4. A further generalization which is also possible, will be dealt with in the next section.

5.12. Method of auxiliary source. Probability flux

Inside a region S with boundary L , consider two harmonic functions. The first function, $G(x, y)$, has the meaning of the Girinsky potential and satisfies the equation

$$\frac{\partial^2 G}{\partial x^2} + \frac{\partial^2 G}{\partial y^2} = 0 \quad (5.12.1)$$

This is clearly a continuous function inside the region S , which, moreover, takes finite values there. At a certain point $M(x, y)$, the value of the function is G_M ; this is a mean of the values of G on a circle of small radius r centred at the point M :

$$G_M = \frac{1}{2\pi r} \int_{\Delta'} G dL' \quad (5.12.2)$$

Other circles lie inside the small one, and the above remarks apply to all of them. Hence

$$G_M = \frac{1}{\pi r^2} \int_{S'} G dS' = \frac{1}{S'} \int_{S'} G dS' \quad (5.12.3)$$

The value of G at the point (x, y) is the mean of all its values over the circular area of small radius r .

We shall further assume that the boundary L is divided into a definite number of segments. Let $G_i = \text{const.}$ in the i -th segment.

The function $G(x, y)$ is related to the stream function $\Psi(x, y)$ which is defined by

$$\frac{\partial^2 \Psi}{\partial x^2} + \frac{\partial^2 \Psi}{\partial y^2} = 0 \quad (5.12.4)$$

Suppose that at a point $M(x, y)$ a source of strength Q operates inside the region S independently of G . The flow is described in terms of the velocity potential $\varphi(x, y)$ and the stream function related to it, $\psi(x, y)$:

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0 \quad (5.12.5)$$

$$\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} = 0 \quad (5.12.6)$$

Eqn. (5.12.6) applies everywhere inside the region S , except at the point M , at which point the value of φ is infinite (a singular point). Eqn. (5.12.6) was obtained by substituting the velocity potential in the equation of continuity.

At a singular point the velocity increments in the directions of the coordinate axes must be equal to the strength of the source, Q . For a very small circular area

of centre at the point $M(x, y)$ we have

$$\int_{S'} \left(\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} \right) dS' = \int_{S'} \left(\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} \right) dS' \quad (5.12.7)$$

where

Q — strength of the source,

v_x, v_y — velocity components.

Hence in the immediate neighbourhood of the point $M(x, y)$ we can assume inside a circle of a small radius, that

$$\int_{S'} G \left(\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} \right) dS' = QG(x_M, y_M) \quad (5.12.8)$$

Far away from the point M , each curve is, of course, defined by eqn. (5.12.6). Assume that we have prescribed $\varphi = 0$ on the boundary L . The quantity which flows in the direction of the normal to the boundary curve, is equal to the strength of the source. Divide the curve L into a definite number of segments. Through each segment flows a definite part Q_i of the total strength of the source,

$$Q_i = \int_{L_i} \frac{\partial \varphi}{\partial n} dL \quad (5.12.9)$$

where the symbol n indicates that the derivative is in the direction of the normal to the boundary curve (the normal points away from the interior of the region).

The functions $G(x, y)$ and $\varphi(x, y)$ are defined in a way which makes it possible to use Green's second theorem

$$\int_S \left[G \left(\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} \right) - \varphi \left(\frac{\partial^2 G}{\partial x^2} + \frac{\partial^2 G}{\partial y^2} \right) \right] dS = \int_L \left(G \frac{\partial \varphi}{\partial n} - \varphi \frac{\partial G}{\partial n} \right) dL \quad (5.12.10)$$

Since it was assumed that $\varphi = 0$ on the boundary, the second term on the right-hand side of eqn. (5.12.10) is equal to zero. Moreover, we have that $G_i = \text{const.}$ in each i -th segment of the boundary. Hence

$$\int_L G \frac{\partial \varphi}{\partial n} dL = \sum_{i=1}^I G_i \int_{L_i} \frac{\partial \varphi}{\partial n} dL = \sum_{i=1}^I G_i Q_i \quad (5.12.11)$$

In view of eqn. (5.12.1), the second term on the left-hand side of eqn. (5.12.10) is equal to zero; we are thus left with

$$\int_S G \left(\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} \right) dS = \sum_{i=1}^I G_i Q_i \quad (5.12.12)$$

Since the right-hand side of eqn. (5.12.12) has a finite value, we must also have on the left-hand side

$$\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} \neq 0$$

In the region S , this condition is only satisfied by a small part S' in the neighbourhood of the point $M(x, y)$. Quantitatively S' is clearly representative of the whole of S . It is, therefore, enough to use eqn. (5.12.8), which leads to the simple relation

$$Q G(x_M, y_M) = \sum_{i=1}^I G_i Q_i$$

or,

$$G(x_M, y_M) = \sum_{i=1}^I G_i \frac{Q_i}{Q} \quad (5.12.13)$$

where $G(x_M, y_M)$ is the value of the G -potential at the point $M(x, y)$ where we have placed the source.

Consider now the problem in which we are required to find the value $G(x, y)$ in the region S , with a boundary divided into segments having prescribed values G_i . In the solution we shall make use of a source of strength Q . We shall measure the quantities of flow Q_i through each segment of the boundary, and substitute the result in eqn. (5.12.13).

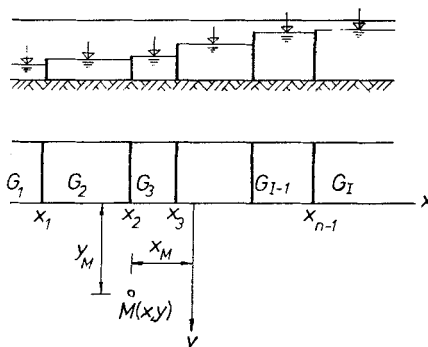


Fig. 5.15. Seepage from a cascade of reservoirs.

The value of G at another point is obtained in a similar way using another source. This will give some idea of the distribution of the function G in the entire field.

In the region S , there can exist segments of the boundary L along which $\partial G / \partial n = 0$ (impervious segments). We can assume that the distribution of the values $G(x, y)$ was obtained by the superposition or composition of partial results, so that we should, in fact, take into consideration a broader region in which the impervious segments are no longer present. We note, however, that we shall use an auxiliary source in this broader region also. It will, therefore, suffice to assume that $\partial \varphi / \partial n = 0$ on the segments of the boundary L where $\partial G / \partial n = 0$.

To illustrate the application of the method of auxiliary source, consider the plane flow in the shore zone of a canal provided with baffles which divide the longitudinal profile into a number of reservoirs with nearly horizontal levels. According to Fig. 5.15 a value of G_i corresponds to each (i -th) level. The system x, y is chosen at will. The positions of the reservoirs on the x axis are described by the abscissae x_i .

We are required to determine the value of G at the point $M(x, y)$. The auxiliary source placed at this point, operates on the assumption that $\varphi = 0$ at $x = 0$. Theoretically speaking, we are studying a dipole, i.e. a case which we have already discussed. In the neighbourhood of the source, the velocity potential is given by the expression

$$\varphi = \frac{Q}{4\pi} \ln \frac{(x - x_M)^2 + (y + y_M)^2}{(x - x_M)^2 + (y - y_M)^2} \quad (5.12.14)$$

Along the x axis we have

$$\frac{\partial \varphi}{\partial y} = \frac{Q}{\pi} \frac{y_M}{(x - x_M)^2 + y_M^2} \quad (5.12.15)$$

The partial discharges into the individual segments of the x axis are obtained by integration, viz.

$$Q_i = \int_{x_{i-1}}^{x_i} \frac{\partial \varphi}{\partial y} dx = \frac{Q}{\pi} \left(\arctg \frac{x_i - x_M}{y_M} - \arctg \frac{x_{i-1} - x_M}{y_M} \right) \quad (5.12.16)$$

Hence eqn. (5.12.13) becomes

$$G(x_M, y_M) = \frac{1}{\pi} \sum_{i=1}^I G_i \left(\arctg \frac{x_i - x_M}{y_M} - \arctg \frac{x_{i-1} - x_M}{y_M} \right) \quad (5.12.17)$$

Since the position of the point M was chosen arbitrarily, the resulting equation holds generally. We can therefore write directly $x_M = x$, $y_M = y$ in eqn. (5.12.17).

Another interesting interpretation of the method of auxiliary source stems from the work of Petrowsky who studied random motions. Consider the initial point $M(x, y)$ inside the region S with boundary L , and advance from it by a certain, relatively small length r , where the direction of the advance is chosen at random. All points of the circle L' of a radius equal to the chosen length centred at the point M , can be reached with the same probability denoted by

$$\frac{1}{2\pi r} \varrho_{L'} \quad (5.12.18)$$

Imagine now that we have reached a certain point of the circle and are continuing in random motion along the same path as before. The probability that we shall arrive

again at the initial point $M(x, y)$ is equal to (5.12.18). On the other hand, if we had started from all points of the circle, the total probability that we shall arrive again at the point $M(x, y)$ is given by the formula

$$\varrho(x, y) = \frac{1}{2\pi r} \int_{L'} \varrho_{L'} dL' \quad (5.12.19)$$

which resembles the formulation of the principle of mean value, eqn. (5.12.2). We assume, therefore, that there exists a harmonic function $\varrho(x, y)$ inside L for which

$$\frac{\partial^2 \varrho}{\partial x^2} + \frac{\partial^2 \varrho}{\partial y^2} = 0 \quad (5.12.20)$$

The boundary conditions on L will be determined by examining the advance of the random motion. When moving from the initial point $M(x, y)$, we can expect with certainty to reach some point of a small circle. The probability of reaching this circle is $\varrho = 1$. However, only one of the points is actually reached. Imagine a circle drawn at that point with radius equal to that of the former circle, and continue moving. Upon reaching one of the points of this new circle, repeat the procedure until the boundary L is reached somewhere. Since the point can move no farther, the probability of reaching some other point is zero. On the boundary L , $\varrho = 0$.

The motion can be thought of as a sequence of jerky advances which become continuous as we make the lengths of the paths get smaller and smaller. Connection with a previous motion is absent at the initial point $M(x, y)$ which was originally at rest. Except at this point we assume the function $\varrho(x, y)$ which characterizes the probability that the point of coordinates (x, y) will be reached in random motion from the initial point with coordinates (x_M, y_M) , to be continuous. Inside the region S (with the exception of the point M) we can consider the total differential

$$d\varrho = \frac{\partial \varrho}{\partial x} dx + \frac{\partial \varrho}{\partial y} dy \quad (5.12.21)$$

The partial derivatives $\partial \varrho / \partial x$, $\partial \varrho / \partial y$ stand for the increments of probability in the directions of the coordinate axes.

Among the whole set of paths which lead from the point M we shall find one for which the expectation that a point inside S , specified in advance, will be reached from the point M , is the greatest. The probability of reaching other points naturally varies along this path: it falls off from a maximum in the immediate vicinity of the point M , to a minimum on the boundary L . Select now a complete set of paths of greatest expectation. On these paths lie points which can be reached with the same probability. The isolines $\varrho = \text{const.}$ are orthogonal to the curves of the paths of greatest expectation, i.e. the curves which we may describe by $\sigma = \text{const.}$ The con-

dition of orthogonality is

$$\frac{\partial \varrho}{\partial x} \frac{\partial \sigma}{\partial x} + \frac{\partial \varrho}{\partial y} \frac{\partial \sigma}{\partial y} = 0 \quad (5.12.22)$$

$$\frac{\partial \varrho}{\partial x} = \frac{\partial \sigma}{\partial y} \quad (5.12.23)$$

$$\frac{\partial \varrho}{\partial y} = - \frac{\partial \sigma}{\partial x} \quad (5.12.24)$$

The derivatives of the function $\sigma(x, y)$ in the directions of the coordinate axes give uniquely the components of the probability variations in the directions of opposite axes. Eqns. (5.12.23) and (5.12.24) imply that, because of the continuity of $\sigma(x, y)$, the total differential

$$d\sigma = \frac{\partial \sigma}{\partial x} dx + \frac{\partial \sigma}{\partial y} dy \quad (5.12.25)$$

exists.

Eqn. (5.12.25) is, of course, invalid at the initial point M .

The total differential suggests the probability of reaching an element of some line inside S , which does not pass through the point M . Along a definite line starting at the point C and terminating at the point D , we have

$$\Sigma_{CD} = \oint_{CD} d\sigma = \sigma_D - \sigma_C \quad (5.12.26)$$

Since the path of integration is immaterial, the shape of the curve between C , D is also immaterial. Along a closed curve ($C \equiv D$), $\Sigma_{CD} = 0$.

Consider now a fixed point C inside S . Between the point C and some other point

$$\Sigma = \sigma - \sigma_C \quad (5.12.27)$$

Substituting in eqns. (5.12.23), (5.12.24) we obtain

$$\frac{\partial \varrho}{\partial x} = \frac{\partial \Sigma}{\partial y} \quad (5.12.28)$$

$$\frac{\partial \varrho}{\partial y} = - \frac{\partial \Sigma}{\partial x} \quad (5.12.29)$$

On substituting in eqn. (5.12.25) we find that

$$d\sigma = d\Sigma = \frac{\partial \Sigma}{\partial x} dx + \frac{\partial \Sigma}{\partial y} dy = - \frac{\partial \varrho}{\partial y} dx + \frac{\partial \varrho}{\partial x} dy \quad (5.12.30)$$

Since along the curve L where is prescribed $\varrho = 0 = \text{const.}$, the lines $\sigma = \text{const.}$ are normal to the boundary, we have there

$$d\Sigma = d\sigma = - \frac{\partial \varrho}{\partial n} \quad (5.12.31)$$

$$\Sigma = \int_L d\sigma \, dL \quad (5.12.32)$$

and also

$$\Sigma = - \frac{\partial \varrho}{\partial n} dL \quad (5.12.33)$$

The minus sign in eqn. (5.12.33) implies that the derivative is in the direction of the normal to L , which points away from the interior of S . This is a mathematical detail which is of no great importance in the case being considered.

Since we have found no appropriate term for the function $\Sigma(x, y)$ in the literature, we call it — for the time being — the probability flux.

Suppose further that on some segments of the boundary L the condition

$$\frac{\partial \varrho}{\partial n} = 0 \quad (5.12.34)$$

is imposed.

It is thus not allowed that the motion from the point M should terminate in places where eqn. (5.12.34) holds. After the boundary is reached, a recoil takes place towards the interior of S . Evidently, $\sigma = \text{const.}$, $\Sigma = \text{const.}$ in segments on which eqn. (5.12.34) holds.

A random motion from the point M can clearly be described similarly as a flow from the auxiliary source. The formal relations implementing the correspondence are

$$\begin{aligned} \varphi(x, y) &\leftrightarrow \varrho(x, y) \\ \psi(x, y) &\leftrightarrow \sigma(x, y) \\ Q_i(x, y) &\leftrightarrow \Sigma_i(x, y) \end{aligned} \quad (5.12.35)$$

Hence we can also make use of the proportion

$$\frac{\Sigma_i}{\Sigma} = \frac{Q_i}{Q} \quad (5.12.36)$$

where

Σ_i is the probability flux along the i -th segment of the boundary L , and Σ is the total probability flux.

The probabilistic approach can also be applied to the solution of the problem being considered. Indeed, in place of eqn. (5.12.13) we can write that

$$G(x_M, y_M) = \sum_{i=1}^I G_i \frac{\Sigma_i}{\Sigma} \quad (5.12.37)$$

This conclusion can be put to good use when quick information is required about the value of the function G at some point of a given field. The flow region is drawn on paper, and the impervious segments of the boundary are marked out with rulers. Random and zig-zag lines are then drawn by hand from a chosen point until the boundary is reached. The rulers preclude the crossing of the segments which should not be crossed. If a sufficient number, N , of such experiments is made, we find that N_i lines have passed through the i -th boundary, and we have, approximately, that

$$\frac{N_i}{N} \approx \frac{\Sigma_i}{\Sigma} \quad (5.12.38)$$

$$(Gx_M, y_M) = \sum_{i=1}^I G_i \frac{N_i}{N} \quad (5.12.39)$$

Eqn. (5.12.39) enables us to calculate the required value of G at the initial point M .

The procedure has the merit that it determines G at selected points of the region S . It is not necessary to analyze the flow in the whole field all at once. What matters, is, of course, the character of the problem in question. In many cases the point of interest is the overall picture which can be obtained by one of the methods described in the preceding sections, or by means of an analytical interpretation of the method of auxiliary sources. From the theoretical viewpoint the latter is equivalent to the method of Green's function (cf. Chapter 6). However, even in the solutions based on that method, the procedure of dividing the region of flow into fragments, analyzing the fragments separately and carrying out a synthesis of the results, can sometimes be applied to advantage.

5.13. Application of the method of fragments to plane flow

In plane flow, the flow field is defined by the functions G, Ψ which are harmonic. The steady state may, therefore, be described by a flow net containing a family of lines $G = \text{const.}$, $\Psi = \text{const.}$ The principles of the method of fragments explained in Section 4.3 will apply here in a somewhat modified form.

The modulus of resistance, Φ , is defined by

$$\Phi_i = \frac{1}{(Q_r)_i}, \quad (Q_r)_i = \frac{Q_i}{\Delta G_i} \quad (5.13.1)$$

where

Q_i — the volume flow in a certain, i -th part of the flow field,

Q_r — the reduced volume flow,

ΔG_i — the difference between the values of the potential G on the edges of the i -th part of the flow field.

Assume that there are $i = I, II, \dots, N$ parts in a given flow field. In the i -th part (fragment) we have

$$\Delta G_i = \frac{(G_1 - G_2)}{\sum_{i=I}^N \Phi_i} \quad (5.13.2)$$

where G_1, G_2 are the values of the potential G on the edges of the flow field. On the interfaces of the fragments

$$G_{i-I,i} = G_2 + (G_1 - G_2) \frac{\sum_{i=I}^{i-I} \Phi_i}{\sum_{i=I}^N \Phi_i} \quad (5.13.3)$$

The total volume flow Q is found from the formula

$$Q = \frac{G_1 - G_2}{\sum_{i=1}^N \Phi_i} \quad (5.13.4)$$

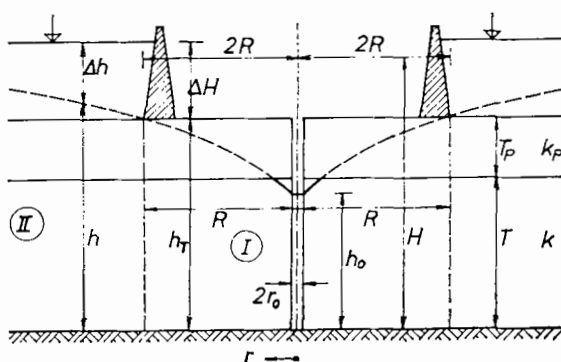


Fig. 5.16. Protection of a well in inundation.

As an example of the application of the method of fragments, consider a problem of sanitary engineering. Fig. 5.16 shows a well operating in extensive inundation, protected from the effect of surface water by an embankment of circular ground plan. It is required that the pressure line should pass underneath the embankment as well as inside the protected space, under the terrain level. We shall solve the problem for the least favourable situation when the pressure line passes just through the upstream

toe of the embankment. Thus a combination of plane flow (protected region) and quasi-plane flow (inundation) is considered in the scheme indicated.

We first note the situation in the inundated part of the site. On substitution of eqn. (5.10.3) the fundamental differential equation (5.10.5) yields the general solution (5.10.6). The values of the constants C_1 , C_2 are calculated from the given boundary conditions. Since we know that $\Delta h < \infty$ at $r = \infty$, we choose $C_1 = 0$ and obtain

$$\Delta h = C_2 K_0(\beta r) \quad (5.13.5)$$

At the point $r = R$, we take $\Delta h = \Delta H$ so that

$$\begin{aligned} \Delta H &= C_2 K_0(\beta R) \\ C_2 &= \frac{\Delta H}{K_0(\beta R)} \\ \Delta h &= \Delta H \frac{K_0(\beta r)}{K_0(\beta R)} \end{aligned} \quad (5.13.6)$$

The volume flow through a cylinder of radius r is

$$Q = 2\pi r k T \frac{\partial(\Delta h)}{\partial r} = -2\pi r k T \beta \Delta h \frac{K_1(\beta r)}{K_0(\beta R)} \quad (5.13.7)$$

on the upstream toe of the embankment, $r = R$; hence

$$Q = Q_R = -2\pi R k T \Delta H \frac{K_1(\beta R)}{K_0(\beta R)} \quad (5.13.8)$$

The overburden is assumed to be of low permeability. The horizontal component of velocity is neglected; therefore, from the point of formulation of the potential G , we assume that $k_p = k_{n+1} = 0$. According to eqn. (1.4.83) we obtain for the case being considered

$$G = -khT + \frac{1}{2}kT^2 \quad (5.13.9)$$

For the water level upstream of the embankment

$$G_H = -k(h_T T + \Delta H) T + \frac{1}{2}kT^2 \quad (5.13.10)$$

For the level at the surface of the terrain

$$G_T = -kh_T T + \frac{1}{2}kT^2 \quad (5.13.11)$$

The difference between the values of the potential G at the edges of part II (Fig. 5.14) is given by

$$\begin{aligned} \Delta G_{II} &= G_H - G_T \\ \Delta G_{II} &= -kh_T T - k \Delta H T + \frac{1}{2}kT^2 + kh_T T - \frac{1}{2}kT^2 = -kT \Delta H \end{aligned} \quad (5.13.12)$$

Eqn. (5.13.8) may, therefore, be written as follows:

$$Q_R = 2\pi R\beta \Delta G_{II} \frac{K_1(\beta R)}{K_0(\beta R)} \quad (5.13.13)$$

In keeping with the definition of the modulus of resistance – eqn. (5.13.1) – we obtain from this for part *II*

$$\Phi_{II} = \frac{K_0(\beta R)}{2\pi R\beta K_1(\beta R)} \quad (5.13.14)$$

For the inner part *I*, we make use of eqn. (5.3.10) which gives

$$\Phi_I = \frac{1}{Q_r} = \frac{1}{2\pi} \ln \frac{R}{r_0} \quad (5.13.15)$$

In the flow field, we now have the potentials $G_2 = G_0$, $G_1 = G_H$ on the edges, and the potential $G_{I,II} = G_T$ on the dividing line between the parts *I*, *II*. Referring to eqn. (5.11.3) we obtain

$$G_T = G_0 + (G_H - G_0) \frac{(1/2\pi) \ln (R/r_0)}{(1/2\pi) \ln (R/r_0) + [K_0(\beta R)/2\pi R\beta K_1(\beta R)]} \quad (5.13.16)$$

and from there

$$\frac{G_H - G_0}{G_T - G_0} = 1 + \frac{K_0(\beta R)}{\beta R K_1(\beta R) \ln (R/r_0)} \quad (5.13.17)$$

Eqn. (5.13.17) is used for calculating the value G_0 corresponding to the level in the well. Once G_0 is known, it is easy to find the corresponding h_0 .

The discharge Q is obtained from eqn. (5.13.4):

$$Q = \frac{2\pi(G_H - G_0)}{\ln (R/r_0) + [K_0(\beta R)/\beta R K_1(\beta R)]} \quad (5.13.18)$$

5.14. Flow in a plane non-homogeneous region

The problem solved in Section 5.13 can also be solved without the application of the method of fragments. It suffices to use the equation of continuity of flow on a cylinder of radius $r = R$. According to it, eqn. (5.13.13) can be equated with eqn. (5.3.10) so that

$$\beta R \Delta G_{II} \frac{K_1(\beta R)}{K_0(\beta R)} = \frac{G_T - G_0}{\ln (R/r_0)} \quad (5.14.1)$$

Hence

$$G_0 = G_T - \beta R \Delta G_{II} \frac{K_1(\beta R)}{K_0(\beta R)} \ln \frac{R}{r_0} \quad (5.14.2)$$

The validity of this step can be verified by rearranging eqn. (5.14.2) to

$$G_T - G_0 = [(G_H - G_0) - (G_T - G_0)] \beta R \frac{K_1(\beta R)}{K_0(\beta R)} \ln \frac{R}{r_0}$$

from which we obtain

$$(G_T - G_0) \left[1 + \frac{1}{[\beta R K_1(\beta R)/K_0(\beta R)] \ln (R/r_0)} \right] = G_H - G_0$$

This expression leads directly to eqn. (5.13.17).

Similarly, it is not necessary to calculate the discharge into the well by the method of fragments, for we know that inside the protected area, $Q = Q_R$, and therefore, eqn. (5.13.13) will be adequate for the purpose.

The advantages gained by direct application of the equation of continuity to steady plane flow become particularly apparent whenever two-dimensional non-homogeneities with edges coinciding in form with equipotential lines, are found in the region. Another formulation of the potential G may apply inside the boundary of the non-homogeneous sub-region.

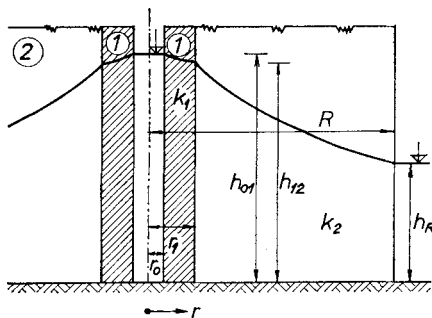


Fig. 5.17. Flow in a well fill.

As a simple example consider the seepage from a well provided with a fill. Assume that both the natural material and the soil of the fill make it possible to apply the linear law of flow. Because of the axial symmetry, the equipotential lines are circular. According to the scheme shown in Fig. 5.17 we obtain the following values:

$$\text{in the range } r_0 \leq r \leq r_1, \quad G_{01} \leq G_1 \leq G_{12}$$

$$\text{in the range } r_1 \leq r \leq R, \quad G_{12} \leq G_2 \leq G_R$$

According to eqn. (5.3.10) we have separately in each layer

$$Q = \frac{2\pi(G_{01} - G_{12})}{\ln(r_1/r_0)} = - \frac{\pi k_1(h_{01}^2 - h_{12}^2)}{\ln(r_1/r_0)}$$

$$Q = \frac{2\pi(G_{12} - G_R)}{\ln(R/r_1)} = - \frac{\pi k_2(h_{12}^2 - h_R^2)}{\ln(R/r_1)}$$

On the interface between the layers 1, 2 the equation of continuity leads to the equation

$$\frac{k_1(h_{01}^2 - h_{12}^2)}{\ln(r_1/r_0)} = \frac{k_2(h_{12}^2 - h_R^2)}{\ln(R/r_1)}$$

from which it follows that

$$h_{12}^2 = \frac{a_2 h_R^2 + a_1 h_{01}^2}{a_1 + a_2}$$

$$a_1 = k_1 \ln \frac{R}{r_1}, \quad a_2 = k_2 \ln \frac{r_1}{r_0}$$

Knowing the height h_{12} we can finish the solution in each layer separately.

The value of h_{12} also depends on the ratio between the coefficients k_1, k_2 . For a large enough discharge we can expect the flow in the coarse material of the fill to be turbulent, characterized by the coefficient k_t . According to Section 1.5

$$Q = 2\pi r \sqrt{\frac{\partial G_t}{\partial t}} \quad (5.14.3)$$

where G_t is the function defined by eqn. (1.5.48) in which we substitute $k_c = k_t$.

Eqn. (5.14.3) is readily solved by direct integration. On calculating the constants for $h = h_{01}$ at the point $r = r_0$, and $h = h_{12}$ at the point $r = r_1$, we obtain

$$Q = 2\pi k_t \sqrt{\left(\frac{h_{01}^3 - h_{12}^3}{3} \frac{r_0 r_1}{r_1 - r_0} \right)} \quad (5.14.4)$$

For $r_0 \leq r \leq r_1$, the solution gives

$$h = \sqrt[3]{\left[(h_{01}^3 - h_{12}^3) \frac{r_1}{r} \frac{r - r_0}{r_1 - r_0} + h_{12}^3 \right]} \quad (5.14.5)$$

The course of the free surface in the fill is no longer the same as that in the case in which the law of flow was assumed to be linear. Of interest is the boundary value of h_{12} for which we shall set up the appropriate equation. The equation of continuity leads to

$$\frac{\pi k_2(h_{12}^2 - h_R^2)}{\ln(R/r_1)} = 2\pi k_t \sqrt{\frac{h_{01}^3 - h_{12}^3}{3} \frac{r_0 r_1}{r_1 - r_0}} \quad (5.14.6)$$

The biquadratic equation in h_{12} obtained from eqn. (5.14.6) can be rearranged by substitution as a reduced equation. The solution of the latter by means of the so-called cubic resolvent was suggested by Euler.

We shall not perform the necessary operations here. We merely state in conclusion that in cases when the shape of the equipotential lines does not coincide

with the shape of the contour of the region, the solution of plane flow in a non-homogeneous medium is not as straightforward as in the case just discussed. In cases of that sort we usually rely on an experimental solution or a drawn flow net.

5.15. Pseudo-plane flow

By its nature, the theory of plane flow is approximate. It can be expected to be successful whenever its initial assumptions are satisfied. The most serious is the requirement that the equipotential lines should be vertical. However, the basic idea of the theory can be applied even to cases in which the equipotential lines are not vertical, provided that it is possible to neglect the effect of flow resistance in some, not necessarily the vertical, direction. Although the approach to the solution differs somewhat from that corresponding to the classical theory of plane seepage, the method of solution is analogous. We thus speak of pseudo-plane flows.

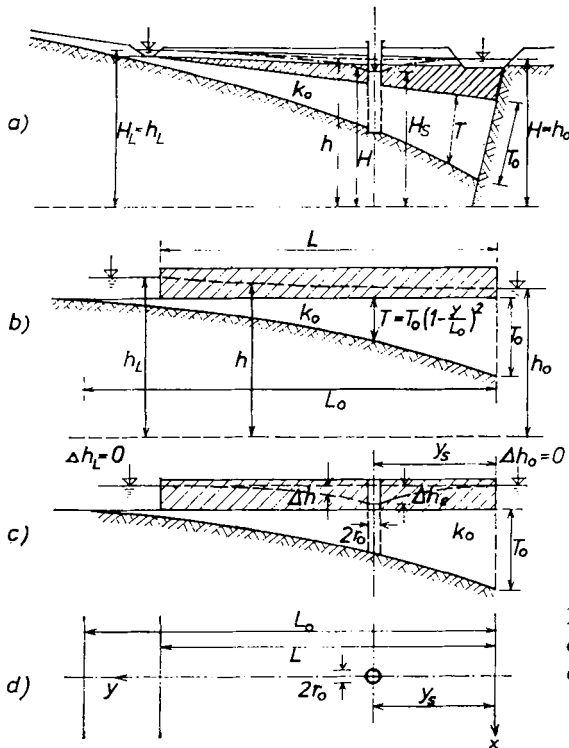


Fig. 5.18. Collection of water from a layer of varying thickness.

To illustrate we consider the example shown schematically in Fig. 5.18a.

On the surface of a relatively impervious rock base lies a permeable layer of tertiary gravel sand overlain by impervious tertiary clays. The surface of the base is curvilinear, and the scheme shows the position of the line of fault. The quaternary

formations lie on clays, and water flows through them as well. On the boundaries are two streams, the levels of which co-determine the motion in the tertiary and the quaternary. On the contact between the clays and the base, the line of fault is not impervious everywhere. In the tertiary, the inclination of the pressure line is from the left-hand stream (a brook) to the right-hand one (a river). The position of the path along which the water enters the river bed from below, is not known exactly. We know, however, that the tertiary formations receive water from the slope inflow and seepage from the brook. The water is of good quality, suitable for water supply purposes. The collecting well has a perforated section in the tertiary.

The waters from the quaternary and the river are of poor quality, unsuitable for collection, and so the well is leak-proof in the upper sections. We are required to prevent the water from infiltrating downward through the fault. We must establish such conditions of operation that only the water percolating from the left will be collected. We require that the pressure line along the right-hand boundary should be inclined toward the river under all circumstances.

The tertiary formation has a roof inclined to the horizontal and is bounded from below by a curved line. Hence the equipotential surfaces cannot have vertical directrices. However, we shall assume that they can be replaced approximately by parts of cylindrical surfaces whose directrices are perpendicular to the line of the roof. The course of the stretch $H(x, y)$ along the roof also determines the distribution of pressure during the draw-off.

We suppose that the situation which will arise in the operation of the well, can be described by superposition of the natural flow [the stretch of $h(x, y)$] and the flow induced by the action of the collecting equipment [the stretch $\Delta h(x, y)$].

Figs. 5.18b, c show the way in which we have chosen the partial flows, and explain the symbols used in the solution.

The thickness T of the tertiary with a coefficient of permeability k_0 , varies according to the quadratic law

$$T = T_0 \left(1 - \frac{y}{L_0}\right)^2 \quad (5.15.1)$$

To simplify, the right-hand boundary of the region is assumed to be an equipotential surface. The left-hand boundary equipotential surface passes through the end of the wedge of the tertiary impervious clays (in the position $y = L$). The ordinate of the line of intersection of the tertiary roof and the surface of the impervious base is $y = L_0$.

In the system shown in Fig. 5.18b, we solve the equation

$$\frac{\partial}{\partial x} \left(k_0 T \frac{\partial h}{\partial x} \right) + \frac{\partial}{\partial y} \left(k_0 T \frac{\partial h}{\partial y} \right) = 0 \quad (5.15.2)$$

under the condition that $h = h_0$ at the points $(x, 0)$, and $h = h_L$ at the points (x, L) .

We can easily integrate eqn. (5.15.2) to obtain

$$h = (h_L - h_0) \frac{L_0 - L}{L} \frac{y}{L_0 - y} + h_0 \quad (5.15.3)$$

In the system shown in Fig. 5.18c, we solve the equation

$$\frac{\partial}{\partial x} \left[k_0 T \frac{\partial(\Delta h)}{\partial x} \right] + \frac{\partial}{\partial y} \left[k_0 T \frac{\partial(\Delta h)}{\partial y} \right] = 0 \quad (5.15.4)$$

on the assumption (5.15.1) and the condition that $\Delta h = 0$ at the points $(x, 0)$, (x, L) , and $\Delta h = \Delta h_s$ at the point $y = y_s$, $x = r_0$. Theoretically, we consider a singularity at the point $x = 0$, $y = y_s$ and assume that a flow close to an axi-symmetrical one exists in its immediate neighbourhood. The contour of the well coincides approximately with one of the assumed equipotential surfaces.

Since the thickness T of the stratum is a quadratic function of y , we replace eqn. (5.15.4) by the relation

$$\begin{aligned} \frac{\partial}{\partial x} \left[f(x, y) \frac{\partial(\Delta h)}{\partial x} \right] + \frac{\partial}{\partial y} \left[f(x, y) \frac{\partial(\Delta h)}{\partial y} \right] &= 0 \\ f(x, y) &= \left(1 - \frac{y}{L} \right)^2 \end{aligned} \quad (5.15.5)$$

Referring to Section 4.11, we see that we can use the auxiliary function

$$F = k_0 T_0 \Delta h \sqrt{f} \quad (5.15.6)$$

Substituting this function in eqn. (5.15.5) we obtain the equation

$$\frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial y^2} = 0 \quad (5.15.7)$$

which we solve under the condition that $F = 0$ at the points $(x, 0)$, (x, L) , and also under the condition that $F = F_s$ at the point $y = y_s$, $x = r_0$. Using the method of mirror representation we obtain the resulting integral

$$F = c \ln \frac{\cosh^2 (\pi x/L) - \cos^2 [\pi(y - y_s)/2L]}{\cosh^2 (\pi x/L) - \cos^2 [\pi(y + y_s)/2L]} \quad (5.15.8)$$

where y_s is the ordinate of the well axis.

The value of the constant c is determined from the condition laid down on the well contour:

$$\Delta h = \Delta h_s \frac{L_0 - y_s}{L_0 - y} \frac{\ln \frac{\cosh^2(\pi x/2L) - \cos^2[\pi(y + y_s)/2L]}{\cosh^2(\pi x/2L) - \cos^2[\pi(y - y_s)/2L]}}{\ln \frac{\cosh^2(\pi r_0/2L) - \cos^2(\pi y_s/L)}{\sinh^2(\pi r_0/2L)}} \quad (5.15.9)$$

The calculation of the strength of the well starts from the assumption that the inflow is axi-symmetrical; hence

$$Q = 2\pi r_0 k_0 T_0 \left(1 - \frac{y_s}{L_0}\right)^2 \frac{\partial(\Delta h)}{\partial x} \Big|_{x=r_0, y=y_s} \quad (5.15.10)$$

where Q is the quantity of water drawn from the well.

Differentiating eqn. (5.15.9) partially with respect to x and substituting the result in eqn. (5.15.10) we obtain

$$Q = \frac{2\pi^2 r_0 k_0 T_0}{AL} \left(1 - \frac{y_s}{L_0}\right)^2 \Delta h_s \frac{\sin^2(\pi y_s/L)}{[\operatorname{tgh}(\pi r_0/2L)][\cosh^2(\pi r_0/2L) - \cos^2(\pi y_s/L)]} \quad (5.15.11)$$

where

$$A = \ln \frac{\sinh^2(\pi r_0/2L)}{\cosh^2(\pi r_0/2L) - \cos^2(\pi y_s/L)} \quad (5.15.12)$$

For $r_0 \ll L$, eqn. (5.15.11) can be simplified by replacing some of the hyperbolic and trigonometrical functions by approximate expressions. This step leads to

$$Q = 2\pi k_0 T_0 \left(1 - \frac{y_s}{L_0}\right)^2 \Delta h_s \frac{1}{\ln[\pi r_0/2L \sin(\pi y_s/L)]} \quad (5.15.13)$$

Partial differentiation of eqn. (5.15.9) with respect to y makes it possible to calculate the gradient of the function along the boundaries. Thus, for example, along $y = 0$ we have

$$\frac{\partial(\Delta h)}{\partial y} \Big|_{y=0} = \frac{\Delta h_s (L_0 - y_s) \pi}{LL_0 A} \frac{-\sin(\pi y_s/L)}{\cosh^2(\pi x/2L) - \cos^2(\pi y_s/2L)} \quad (5.15.14)$$

The gradient is largest at the origin of the coordinate system:

$$\frac{\partial(\Delta h)}{\partial y} \Big|_{x=0, y=0} = \frac{-2\pi \Delta h_s (L_0 - y_s)}{ALL_0 \operatorname{tg}(\pi y_s/2L)} \quad (5.15.15)$$

The position of the resulting pressure surface is obtained by the superposition of the partial results:

$$H = h - \Delta h \quad (5.15.16)$$

Downward flow from the river into the tertiary will be precluded under the condition that

$$\left. \frac{\partial h}{\partial y} \right|_{y=0} - \left. \frac{\partial(\Delta h)}{\partial y} \right|_{x=0, y=0} \geq 0 \quad (5.15.17)$$

The value of the first term on the left-hand side of the inequality (5.15.17) is easily determined by differentiating eqn. (5.15.3) with respect to y . For the limiting case we obtain

$$\Delta h_s = - \frac{A}{2\pi} \frac{L_0 - L}{L_0 - y_s} (h_L - h_0) \operatorname{tg} \frac{\pi y_s}{2L} \quad (5.15.18)$$

The admissible draw-off from the well is calculated by substituting eqn. (5.15.18) in eqn. (5.15.13). The procedure clearly guarantees safety of operation because it neglects the effect of outlet resistances in the line of fault and its neighbourhood.

The example demonstrated the possibilities of an approximate solution of problems of three-dimensional groundwater flow. However, the simplification might not always be acceptable. More complicated problems are then solved by methods whose principles will be explained in the next chapter.

CHAPTER 6

SOME PARTIAL PROBLEMS OF THREE-DIMENSIONAL FLOW

Although the theory of plane groundwater flow provides satisfactory solutions to a number of practical problems, the restricted applicability of the assumptions on which this theory is founded should be appreciated. Take, for example, the case of the free surface described by eqn. (1.4.45). In steady-state flow, the equation takes the form

$$\left(\frac{\partial h}{\partial x}\right)^2 + \left(\frac{\partial h}{\partial y}\right)^2 + \frac{\partial h}{\partial z} \left(\frac{\partial h}{\partial z} - 1\right) = 0$$

If in the limiting case

$$\frac{\partial h}{\partial x} = \frac{\partial h}{\partial y} \rightarrow 0 \quad \text{then} \quad \frac{\partial h}{\partial z} \rightarrow 1$$

Hence the maximum gradient at which water particles can move on the free surface, is equal to unity. Full account is not always taken of this fact in the theory of plane ground-water flow. As an example, consider the discharge into the well

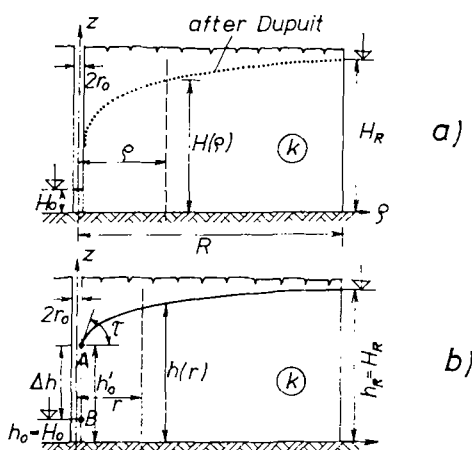


Fig. 6.1. Axially symmetrical flow to an hydraulically perfect well.

shown in Fig. 6.1. At very low levels, the simplified theory yields extremely high inlet hydraulic gradients on the well lining. Actually, however, the hydraulic gradient of the free surface on the well contour can never be greater than one; therefore, the

depression curve does not connect with the level inside the object. Consequently, there arises a so-called surface of seepage of height Δh . Within its range, water seeps out of the soil and flows along the wall of the well.

Scrutiny of the boundary conditions shows that the connection of the depression curve onto the well lining is tangential. The height of the surface of seepage is determined by the point A and by the point of intersection, B , of the level in the well and the well lining (Fig. 6.1b). At the point B , the water moves with infinite velocity (a singular point).

Since in three-dimensional flow all velocity components come into play, the anisotropy of the permeable medium takes on a special significance.

6.1. Surface of seepage on the well lining. Charny's proof

As Charny has shown, the origin of a surface of seepage does not challenge the correctness of Dupuit's equation (5.3.10) which we use to calculate the volume flow Q . The proof is based on the existence of axial symmetry. Through a cylinder of an elementary height dz (Fig. 6.1b), there flows the quantity

$$dQ = 2\pi r \frac{\partial \varphi}{\partial r} dz$$

where φ is the velocity potential, $\varphi = -kh = -k(p/\gamma + z)$. For a height h we obtain

$$Q = 2\pi \int_0^h r \frac{\partial \varphi}{\partial r} dz = 2\pi \int_0^h r \frac{\partial \varphi}{\partial(\ln r)} \frac{\partial(\ln r)}{\partial r} dz = 2\pi \int_0^h \frac{\partial \varphi}{\partial(\ln r)} dz \quad (6.1.1)$$

We shall in what follows make use of the integral I

$$I = \int_0^h \varphi(r, z) dz \quad (6.1.2)$$

For the well contour where $\varphi = -kh$, $h = h_0 + \Delta h$, the integral becomes

$$\begin{aligned} I(r_0) &= \int_0^{h_0 + \Delta h} \varphi dz = -kh_0 \int_0^{h_0} dz - k \int_{h_0}^{h_0 + \Delta h} z dz = \\ &= -kh_0^2 - k \frac{(h_0 + \Delta h)^2}{2} + k \frac{h_0^2}{2} = -\frac{kh_0^2}{2} - \frac{k(h_0 + \Delta h)^2}{2} \end{aligned} \quad (6.1.3)$$

On a cylinder of radius R , $\varphi = -kH_R$, $h = H_R$, and the integral is

$$I(R) = -kH_R \int_0^{H_R} dz = -kH_R^2 \quad (6.1.4)$$

On differentiating eqn. (6.1.2) with respect to $\ln r$ and remembering that $h = h(r)$, $\varphi = -kh$, we have

$$\begin{aligned}\frac{dI}{d(\ln r)} &= \int_0^h \frac{\partial \varphi}{\partial (\ln r)} dz + \varphi \frac{dh}{d(\ln r)} = \int_0^h \frac{\partial \varphi}{\partial (\ln r)} dz - \frac{d(\frac{1}{2}kh^2)}{d(\ln r)} = \\ &= \frac{Q}{2\pi} - \frac{d(\frac{1}{2}kh^2)}{d(\ln r)}\end{aligned}$$

and hence

$$\frac{Q}{2\pi} d(\ln r) = dI + d\left(\frac{kh^2}{2}\right)$$

Integration between the limits r_0 and R gives

$$-\frac{Q}{2\pi} (\ln R - \ln r_0) = I(r_0) - I(R) + \frac{k(h_0 + \Delta h)^2}{2} - \frac{kH_R^2}{2} \quad (6.1.5)$$

and substituting eqns. (6.1.3), (6.1.4) we have

$$-\frac{Q}{2\pi} \ln \frac{R}{r_0} = -\frac{kh_0^2}{2} + kH_R^2 - \frac{kH_R^2}{2} = -\frac{kh_0^2}{2} + \frac{kH_R^2}{2} = G_0 - G_R$$

The above leads to Dupuit's equation

$$Q = \frac{2\pi(G_R - G_0)}{\ln(R/r_0)} \quad (6.1.6)$$

which is clearly an exact solution.

The theory of plane seepage is evidently not suitable for calculating the free surface in the vicinity of a well. Unfortunately, no complete and exact analytical solution of the problem is available at present. On the other hand, there are numerous studies in which the subject is treated experimentally [15].

According to Fig. 6.1b, the experimentally obtained representations of the free surface terminate on the line of the outer contour of the well, and the angle τ made by the tangent to the surface at the point A is usually not equal to the $\pi/2$ predicted by the theory.

We believe that the reason for the discrepancy between theory and experiments should be sought in the fact that within the range of the surface of seepage, water (or any other liquid used in hydraulic models) emerges from the permeable medium and flows down along the model of the well lining. Hence the phenomenon which is being studied theoretically on the contour, is the passage of a flow of water across the boundary between two media with diametrically different permeabilities. Moreover, in hydraulic analogues (slot analogues with a slot of variable width) the effect of the changes in the velocity field in the region of emergence is noticeable. Work

with electrical analogues is of limited accuracy, and numerical procedures a priori assume discretization. Difficulties are also likely to arise when identifying the singularity at the point *B* (Fig. 6.1b).

Published results, while explicitly acknowledging the existence of the surface of seepage, overestimate its height to a greater or lesser degree. One of the reasons for the differences is the straightening of the depression curve in the proximity of the well lining.

We now attempt to present an acceptable definition of the position of the point *A*, basing our approach on the hydraulic theory which yields directly applicable results in the form of mathematical relationships.

We shall define the region with the help of the system (ϱ, z) (Fig. 6.1a). In the space (r, z) , on the other hand, we shall concentrate on the actual pattern of discharge to the well (Fig. 6.1b). A free surface is assumed to exist in both systems, and the relation

$$h(r) = H(\varrho) \quad (6.1.7)$$

is assumed to apply for certain corresponding abscissae r, ϱ .

The two basic theoretical conceptions proceed from different definitions of the surface velocity. In the region of actual flow, the velocity is determined by the derivative of the potential in the direction of the depression curve, while the hydraulic theory starts from the assumption that in steady-state flow, the surface velocity has only a horizontal component. Since the coefficient of permeability is the same in both regions, we expect it to hold that

$$\left. \frac{\partial h}{\partial s} \right|_r = \left. \frac{\partial H}{\partial \varrho} \right|_\varrho \quad (6.1.8)$$

in the corresponding vertical lines, where ds is the differential of the line of the free surface according to Fig. 6.1b.

The crucial point of the problem is evidently a theoretical description of the relationships between the positions of the corresponding vertical lines. We undertake a quantitative analysis of the situation. In the (r, z) region, eqn. (6.1.1) can be replaced by the equation

$$Q = 2\pi r \int_0^h \frac{\partial \varphi}{\partial r} dz = 2\pi r \left[\frac{\partial}{\partial r} \int_0^h \varphi(r, z) dz + kh \frac{\partial h}{\partial r} \right] \quad (6.1.9)$$

where Q is the quantity of water which has passed through the cylinder; (if the water passes into the well, this quantity is negative).

On integrating we obtain

$$C_1 = \frac{Q}{2\pi} \ln r - \int_0^h \varphi(r, z) dz - k \frac{h^2}{2} \quad (6.1.10)$$

where C_1 is the constant of integration.

The quantity defined by the hydraulic theory is, on the other hand,

$$Q = -k 2\pi \varrho H \frac{\partial H}{\partial \varrho} \quad (6.1.11)$$

and on integration

$$C_2 = \frac{Q}{2\pi} \ln \varrho + k \frac{H^2}{2} \quad (6.1.12)$$

where C_2 is the constant of integration.

Assume now that there exist cylinders, generally of different radii r , ϱ , on which eqn. (6.1.7) holds. The outer cylindrical boundaries of the regions, which are sufficiently remote from the well axis and for which $r = \varrho = R$ as well as $\varphi(R, z) = -kH_R = -kh_R$, are a special case. Since the quantity Q which passes into the wells, is the same in both regions, the constants C_1 , C_2 are calculated from identical boundary conditions. On comparing eqn. (6.1.10) with eqn. (6.1.12) we find that

$$\frac{Q}{2\pi} \ln \frac{\varrho}{r} + \int_0^h \varphi(r, z) dz + \frac{k}{2} (H^2 + h^2) = 0 \quad (6.1.13)$$

Eqn. (6.1.13) suggests the required relationship between the positions of the corresponding vertical lines. Its direct application is complicated by the difficulties which arise in connection with the evaluation of the integral on the left-hand side. An analysis of the situation in the vertical line on the well lining in the actual flow is somewhat easier since along the surface of seepage we have $\varphi = -kz$. On the part of the lining which is under water, we simultaneously have $\varphi = -kh_0$ and therefore,

$$\int_0^{h'_0} \varphi(r_0, z) dz = -\frac{k}{2} (h_0'^2 + h_0'^2) \quad (6.1.14)$$

where h'_0 is the ordinate of the upper boundary of the surface of seepage.

To the lining of the well of radius r_0 in the (r, z) system, there corresponds in the (ϱ, z) system the lining of a well of radius ϱ_0 . Since also $\varrho_0 \geq r_0$, $H_0 = h'_0$, we obtain, according to eqn. (6.1.13) and in agreement with Charny's proof, that

$$\frac{Q}{\pi} \ln \frac{\varrho_0}{r_0} = -k(h_0'^2 - h_0^2) \quad (6.1.15)$$

with $h'_0 = H$, $h_0 = H_0$

$$H = \sqrt{\left(-\frac{Q}{\pi k} \ln \frac{\varrho_0}{r_0} + H_0^2 \right)} \quad (6.1.16)$$

It goes without saying that Dupuit's equation (6.1.6) also applies.

The assumptions (6.1.7) and (6.1.8) lead to a graphical construction which enables us to determine the position of the point A . At an angle of 45° with the positive direction of the axis of abscissae, a tangent is drawn to the depression curve obtained on the basis of calculations according to the hydraulic theory. Horizontal projection of the point of contact onto the well lining gives the position of the point A (Fig. 6.2, top).

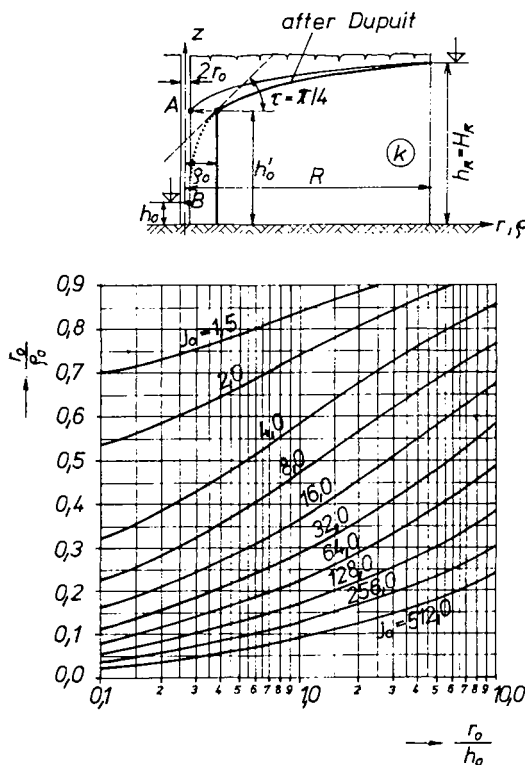


Fig. 6.2. Graphical construction of the upper boundary of a surface of seepage.

A numerical procedure for calculating the upper boundary of a surface of seepage is also possible. By working out a number of examples we obtained data for drawing the diagram shown in Fig. 6.2, bottom. This diagram represents the functional relation

$$\frac{r_0}{h_0} = f\left(J_0, \frac{r_0}{h_0}\right) \quad (6.1.17)$$

where

$$J_0 = - \frac{Q}{2\pi r_0 k h_0} \quad (6.1.18)$$

The letter J_0 denotes the gradient on the inlet to the well, calculated on the basis of the hydraulic theory.

For specified J_0 , r_0/h_0 , the ratio r_0/ϱ_0 is found from the diagram, and the ordinate h'_0 of the point A is obtained from the formula

$$h'_0 = \sqrt{\left[h_0 \left(h_0 + 2J_0 r_0 \ln \frac{\varrho_0}{r_0} \right) \right]} \quad (6.1.19)$$

It is evident from the foregoing that in our conception a surface of seepage can exist only in the case when the tangent to the depression curve calculated using the hydraulic theory, has a slope equal or greater than unity. Otherwise, the depression curve connects directly onto the level in the well and this connection is tangential to the wall.

6.2. A point source in space

The simplest case of three-dimensional flow is that of inflow to a point source or sink situated in an unbounded permeable space. The distribution of the potential $\varphi(x, y, z)$ is established directly by solving eqn. (1.4.24a); by symmetry, this equation takes the form

$$\frac{\partial}{\partial r} \left(r^2 \frac{\partial \varphi}{\partial r} \right) = 0 \quad (6.2.1)$$

Integrating eqn. (6.2.1) twice, we obtain

$$r^2 \frac{\partial \varphi}{\partial r} + c_1 = 0, \quad \varphi = \frac{c_1}{r} + c_2$$

where c_1, c_2 are constants which are determined from the boundary conditions.

Since the equipotential surfaces are spherical, the volume flow through a sphere of radius r is given by

$$Q = 4\pi r^2 \frac{\partial \varphi}{\partial r} = -4\pi r^2 \frac{c_1}{r^2} = -4\pi c_1 \quad (6.2.2)$$

From this equation we obtain

$$c_1 = -\frac{Q}{4\pi}$$

Depending on whether an inflow to a sink or an outflow from a source is being considered, the value of Q is taken as either positive or negative.

$$\varphi = -\frac{Q}{4\pi r} + c_2 \quad (6.2.3)$$

Assume now that a certain equipotential surface has a radius $r = r_0$ and $\varphi = \varphi_0$. In this case eqn. (6.2.3) gives

$$\begin{aligned} c_2 &= \varphi_0 + \frac{Q}{4\pi r_0} \\ \varphi &= -\frac{Q}{4\pi} \left(\frac{1}{r} - \frac{1}{r_0} \right) + \varphi_0 \end{aligned} \quad (6.2.4)$$

If $\varphi = \varphi_R$ on another equipotential surface of radius $r = R$, then

$$\begin{aligned} \varphi_R &= -\frac{Q}{4\pi} \left(\frac{1}{R} - \frac{1}{r_0} \right) + \varphi_0 \\ Q &= \frac{4\pi(\varphi_R - \varphi_0)}{\frac{1}{r_0} - \frac{1}{R}} = \frac{4\pi r_0 R(\varphi_R - \varphi_0)}{R - r_0} \end{aligned} \quad (6.2.5)$$

6.3. Superposition of the effects of spatial sources (sinks)

Since the potential in the neighbourhood of a point sink (source) is a function of the coordinates, one can apply the principle of superposition. A special case of considerable significance is the composition of the effects of sinks arranged along a spatial curve at an infinitesimal spacing. Each sink has an elementary strength, and the total seepage into the whole line is obtained as the sum of elementary quantities of seepage.

As a practical application consider the example shown schematically in Fig. 6.3a. The surface of a relatively impervious subsoil is overlain by a permeable tertiary stratum which comes up to the terrain surface on a sloping terrace. In the valley, the stratum is covered with impervious clay.

Owing to the effect of precipitations, a space containing water under pressure is formed beneath the impervious floor. We locate the system of coordinates x, y, z as shown in Figs. 6.3a, b, and at a distance $y = R_0$ consider a well of radius r_0 , the perforated portion of which is situated within the range of the tertiary. Water from the well is to be drawn by gravity, without pumping. The pressure line which is co-determined by the distribution of the piezometric heads $h(x, y, z)$ along the lower boundary of the floor, passes above the surface of the terrain (shown in dashed lines in Fig. 6.3a). In the well, $h = h_0 = \text{const}$.

The wedge containing the tertiary sediments is bounded by two planes at an angle ϑ . A theoretical description is facilitated by replacing the straight well by a portion of a torus, the axis of which we shall call the toroidal thread. We first assume that water flows symmetrically from an approximately spherical equipotential surface $R \gg R_0$ to the thread.

Because of the symmetry, we can start by examining the flow to the whole thread. The total inflow, Q_a , is the result of the actions of elementary sinks with elementary inflows dQ_a . The distribution of the elementary inflows along the thread is uniform

$$dQ_a = \frac{Q_a}{2\pi} d\delta = q d\delta \quad (6.3.1)$$

where

q is the quantity of flow per unit angle,

δ is the angle shown in Fig. 6.3b.

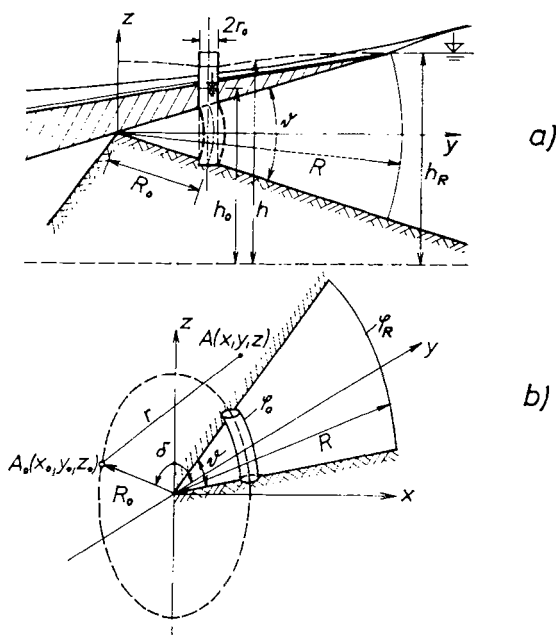


Fig. 6.3. Superposition of spatial sources.

As noted in Sect. 6.2, we have simultaneously for each of the elementary sinks,

$$d(\Delta\varphi) = \frac{dQ_a}{4\pi r}$$

where (see Fig. 6.3)

$$\Delta\varphi = \varphi_R - \varphi, \quad \varphi_R = -kh_R, \quad \varphi = -kh$$

$$\Delta h = h_R - h$$

$$\Delta\varphi = -k(h_R - h) = -k \Delta h$$

Between any point of the thread, for example, the point $A(x_0, y_0, z_0)$ and an arbitrarily chosen point $A(x, y, z)$ in the space, the relation

$$r = \sqrt{[(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2]} \quad (6.3.2)$$

holds with

$$x_0 = 0, \quad y_0 = R_0 \cos \delta, \quad z_0 = R_0 \sin \delta \quad (6.3.3)$$

A plane drawn through the point $A(x, y, z)$ and the origin, at right angles to the thread, divides the space into two parts. The effect of all the elementary sinks is superposed in one hemisphere, and the result is multiplied by two. In this way we obtain the basic relation for the total effect, viz.

$$\Delta\varphi = \frac{q}{2} \int_{\delta_1}^{\delta_2 = \pi + \delta_1} \frac{d\delta}{\sqrt{[x^2 + y^2 + z^2 + R_0^2 - 2R_0(z \sin \delta + y \cos \delta)]}}, \quad \delta = \arctg \frac{z}{y} \quad (6.3.4)$$

Evaluation of the integral (6.3.4) leads to the formula

$$\Delta\varphi = \frac{q}{\sqrt{\{2R_0[a + \sqrt{(y^2 + z^2)}]\}}} K \left[\frac{2}{1 + a/\sqrt{(y^2 + z^2)}} \right] \quad (6.3.5)$$

where

$$a = \frac{x^2 + y^2 + z^2}{2R_0} + \frac{R_0}{2} \quad (6.3.6)$$

$K(\zeta)$ is the complete elliptic integral of the first kind with modulus ζ .

We have assumed that about the toroidal thread, the equipotential surfaces have the shape of a torus. On this torus lies the point $x = r_0, y = R_0, z = 0$ where it is specified that $\varphi_0 = -kh_0, \Delta\varphi_0 = -k(h_R - h_0) = -k \Delta h_0$

$$-k \Delta h_0 = \frac{q}{2R_0 \sqrt{[1 + (r_0/2R_0)^2]}} K \left[\frac{1}{1 + (r_0/2R_0)^2} \right]$$

so that

$$q = - \frac{2\Delta h_0 k R_0 \sqrt{[1 + (r_0/2R_0)^2]}}{K \left[\frac{1}{1 + (r_0/2R_0)^2} \right]} \quad (6.3.7)$$

Since in the case being considered we examine seepage through a segment, the quantity Q which passes into the curvilinear well will be a proportion of the inflow to the whole torus, i.e.

$$Q = q\vartheta$$

where ϑ is the angle between the plane of the surface of the relatively impervious subsoil and the plane of the impervious floor.

On substituting eqn. (6.3.7) in eqn. (6.3.5) we obtain generally for $\varphi = -kh$, $\Delta\varphi = -k(h_R - h) = -k \Delta h$

$$\Delta h = \frac{\Delta h_0 \sqrt{(4R_0^2 + r_0^2)}}{\sqrt{\{2R_0[a + \sqrt{(y^2 + z^2)}]\}}} \frac{K \left[\frac{2}{1 + a/\sqrt{(y^2 + z^2)}} \right]}{K \left[\frac{1}{1 + (r_0/2R_0)^2} \right]} \quad (6.3.8)$$

Along the x -axis

$$\Delta h = \frac{\Delta h_0 \pi \sqrt{(4R_0^2 + r_0^2)}}{2 \sqrt{(x^2 + R_0^2)}} \frac{1}{K \left[\frac{1}{1 + (r_0/2R_0)^2} \right]} \quad (6.3.9)$$

It can readily be shown by an analysis of eqn. (6.3.8) that at large distances from the origin the equipotential surfaces are spherical.

At the origin of the coordinate system where $x = y = z = 0$ we obtain, according to eqn. (6.3.9), that

$$\Delta h = \frac{\Delta h_0 \pi \sqrt{[1 + (r_0/2R_0)^2]}}{K \left[\frac{1}{1 + (r_0/2R_0)^2} \right]} \quad (6.3.10)$$

The principle of superposition is, of course, equally applicable to simpler cases. As an example, we shall present a solution of the case of discharge into an hydraulically imperfect well, i.e. a well the active part of which does not persist throughout the whole saturated medium. The theory of the case of discharge into a well in a half-space, i.e. the case in which the hydraulic imperfection is especially marked, was expounded by Girinski [13].

Girinski assumed that along a line segment with end points $(0, 0, l)$, $(0, 0, -l)$ in the space x, y, z , there lies a series of elementary sinks of equal strength dQ . The distribution of the sinks along the line segment is uniform.

Since the flow is axially symmetrical, we make use of the coordinates (r, z) where $r = \sqrt{(x^2 + y^2)}$. Referring to Fig. 6.4 and Sect. 6.2, we can write for each sink and each point in space that

$$\partial(\Delta\varphi) = \frac{dQ}{4\pi \sqrt{[(z - \zeta)^2 + r^2]}} \quad (6.3.11)$$

Another matter of interest is the strength Q of one half of the sinks. Noting that

$$\frac{dQ}{d\zeta} = \frac{2Q}{2l}, \quad dQ = Q \frac{d\zeta}{l}$$

we find that eqn. (6.3.11) becomes

$$d(\Delta\varphi) = \frac{Q}{4\pi l} \frac{d\zeta}{\sqrt{[(\zeta - z)^2 + r^2]}} \quad (6.3.12)$$

and integrating between the limits $-l$ to $+l$ we obtain

$$\begin{aligned}\Delta\varphi &= \frac{Q}{4\pi l} \int_{-l}^{+l} \frac{d\zeta}{\sqrt{[(\zeta - z)^2 + r^2]}} = \frac{Q}{4\pi l} \int_{-l}^{+l} \frac{d[(\zeta - z)/r]}{[(\zeta - z)/r]^2 + 1} = \\ &= \frac{Q}{4\pi l} \left(\operatorname{argsinh} \frac{l - z}{r} + \operatorname{argsinh} \frac{l + z}{r} \right) \quad (6.3.13)\end{aligned}$$

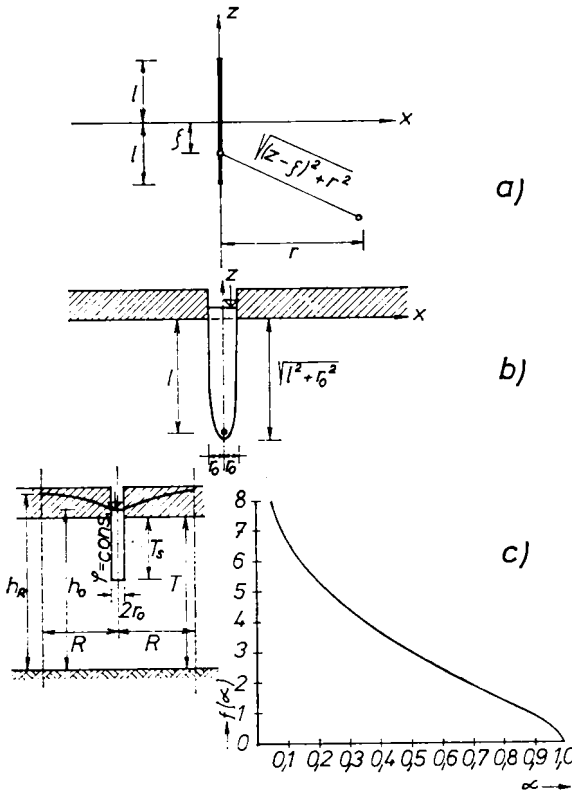


Fig. 6.4. Schematic solution of the discharge into a hydraulically imperfect well.

Using familiar relations we can put eqn. (6.3.13) into the following form:

$$\Delta\varphi = \frac{Q}{4\pi l} \ln \frac{\sqrt{\{(l - z)/r\}^2 + 1} + (l - z)/r}{\sqrt{\{(l + z)/r\}^2 + 1} - (l + z)/r} \quad (6.3.14)$$

As eqn. (6.3.14) suggests, $\varphi = 0$ for r approaching infinity. The description of the flow assumes a zero value for the potential at infinitely remote points.

The equipotential surfaces have the shape of ellipsoids of revolution symmetrical about the x, y plane. Hence the x, y plane is regarded as the plane which forms the lower boundary of the overlying, less permeable layer. On the x, y plane, $z = 0$

so that

$$\Delta\varphi = \frac{Q}{2\pi l} \operatorname{argsinh} \frac{l}{r} \quad (6.3.15)$$

According to Fig. 6.4b, the potential on the contact between the well lining and the (x, y) plane is given by

$$\Delta\varphi_0 = \frac{Q}{2\pi l} \operatorname{argsinh} \frac{l}{r_0}$$

and hence

$$Q = \frac{2\pi l \Delta\varphi_0}{\operatorname{argsinh} (l/r_0)} \quad (6.3.16)$$

In calculating the strength of a cylindrical well passing through a portion of a saturated stratum of finite thickness, Muskat proceeded from an equation of the type of eqn. (6.3.11). When integrating the equation, he varied the position of the elementary sinks along z so as to obtain $\varphi = \text{const.}$ on the well lining (Fig. 6.4c). By using the principle of mirror representation he succeeded in satisfying the condition on $\partial\varphi/\partial r$ along the surface of the relatively impervious subsoil. Since a calculation made with the help of the resulting relations was very tedious, the author confined his study to the evaluation of two specific cases. After generalizing the results he obtained the approximate equation

$$Q = - \frac{2\pi k T (h_R - h_0)}{\frac{1}{2\alpha} [2 \ln (4T/r_0) - f(\alpha)] - \ln (4T/R)} \quad (6.3.17)$$

where $\alpha = T_s/T$.

Fig. 6.4c shows a diagram which we have drawn of the function $f(\alpha)$, and explains the meaning of the symbols used in eqn. (6.3.17).

6.4. A theorem on the mean value of the potential

Because of difficulties encountered in theoretical analyses of the problem of the flow to hydraulically imperfect wells, many authors have adopted strategies which enabled them to obtain a theoretical solution by approximate methods. Thus, for example, Verigin [44] was of the opinion that in the case of wells actually shaped like cylinders, the assumed uniform distribution of the sinks along the well axis led to an incorrect solution because — at variance with the specified requirement — no constant value of the potential was obtained on the flow field boundary. The results tend to approach reality if a theoretical analysis of hydraulically imperfect wells

employs the mean value of the potential, viz.

$$\varphi_a = \frac{1}{l} \int_0^l \varphi(z) dz \quad (6.4.1)$$

where $\varphi(z)$ is the value of the potential along a vertical line of length l .

The theorem on the mean value of the potential has also found useful application in other problems. Consider, for example, the flow into the bottom of a canal of finite length l . Assume that the canal cuts through a layer of relatively poor permeability, and that its semi-circular bottom lies in a permeable subsoil (Fig. 6.5). Applying the principle of mirror representation (Chapter 4), we shall introduce into the scheme a second, auxiliary, drain placed symmetrically about the boundary of the earth massif.

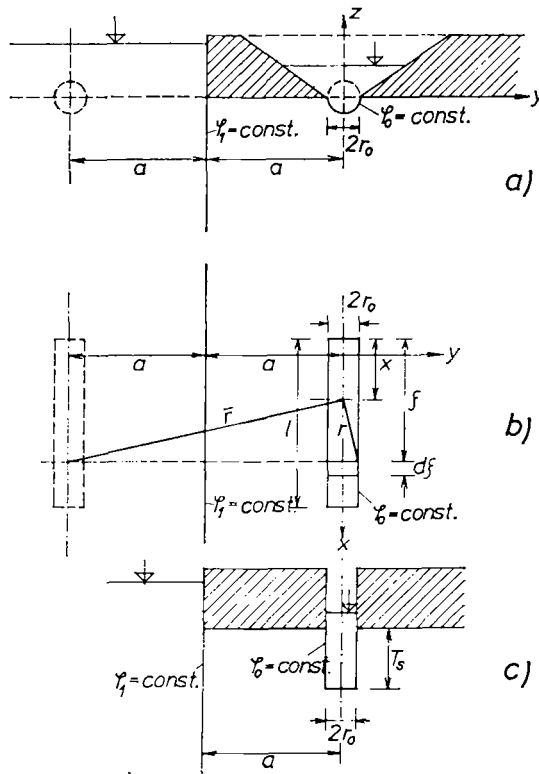


Fig. 6.5. Application of the theorem on the mean value of the potential.

Assume a system of sinks to be distributed uniformly along the drainage axis. In this case, the specific volume flow per unit length is Q/l . On the auxiliary drain we have a system of uniformly distributed sources of equal strength. We know beforehand that on account of the symmetry of the image of the flow field, the specified boundary of the earth massif will be the line $\varphi = \text{const}$. Accordingly, we concentrate

on the situation on the surface of the drainage. We introduce the substitution

$$\Delta\varphi = \varphi_1 - \varphi \quad (6.4.2)$$

According to Sect. 6.3, we have

$$dQ = \frac{2Q}{l} d\zeta \quad (6.4.3)$$

and for a strip of elementary width $d\zeta$ on the drain surface we obtain

$$d(\Delta\varphi) = \frac{Q}{l} \frac{d\zeta}{2\pi} \left(\frac{1}{r} - \frac{1}{\bar{r}} \right) \quad (6.4.4)$$

with

$$r = \sqrt{[r_0^2 + (x - \zeta)^2]} \\ \bar{r} = \sqrt{[4a^2 + (x - \zeta)^2]}$$

Integrating between the limits 0 and l we have

$$\Delta\varphi = \frac{Q}{2\pi l} \int_0^l \left[\frac{1}{\sqrt{[r_0^2 + (x - \zeta)^2]}} - \frac{1}{\sqrt{[4a^2 + (x - \zeta)^2]}} \right] d\zeta \quad (6.4.5)$$

$$\Delta\varphi = \frac{Q}{2\pi l} \left(\operatorname{argsinh} \frac{x}{r_0} + \operatorname{argsinh} \frac{l-x}{r_0} - \operatorname{argsinh} \frac{x}{2a} - \operatorname{argsinh} \frac{l-x}{2a} \right) \quad (6.4.6)$$

We have specified that φ_0 should be constant on the drain perimeter. In this case, the theorem on the mean value of the potential leads to the equations

$$\Delta\varphi_0 = \frac{1}{l} \int_0^l \Delta\varphi(x) dx \quad (6.4.7)$$

$$\begin{aligned} \Delta\varphi_0 = \frac{Q}{2\pi l^2} \int_0^l & \left(\operatorname{argsinh} \frac{x}{r_0} + \operatorname{argsinh} \frac{l-x}{r_0} - \right. \\ & \left. - \operatorname{argsinh} \frac{x}{2a} - \operatorname{argsinh} \frac{l-x}{2a} \right) dx \\ \Delta\varphi_0 = \frac{Q}{2\pi l^2} & \left\{ 2r_0 \left[\frac{l}{r_0} \operatorname{argsinh} \frac{l}{r_0} - \sqrt{\left(1 + \frac{l^2}{r_0^2}\right) + 1} \right] - \right. \\ & \left. - 4a \left[\frac{l}{2a} \operatorname{argsinh} \frac{l}{2a} - \sqrt{\left(1 + \frac{l^2}{4a^2}\right) + 1} \right] \right\} \quad (6.4.8) \end{aligned}$$

$$\begin{aligned} \Delta\varphi_0 = \frac{Q}{\pi l} & \left\{ \operatorname{argsinh} \frac{l}{r_0} - \operatorname{argsinh} \frac{l}{2a} + \frac{r_0}{l} \left[1 - \sqrt{\left(1 + \frac{l^2}{r_0^2}\right)} \right] - \right. \\ & \left. - \frac{2a}{l} \left[1 - \sqrt{\left(1 + \frac{l^2}{4a^2}\right)} \right] \right\} \quad (6.4.9) \end{aligned}$$

From these equations follow the relations which define the total volume flow as

$$Q = \frac{\pi l \Delta \varphi_0}{\operatorname{argsinh} \frac{l}{r_0} - \operatorname{argsinh} \frac{l}{2a} + \frac{r_0}{l} \{1 - \sqrt{[1 + (l^2/r_0^2)]}\} - \frac{2a}{l} \{1 - \sqrt{[1 + (l^2/4a^2)]}\}} \quad (6.4.10)$$

It is clearly seen from the discussion that the solution just outlined can also be applied to analyses of the operation of hydraulically imperfect wells situated close to the boundary of an earth massif. On setting $l = 2T_s$, we obtain for the scheme shown in Fig. 6.5c that

$$Q = \frac{2\pi T_s \Delta \varphi_0}{\operatorname{argsinh} (2T_s/r_0) - \operatorname{argsinh} (T_s/a) + (r_0/2T_s) \{1 - \sqrt{[1 + (4T_s^2/r_0^2)]}\} - (a/T_s) \{1 - \sqrt{[1 + (T_s^2/a^2)]}\}}$$

For $T_s \gg r_0$, $a \gg r_0$, and assuming that

$$\operatorname{argsinh} x = \ln [x + \sqrt{(1 + x^2)}]$$

this becomes

$$Q = \frac{2\pi T_s \Delta \varphi_0}{\ln \frac{4T_s/r_0}{(T_s/a) + \sqrt{[1 + (T_s^2/a^2)]}} + 1 - \frac{a}{T_s} \{1 - \sqrt{[1 + (T_s^2/a^2)]}\}} \quad (6.4.11)$$

6.5. Flow net in space

A general three-dimensional flow field is defined by a system of equipotential surfaces, with streamlines at right angles to these surfaces. It is therefore possible to construct a family of stream surfaces which, together with the equipotential surfaces, divide the space into a system of curved-edged but right prisms (Fig. 6.6a). Imagine that in each of the systems of curvilinear streams thus constructed, there is the same volume flow ΔQ . The cross-sectional area of the streams is $\Delta a \Delta b$. Hence

$$\Delta Q = \Delta a \Delta b \frac{\Delta \varphi}{\Delta l}$$

If there are m partial streams in the field, and the difference between the potentials on the boundaries of the whole region is $\varphi_1 - \varphi_2$, then

$$Q = \Delta a \Delta b \frac{\varphi_1 - \varphi_2}{\Delta l} \frac{m}{n} \quad (6.5.1)$$

where n is the number which co-determines the value of the difference between the potentials of two adjoining equipotential surfaces

$$\Delta\varphi = \frac{\varphi_1 - \varphi_2}{n} = \text{const.}$$

Eqn. (6.5.1) can also be written in the form

$$\frac{Q}{\varphi_1 - \varphi_2} = \frac{\Delta a \Delta b m}{n \Delta l} \quad (6.5.2)$$

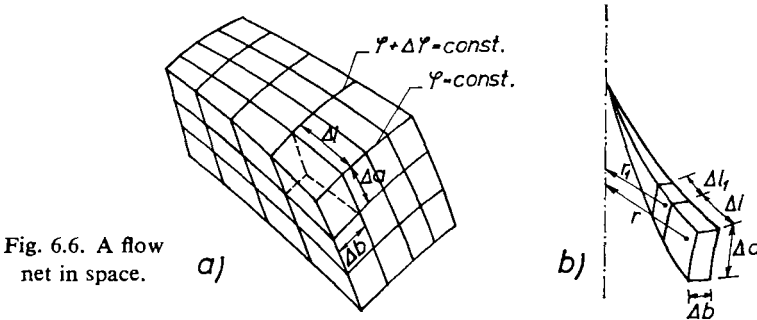


Fig. 6.6. A flow net in space.

The term on the right-hand side of eqn. (6.5.2) represents the reciprocal of the modulus of resistance of the flow field being considered. It is, therefore, possible to divide the flow region in a three-dimensional field by means of suitably chosen equipotential surfaces into several fragments, and to solve a practical problem by the application of the method of fragments.

Consider now a special case, the axi-symmetrical flow. Take a curved-edged prism of the flow (Fig. 6.6b) and denote by r the distance of its centre from the axis of symmetry. Accordingly, there flows through the field per height Δa the volume

$$\Delta Q = 2\pi r \Delta a \frac{\Delta\varphi}{\Delta l} \quad (6.5.3)$$

Assume further that next to the prism being examined there lies another prism with its centre at a distance r_1 from the axis of symmetry. Denote its sides by Δa_1 , Δl_1 . $\Delta\varphi$ is assumed to be constant in all the consecutive prisms. Hence, applying the equation of continuity, we obtain

$$2\pi r \Delta a \frac{\Delta\varphi}{\Delta l} = 2\pi r_1 \Delta a_1 \frac{\Delta\varphi}{\Delta l_1}$$

This leads to

$$\frac{\Delta a}{\Delta a_1} \frac{\Delta l_1}{\Delta l} = \frac{r_1}{r} \quad (6.5.4)$$

Along the streamlines, the ratio between the lengths of the sides of the elements varies depending on the distance from the axis of symmetry. This is why the process of drawing a flow net for an axi-symmetrical flow is more difficult than the graphical construction of patterns of two-dimensional flows.

6.6. Superposition of hydraulic imperfections

In Fig. 6.7 we have drawn in solid lines the pattern of flow to an hydraulically imperfect well. Assume that another, similar, well is situated nearby. If this well were to operate separately, its pattern of flow would be as indicated by the dashed lines. Since the flows are both described by linear partial differential equations, the principle of superposition applies.

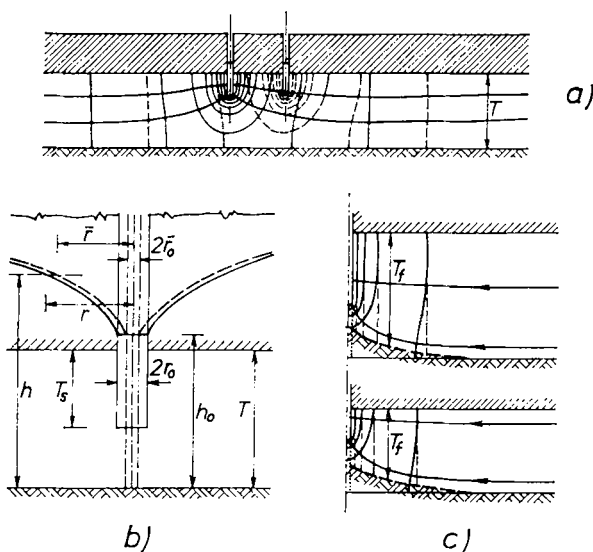


Fig. 6.7. Superposition of hydraulic imperfections.

However, a theoretical solution of the case just outlined is by no means easy, the chief difficulty being the fact that according to our basic conception, each of the wells is solved by synthesizing the effects of the point sinks distributed along its axis. Whenever these sinks are affected by the operation of an adjoining well, a new situation arises. Theoretically, we should choose a new distribution of the sinks, i.e. a distribution that would satisfy the newly-created boundary conditions. We call the phenomenon "interference among hydraulic imperfections".

The solution becomes more feasible if the wells are situated sufficiently far from one another, and their axes lie in those portions of the flow nets in which the equipotential lines are vertical. In such cases the interference among hydraulic imperfections is almost imperceptible, and we can apply simple superposition of the effects of the wells as wholes. There are no theoretical objections to solving systems of wells

by the method of fragments. Interference among hydraulic imperfections comes into play not only in the case of wells but generally in all instances when the effect of the three-dimensional flow of an object extends into the flow field produced by the operation of another object.

The respective solutions usually make use of experimental methods or of simplified theoretical schemes which are easier to handle. As an example, consider the flow into an hydraulically imperfect well of diameter $2r_0$, operating under a pressure regime (Fig. 6.7b). Assume that its effect can be replaced by that of an hydraulically perfect well of a smaller diameter, $2\bar{r}_0$. The depression curve has an ordinate h generally at a distance r from the axis of the hydraulically imperfect well. The same ordinate h is obtained with the help of the substitute, hydraulically perfect well, if we choose the distance to be $\bar{r}$. Hence the problem reduces to that of finding the dependence $\bar{r} = f(r)$. Some authors solved the latter problem experimentally and introduced the results into the calculation in the form of a modification of Dupuit's equation for an hydraulically perfect well:

$$Q = -2\pi kT \frac{h - h_0}{\ln(r/r_0) + A} \quad (6.6.1)$$

The value of A is established as a function of T , T_s , r_0 , r , using experimentally constructed diagrams.

Another form of equation is obtained if eqn. (6.6.1) is rearranged as follows:

$$Q = -2\pi kT \frac{h - h_0}{\ln(r/r_0) + \ln \text{Exp } A} = -2\pi kT \frac{h - h_0}{\ln(r \text{ Exp } A_r / r_0 \text{ Exp } A_0)}$$

With the notation

$$r \text{ Exp } A_r = \frac{r}{B} = \bar{r}, \quad r_0 \text{ Exp } A_0 = \frac{r_0}{B_0} = \bar{r}_0$$

the above equation becomes

$$Q = -2\pi kT \frac{h - h_0}{\ln(B_0 r / B r_0)} = -2\pi kT \frac{h - h_0}{\ln(\bar{r} / \bar{r}_0)} \quad (6.6.2)$$

This step enabled us to obtain Dupuit's equation in which the substitute value $\bar{r}$ appears directly. $\bar{r}$ is determined with the help of the coefficient B which we have plotted as a function of T/r in Fig. 6.8.

The superposition of hydraulic imperfections rests on the general principle of superposition of the potentials. This makes it possible to examine the effect of the operation of a whole system of hydraulically imperfect wells. Assume that there are m objects, each with an active portion of length T_{sj} . In the (x, y) plane which coincides

with the lower boundary of the overburden we have

$$h = \frac{1}{2\pi kT} \sum_{j=1}^m \ln \frac{r_j}{B_j} + c = \frac{1}{2\pi kT} \sum_{j=1}^m \ln \bar{r}_j + c \quad (6.6.3)$$

where

c — a constant,

r_j — distance between the axis of the j -th well and a given point x, y ,

B_j — reduction factor corresponding to the length r_j .

Having determined the constant c from the boundary conditions, we proceed just as in the case of a plane problem except that the real lengths r_j are now replaced by the reduced lengths $\bar{r}_j$.

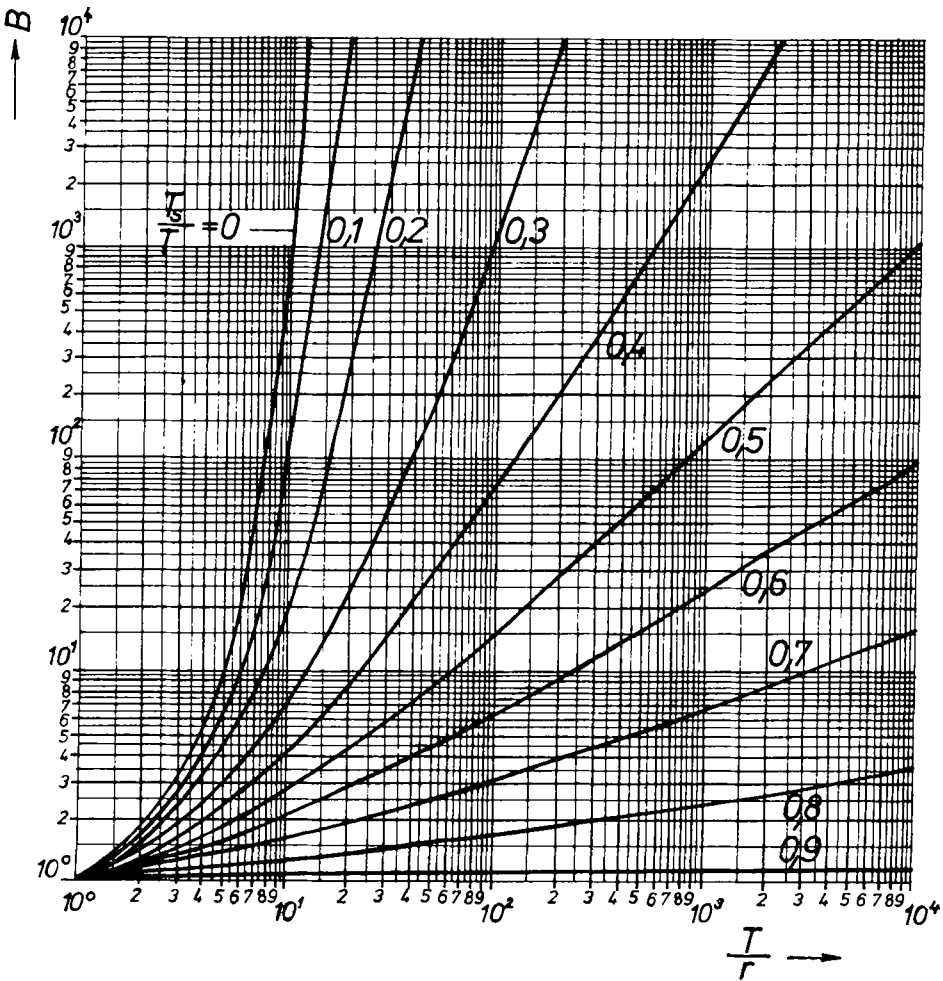


Fig. 6.8. Reduction coefficient B .

We can suppose that on the level of the lower boundary of the overburden there exist the isolines $h = \text{const.}$ which correspond hypothetically to the case of an effect of a system of perfect wells operating in place of the hydraulically imperfect wells. Assume that on the level of the lower boundary of the overburden we follow the actual course of the lines $h = \text{const.}$ and that the hydraulically imperfect wells

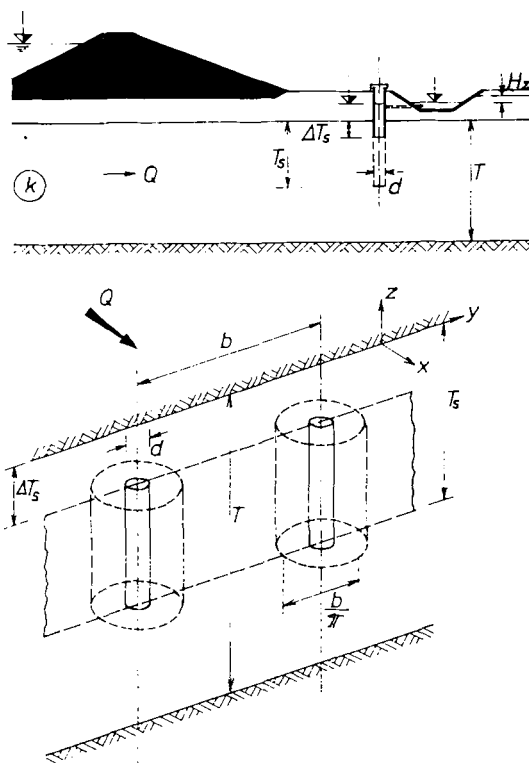


Fig. 6.9. A well system in the hinterland of a dam.

are replaced by perfect wells of diameters equal to those of the real objects. The effect of hydraulic imperfection manifests itself by increased resistance. If the equipotential surfaces are regarded as vertical, it must be taken into account that the thickness of the saturated medium varies from one place to another (the equation of continuity). The flow to hydraulically imperfect wells can be expressed in terms of an equivalent plane flow on the assumption that there exist fictitious curved surfaces which bound the saturated medium two-dimensionally (from below, in the case being considered).

The nature of such surfaces is shown schematically in Fig. 6.7c. The flow net of the actual flow is shown in solid lines, the course of the fictitious cylindrical equipotentials is indicated by vertical dashed lines, and the symbol T_f refers to the thickness of the fictitious saturated stratum.

As the figure suggests, the shape of the fictitious bounding surfaces changes only slightly with increasing length of the well since this makes the object hydraulically more perfect. Essentially, a bounding surface expresses the effect of the curvature of the streamlines at the inlet to the bottom of an imperfect well.

It goes without saying that a description of the flow in the substitute fictitious layer must pay due regard to the effect of the changing thickness of the saturated stratum. The formulation becomes more straightforward in cases when the thickness of the fictitious permeable layer varies relatively little in the direction of flow. The possibility of applying conformal mapping to a fictitious flow defined as simply as indicated above, constitutes a definite advantage.

The idea can also be adopted in a number of improvised approximate solutions of three-dimensional problems. As an example, we shall consider seepage into the drainage system of a protective dam.

Since the problems concerning the solution of drainage systems were discussed in Chapter 4, we do not think it necessary to repeat here the principles of the theoretical approach. We shall concentrate on the question of how to calculate an element of such a system, i.e. a series of hydraulically imperfect wells and a tail canal.

The quantity of flow coming in per unit length of the series of wells and unit time is Q (negative). Deep in the hinterland of the dam, the groundwater level is maintained at a certain depth below the terrain surface. The pipes which connect the wells to a reservoir are situated so that the entrance to them lies below the level in the canal (Fig. 6.9). We are required to determine the height H_z which represents the effect of the losses on the inlet to the wells. In other words we have to find the most desirable position of the level in the canal relative to the groundwater level in the hinterland.

Assume that the three-dimensional flow to the wells can be replaced by a plane flow, noting that — theoretically — fictitious bounding surfaces have been created on the lower as well as on the upper boundary of the permeable medium. Suppose that in the immediate vicinity of the wells, there exists an axi-symmetrical flow which is localized in the cylinder of radius b/π (b is the spacing between the wells). A part of the resistance is produced by the flow inside the annular space; this is characterized by the height H' defined as

$$H' = - \frac{Q_s}{2\pi k(T_s - \Delta T_s)} \ln \frac{b}{\pi d} \quad (6.6.4)$$

where Q_s is the quantity of water entering a well per unit time, i.e.

$$Q_s = Qb \quad (6.6.5)$$

- k — coefficient of permeability of the medium,
- T_s — distance between the lower edge of the well filter and the overburden,
- ΔT_s — distance between the upper edge of the well filter and the overburden,
- d — well diameter,
- b — spacing between the wells.

In the (x, y) plane we transform the space with the vertical bounding cylinders of radius b/π into a space with a bounding straight surface. This step leads to the formation of a continuous strip of width $T_s - \Delta T_s$. From the hydraulic point of view the flow being studied is that into a cut located inside a permeable stratum. The upper and lower fictitious bounding surfaces of the flow region are not plane after the transformation. However, the wrinkling is highly localized and can be neglected. The flow in the area thus simplified can also be solved by the method of conformal mapping in the (x, z) plane.

The idea of fictitious bounding surfaces has merely served us to show that it is justifiable to use a continuous cut for the calculation of the second partial loss, characterized by the height H''

$$H'' = \frac{Q}{2\pi k} \ln \left(\sin^2 \frac{\pi T_s}{2T} - \sin^2 \frac{\pi \Delta T_s}{2T} \right) \quad (6.6.6)$$

Since we have

$$H_z = H' + H'' \quad (6.6.7)$$

we obtain the total height H_z in the form

$$H_z = -\frac{Q}{2\pi k} \left[\frac{b}{T_s - \Delta T_s} \ln \frac{b}{\pi d} - \ln \left(\sin^2 \frac{\pi T_s}{2T} - \sin^2 \frac{\pi \Delta T_s}{2T} \right) \right] \quad (6.6.8)$$

In the case when the series contains hydraulically perfect wells, $T_s = T$, $\Delta T_s = 0$, and this becomes

$$H_z = -\frac{Qb}{2\pi kT} \ln \frac{b}{\pi d} \quad (6.6.9)$$

A more exact solution of this limiting case leads to the equation

$$H_z = -\frac{Qb}{2\pi kT} \ln \frac{1}{2 \sinh(\pi d/2b)} \quad (6.6.10)$$

Eqns. (6.6.10) and (6.6.9) give nearly identical results for

$$2 \sinh \frac{\pi d}{2b} \rightarrow \frac{\pi d}{b}$$

This relation holds good for $d/b \leq 0.05$.

6.7. Problem of interference among surfaces of seepage

An analysis of the conditions conducive to the formation of a surface of seepage, is somewhat less difficult in the case of axial symmetry, i.e. when the flow to the well is assumed to be distributed uniformly along the well perimeter. However, in civil

engineering practice, one is apt to come across systems containing hydraulically perfect wells placed so close to one another that the inflow of water to the individual objects is no longer axi-symmetrical.

We have noted in Sect. 6.1 that the height of a surface of seepage depends on the loading of the well lining. Hence an asymmetrical inflow alters the conditions necessary for the formation of a surface of seepage along an object's perimeter.

Consider, for example, the scheme of two parallel systems of wells shown in Fig. 6.10. On the condition that $a \geq 3d$, $b \geq 3d$, we calculated at the points *I*, *II* of the lining of one of the wells, the inlet gradient J_0 (defined by the hydraulic theory) and obtained

$$J_0 = -\frac{Q}{bkh_0} \left(\frac{1}{\sin(\pi d/b)} \pm \frac{\cosh(\pi a/b)}{\sqrt{[\sinh^2(\pi a/b) + \sin^2(\pi d/b)]}} \right) \quad (6.7.1)$$

where

Q — inflow to a well,

a — distance between the axes of the parallel systems of wells,

b — spacing between the wells,

d — diameter of the outer perimeter of a well,

k — coefficient of permeability.

In eqn. (6.7.1), the minus sign applies to the point *I*, the plus sign to the point *II*.

Using the hydraulic theory, we also calculated the ordinates of the points of the free surface and obtained, for the point *I*

$$H_1 = \sqrt{\left\{ h_0^2 - \frac{2Q}{\pi k} \operatorname{argtgh} \frac{\sinh[(\pi/b)(\frac{1}{2}a - d)]}{\sinh[(\pi/b)(\frac{1}{2}a + d)]} \right\}} \quad (6.7.2)$$

and for the point 2

$$H_2 = \sqrt{\left\{ h_0^2 - \frac{2Q}{\pi k} \operatorname{argtgh} \frac{\cosh[(\pi/b)(\frac{1}{2}a - d)]}{\cosh[(\pi/b)(\frac{1}{2}a + d)]} \right\}} \quad (6.7.3)$$

As these equations suggest, the non-uniformity of discharge along a well lining is negligible for $b/d > 50$. The asymmetry becomes very marked for $b/d < (10 \text{ to } 20)$. It may well happen that the conditions for the formation of a surface of seepage at the point *I* will be more unfavourable by far than those at the point *II*. The upper boundary of the surface of seepage can take the form of a spatial curve. Fig. 6.11 shows a flow net which indicates schematically the configuration of the field in one of the examined examples.

As we have noted, the free surface in the space of interfering wells has a complicated three-dimensional character. The situation becomes especially hard to follow in the case of overlapping effects of the surfaces of seepage of adjoining objects. Although we speak of interference among surfaces of seepage, we realize that the

term lacks something in theoretical clarity, for it describes only the visible consequences of a phenomenon the reasons for which must be sought in general physical laws. We believe, however, that the meaning of the term will be quite clear to practising engineers.

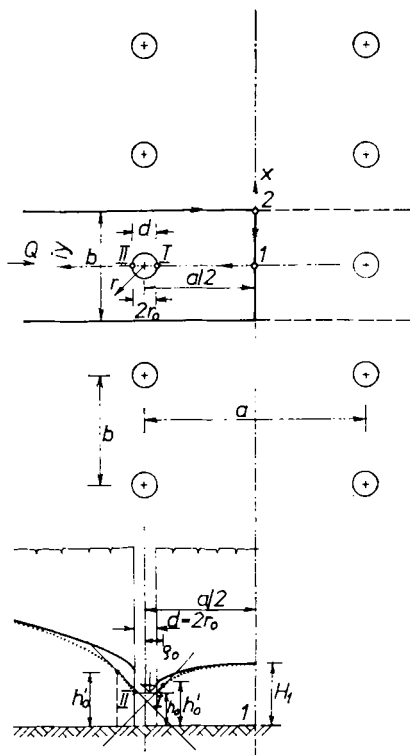


Fig. 6.10. Two parallel well systems.

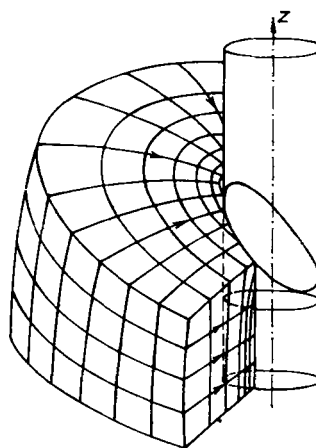


Fig. 6.11. Spatial idea of an asymmetrical surface of seepage.

An analysis of the interference problem has been found especially useful in cases in which a marked lowering of the groundwater level is the principal object. The choice of a suitable spacing between wells (as it depends on the diameter of the wells) makes it possible to eliminate the surfaces of seepage inside spaces protected by systems of wells. In the first estimate of the effect being achieved, use is made of the graphical construction shown schematically in Fig. 6.10, bottom. The course of the free surface obtained with the help of the hydraulic theory (with due consideration given to the actual boundary condition on the well lining) is drawn in dotted lines. The height of the point of contact of a tangent to the depression curve, which forms an angle of 45° with the horizontal, determines the position of the upper boundary of the surface of seepage. A surface of seepage does not form on that portion of the depression curve to which all the tangents have slopes less than one.

Only a restricted class of problems of three-dimensional flow is amenable to a simplified description. The study becomes especially complicated whenever one wishes to obtain a more exact quantitative idea of groundwater motion. In most cases the assumptions of the hydraulic theory are no longer adequate, one must resort to more general methods.

6.8. Green's function. Spherical inversion

Consider the functions $\varphi(x, y, z)$, $\psi(x, y, z)$ which are harmonic in a given region and have continuous second partial derivatives. We make use of the notation

$$\Delta\varphi = \frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2} + \frac{\partial^2\varphi}{\partial z^2}, \quad \Delta\psi = \frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} + \frac{\partial^2\psi}{\partial z^2}$$

The derivatives in the direction of a normal which points into the space V , are

$$\begin{aligned} \frac{\partial\varphi}{\partial n} &= \frac{\partial\varphi}{\partial x} \cos \alpha + \frac{\partial\varphi}{\partial y} \cos \beta + \frac{\partial\varphi}{\partial z} \cos \gamma \\ \frac{\partial\psi}{\partial n} &= \frac{\partial\psi}{\partial x} \cos \alpha + \frac{\partial\psi}{\partial y} \cos \beta + \frac{\partial\psi}{\partial z} \cos \gamma \end{aligned}$$

According to Green's theorem we have that

$$\begin{aligned} \iiint_V \left(\frac{\partial\varphi}{\partial x} \frac{\partial\psi}{\partial x} + \frac{\partial\varphi}{\partial y} \frac{\partial\psi}{\partial y} + \frac{\partial\varphi}{\partial z} \frac{\partial\psi}{\partial z} \right) dx dy dz &= \\ &= - \iiint_V \varphi \Delta\psi dx dy dz - \iint_S \varphi \frac{\partial\psi}{\partial n} dS \end{aligned}$$

where S is the surface area which bounds the space V .

Interchanging the functions φ and ψ , we obtain another, similar, equation. Taking the difference between the two equations yields

$$\iiint_V (\varphi \Delta\psi - \psi \Delta\varphi) dx dy dz = \iint_S \left(\psi \frac{\partial\varphi}{\partial n} - \varphi \frac{\partial\psi}{\partial n} \right) dS \quad (6.8.1)$$

Essentially, Green's formula (6.8.1) applies even in the case when the space V is bounded by several surfaces S . Then the right-hand side of eqn. (6.8.1) contains the sum of the integrals for all the surfaces.

Assume that φ has the meaning of a velocity potential and that

$$\psi = \frac{1}{r}$$

where r is the distance between the point $N(\xi, \eta, \zeta)$ which belongs to the region V with the surface S , and the point $M(x, y, z)$ which can lie either within the surface S or outside the region V . Accordingly, $\varphi(\xi, \eta, \zeta) = \varphi(N)$ at the point N , and $\varphi(x, y, z) = \varphi(M)$ at the point M . Similarly, we also write that $\psi = \psi(M, N)$. If the points M, N are coincident, then $\psi = \infty$.

Assume that the point M lies at the centre of a sphere which has a small radius and a surface S_M . Assume further that we have

$$\begin{aligned} \frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} + \frac{\partial^2 \varphi}{\partial z^2} &= 0, & \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} + \frac{\partial^2 \psi}{\partial z^2} &= 0 \\ \frac{\partial^2 \varphi}{\partial \xi^2} + \frac{\partial^2 \psi}{\partial \eta^2} + \frac{\partial^2 \psi}{\partial \zeta^2} &= 0, & \frac{\partial^2 \psi}{\partial \xi^2} + \frac{\partial^2 \psi}{\partial \eta^2} + \frac{\partial^2 \psi}{\partial \zeta^2} &= 0 \end{aligned}$$

For two harmonic and interchangeable functions, Green's theorem (6.8.1) leads to the following relations for the surfaces S and S_M :

$$\iint_S \left[\frac{1}{r} \frac{\partial \varphi}{\partial n} - \varphi \frac{\partial(1/r)}{\partial n} \right] dS + \iint_{S_M} \left[\frac{1}{r} \frac{\partial \varphi}{\partial v} - \varphi \frac{\partial(1/r)}{\partial v} \right] dS = 0 \quad (6.8.2)$$

where n, v indicate, respectively, that normals are chosen to the surface S and to the surface S_M , the normals pointing into the respective spaces.

We now turn to the second term on the left-hand side of eqn. (6.8.2). Assuming the radius of the surface S_M to be small, we have $\partial \varphi / \partial v \doteq 0$, and

$$\begin{aligned} \lim_{r \rightarrow 0} \iint_{S_M} \frac{1}{r} \frac{\partial \varphi}{\partial v} dS &= 0 \\ \lim_{r \rightarrow 0} \iint_{S_M} \varphi \frac{\partial(1/r)}{\partial v} dS &= \lim_{r \rightarrow 0} \iint_{S_M} \frac{\varphi}{r^2} dS \end{aligned} \quad (6.8.3)$$

For a very small $r = \varrho$, this becomes

$$\lim_{r \rightarrow 0} \iint_{S_M} \varphi \frac{\partial(1/r)}{\partial v} dS \approx \lim_{\varrho \rightarrow 0} \varphi(M) \iint_{S_M} \frac{1}{\varrho^2} dS = \varphi(M) \frac{4\pi \varrho^2}{\varrho^2} = 4\pi \varphi(M)$$

and eqn. (6.8.2) can be written in the form

$$\iint_S \left[\frac{1}{r} \frac{\partial \varphi}{\partial n} - \varphi \frac{\partial(1/r)}{\partial n} \right] dS - 4\pi \varphi(M) = 0,$$

from which we obtain

$$\varphi(M) = \frac{1}{4\pi} \iint_S \left[\frac{1}{r} \frac{\partial \varphi}{\partial n} - \varphi \frac{\partial(1/r)}{\partial n} \right] dS \quad (6.8.4)$$

In the literature, eqn. (6.8.4) goes under the name of Green's integral. Note that ψ is not required to have the physical meaning of the stream function. It only serves in the explanation of the significance of the so-called Green's function $\mathcal{F}(x, y, z, \xi, \eta, \zeta)$ which is defined as follows:

$$\mathcal{F}(x, y, z, \xi, \eta, \zeta) = \frac{1}{r} + \psi(x, y, z, \xi, \eta, \zeta) \quad (6.8.5)$$

$\psi(x, y, z, \xi, \eta, \zeta)$ is a harmonic function inside the region S . It is required that the function $\mathcal{F}$ should be equal to zero on the surface S .

Green's function can be used to advantage in the solution of harmonic problems. It follows from Green's formula (6.8.1) that

$$\begin{aligned} \frac{1}{4\pi} \iint_S \left[\left(\mathcal{F} - \frac{1}{r} \right) \frac{\partial \varphi}{\partial n} - \varphi \frac{\partial [\mathcal{F} - (1/r)]}{\partial n} \right] dS &= 0 \\ \frac{1}{4\pi} \iint_S \left[\mathcal{F} \frac{\partial \varphi}{\partial n} - \frac{1}{r} \frac{\partial \varphi}{\partial n} - \varphi \frac{\partial \mathcal{F}}{\partial n} + \varphi \frac{\partial (1/r)}{\partial n} \right] dS &= 0 \end{aligned} \quad (6.8.6)$$

$$\frac{1}{4\pi} \iint_S \left(\mathcal{F} \frac{\partial \varphi}{\partial n} - \varphi \frac{\partial \mathcal{F}}{\partial n} \right) dS - \frac{1}{4\pi} \iint_S \left[\frac{1}{r} \frac{\partial \varphi}{\partial n} - \varphi \frac{\partial (1/r)}{\partial n} \right] dS = 0 \quad (6.8.7)$$

On substituting Green's integral (6.8.4) for the second term of eqn. (6.8.7) we obtain

$$\varphi(M) = \frac{1}{4\pi} \iint_S \left(\mathcal{F} \frac{\partial \varphi}{\partial n} - \varphi \frac{\partial \mathcal{F}}{\partial n} \right) dS \quad (6.8.8)$$

According to our assumption, the function $\mathcal{F}$ on the surface S is equal to zero; hence eqn. (6.8.8) leads to either

$$\varphi(M) = - \frac{1}{4\pi} \iint_S \varphi \frac{\partial \mathcal{F}}{\partial n} dS \quad (6.8.9)$$

or

$$\varphi(M) = \frac{1}{4\pi} \iint_S \varphi \frac{\partial \mathcal{F}}{\partial n'} dS \quad (6.8.10)$$

where n' denotes the normal pointing out of the region.

Thus if we know Green's function $\mathcal{F}$, we can establish the value of the function φ at any specified point of the field, provided that we have information about the value of φ on the surface S .

It is a difficult problem to find function $\mathcal{F}$ such as to satisfy the prescribed conditions. The following form is quoted in the literature for a sphere of radius R :

$$\begin{aligned} \mathcal{F} = & \frac{1}{\sqrt{[(x - \xi)^2 + (y - \eta)^2 + (z - \zeta)^2]}} - \\ & - \frac{R}{\sqrt{\{(x^2 + y^2 + z^2) [(x' - \xi)^2 + (y' - \eta)^2 + (z' - \zeta)^2]\}}} \end{aligned} \quad (6.8.11)$$

It is assumed in this equation that we have the point $M(x, y, z)$, the point $N(\xi, \eta, \zeta)$ and the auxiliary point $M'(x', y', z')$ which lies on the line connecting the centre of the sphere with the point M . The position of the conjugate points M, M' is defined by the relation

$$R^2 = \varrho_s r_s \quad (6.8.12)$$

The meaning of the symbols is evident from Fig. 6.12a which depicts a section of the sphere. The plane of the section passes through the points M, M', N .

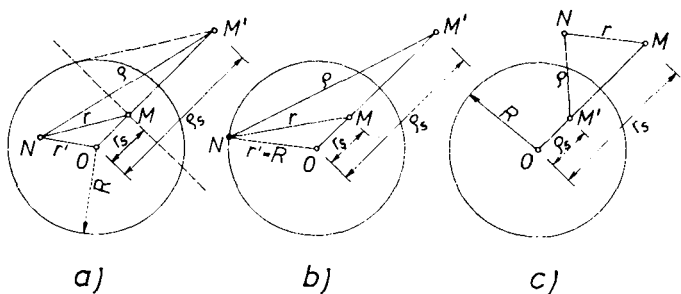


Fig. 6.12. Fundamental geometrical relations of spherical inversion.

As a verification we substitute eqn. (6.7.12) in the equation

$$\frac{\partial^2 \mathcal{F}}{\partial x^2} + \frac{\partial^2 \mathcal{F}}{\partial y^2} + \frac{\partial^2 \mathcal{F}}{\partial z^2} = 0$$

and note that the latter is satisfied by each term of eqn. (6.8.11). In the next step we regard the point N as lying on the surface of the sphere (Fig. 6.12b), and rewrite eqn. (6.8.11) as follows:

$$\mathcal{F} = \frac{1}{r} - \frac{R}{r_s \varrho} \quad (6.8.13)$$

According to eqn. (6.8.12) we obtain

$$\frac{r_s}{R} = \frac{R}{\varrho_s} \quad (6.8.14)$$

As the equality (6.8.14) implies, the triangles $OM'N$ and OMN are similar triangles, and so their other sides must also be in the same ratio. Thus, for example,

$$\frac{r_s}{r} = \frac{R}{\varrho}$$

which leads to

$$R = \frac{\varrho r_s}{r} \quad (6.8.15)$$

Substitution of eqn. (6.8.15) in eqn. (6.8.13) leads to $\mathcal{F} = 0$ so that Green's function (6.8.11) on the sphere surface is indeed equal to zero. It is, of course, necessary that the point N should remain inside the sphere.

The scheme of an analogous problem, i.e. that of determining Green's function for a region outside a sphere, is shown in Fig. 6.12c. Assuming the continued validity of eqn. (6.8.12) we note that eqn. (6.8.11) and hence also eqn. (6.8.13) apply to this case as well. On the strength of this fact we conclude that the effect produced on the sphere surface by a sink or a source situated at a point inside the sphere, is the same as that produced by an equal sink or source situated at the conjugate point. The effects can be interchanged. The resulting spherical inversion is of considerable importance for the theory of experimental studies of three-dimensional problems.

The above exposition confirms the possibility of superposing several effects in spatial conditions. If a source is placed at a point, and a sink at the conjugate point, the resulting flow will be characterized by a constant value of φ on the sphere.

The equipotential surfaces will also be spherical, and their centres will lie on the line connecting the conjugate points.

6.9. Green's function for a half-space

Consider a region V in the form of a sphere with a very large radius. The surface of such a sphere is nearly planar. Suppose that a pair of conjugate points, one with a sink, the other with a source, exist in the vicinity of the sphere surface. On the basis of the results obtained in Sect. 6.2, we have — according to the principle of superposition (Fig. 6.13a)

$$\varphi = \frac{Q}{4\pi} \left(\frac{1}{r} - \frac{1}{\bar{r}} \right) \quad (6.9.1)$$

The expression

$$\mathcal{F} = \frac{1}{r} - \frac{1}{\bar{r}} \quad (6.9.2)$$

can be taken as Green's function of a half-space, for $\mathcal{F} = 0$ on the sphere surface, i.e. in the (ξ, η) plane, and moreover, $\Delta\mathcal{F} = 0$ at all points which lie above the (ξ, η) plane. Hence eqn. (6.9.2) is acceptable for the half-space above the (ξ, η) plane, i.e. for $\zeta > 0$.

On the sphere, i.e. in the (ξ, η) plane, we have according to eqn. (6.8.10)

$$\varphi(M) = \frac{1}{4\pi} \iint_S \varphi \frac{\partial \mathcal{F}}{\partial n'} dS \quad (6.9.3)$$

$$\frac{\partial \mathcal{F}}{\partial n'} = \frac{\partial}{\partial \zeta} \left(\frac{1}{r} \right) - \frac{\partial}{\partial \zeta} \left(\frac{1}{\bar{r}} \right) = -\frac{1}{r^2} \frac{\partial r}{\partial \zeta} + \frac{1}{\bar{r}^2} \frac{\partial \bar{r}}{\partial \zeta} \quad (6.9.4)$$

where

$$r = \sqrt{[(x - \xi)^2 + (y - \eta)^2 + (z - \zeta)^2]}$$

$$\bar{r} = \sqrt{[(x' - \xi)^2 + (y' - \eta)^2 + (z' - \zeta)^2]}$$

$$\frac{\partial r}{\partial \zeta} = - \frac{z - \zeta}{r} \bigg|_{\zeta=0} = - \frac{z}{r}$$

$$\frac{\partial \bar{r}}{\partial \zeta} = \frac{z + \zeta}{r} \bigg|_{\zeta=0} = \frac{z}{r}$$

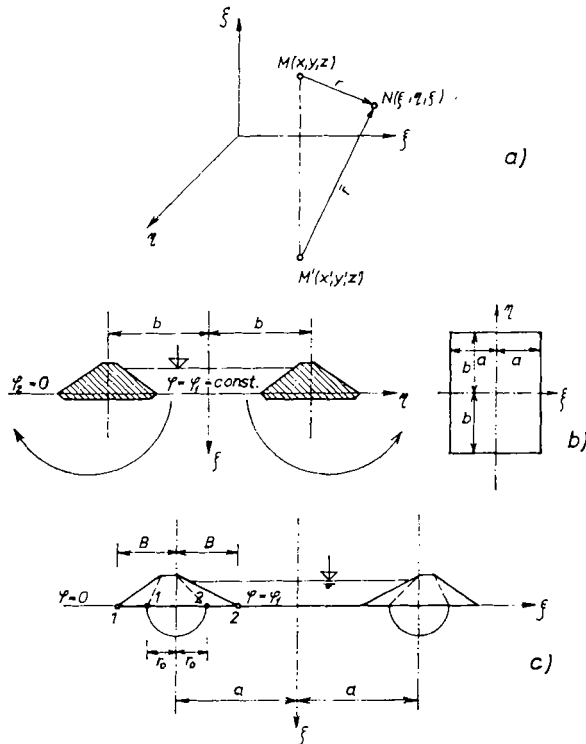


Fig. 6.13. Seepage through the rectangular bottom of a foundation pit.

On substituting in eqn. (6.9.4), we obtain in the (ξ, η) plane for $r = \bar{r}$ that

$$\frac{\partial \mathcal{F}}{\partial n'} = \frac{z}{r^3} - \frac{1}{\bar{r}^2} \frac{z}{r} = \frac{2z}{r^3} \quad (6.9.5)$$

Accordingly,

$$\varphi(M) = \frac{z}{2\pi} \iint_S \frac{\varphi}{r^3} dS \quad (6.9.6)$$

As an example of a practical application of Green's function for a half-space, consider seepage from a fairly large reservoir of rectangular plan. The dimensions of the embankment are regarded as relatively small (Fig. 6.13b), and the boundaries of the reservoir are assumed to lie at the half-length of the footing bottom of the embankment. Water percolates through the bottom of the reservoir, passes underneath the embankments, seeps to the terrain surface behind them and flows away.

Since the potential on the reservoir bottom is $\varphi = \varphi_1 = \text{const.}$, we obtain, according to eqn. (6.9.6) on the assumption that $\varphi = 0$ on the terrain surface, that

$$\varphi = \frac{\varphi_1 z}{2\pi} \iint_S \frac{dS}{r^3}$$

$$r = \sqrt{[(x - \xi)^2 + (y - \eta)^2 + z^2]} \quad (6.9.7)$$

$$\varphi = \frac{\varphi_1 z}{2\pi} \int_{\eta=-b}^{\eta=+b} \int_{\xi=-a}^{\xi=+a} \frac{d\xi d\eta}{[(x - \xi)^2 + (y - \eta)^2 + z^2]^{3/2}} \quad (6.9.8)$$

Making the substitution

$$(x - \xi) = \sigma, \quad (y - \eta) = \tau$$

we have

$$\varphi = \frac{\varphi_1 z}{2\pi} \int_{y-b}^{y+b} \int_{x-a}^{x+a} \frac{d\sigma d\tau}{(\sigma^2 + \tau^2 + z^2)^{3/2}} \quad (6.9.9)$$

The integral in eqn. (6.9.9) is evaluated by repeated integration, viz.

$$\begin{aligned} & \int_{x-a}^{x+a} \frac{d\sigma}{(\sigma^2 + \tau^2 + z^2)^{3/2}} = \\ &= \frac{1}{\tau^2 + z^2} \left[\frac{x+a}{\sqrt{[(x+a)^2 + \tau^2 + z^2]}} - \frac{x-a}{\sqrt{[(x-a)^2 + \tau^2 + z^2]}} \right] \\ \varphi &= \frac{\varphi_1 z}{2\pi} \left\{ (x+a) \int_{y-b}^{y+b} \frac{d\tau}{(\tau^2 + z^2) \sqrt{[(x+a)^2 + \tau^2 + z^2]}} - \right. \\ & \quad \left. - (x-a) \int_{y-b}^{y+b} \frac{d\tau}{(\tau^2 + z^2) \sqrt{[(x-a)^2 + \tau^2 + z^2]}} \right\} \end{aligned}$$

Following the integration we have

$$\begin{aligned} \varphi = \varphi(M) &= \frac{\varphi_1}{2\pi} \left\{ \text{arctg} \frac{x+a}{z} \frac{y+b}{\sqrt{[(x+a)^2 + (y+b)^2 + z^2]}} - \right. \\ & \quad \left. - \text{arctg} \frac{x+a}{z} \frac{y-b}{\sqrt{[(x+a)^2 + (y-b)^2 + z^2]}} + \right. \end{aligned}$$

$$\begin{aligned}
& + \operatorname{arctg} \frac{x-a}{z} \frac{y+b}{\sqrt{[(x-a)^2 + (y+b)^2 + z^2]}} - \\
& - \operatorname{arctg} \frac{x-a}{z} \frac{y-b}{\sqrt{[(x-a)^2 + (y-b)^2 + z^2]}} \} \quad (6.9.10)
\end{aligned}$$

The solution as outlined above assumes the footing bottom to be comparatively narrow, nearly a point. The flow which arises halfway along the sides of the reservoir bottom can be approximated by a two-dimensional flow. In the immediate vicinity of the foundation the streamlines are nearly circular: one such could possibly be thought of as the limiting streamline. Following a theoretical treatment by the method explained in Chapter 4, the calculation procedure could be modified to respect the requirement of a finite width of the footing bottom. Such a method would, of course, fail in the corners of the reservoir bottom where the flow has a three-dimensional character.

A detailed analysis suggests a concentration of flow in the vicinity of the rectangular contacts between the bottom sides. Hence embankments with a curved ground plan, or at least a rounded perimeter for the reservoir, are recommended to satisfy hydraulic requirements.

No such modifications are necessary when the reservoir bottom has the character of an infinitely long strip. For $b \rightarrow \infty$, eqn. (6.9.10) is replaced by the equation

$$\varphi(M) = \frac{\varphi_1}{\pi} \left(\operatorname{arctg} \frac{x+a}{z} - \operatorname{arctg} \frac{x-a}{z} \right) \quad (6.9.11)$$

The velocity of flow in the direction of the z axis is

$$v_z = \frac{\partial \varphi}{\partial z} = \frac{\varphi_1}{\pi} \left[-\frac{x+a}{z^2 + (x+a)^2} + \frac{x-a}{z^2 + (x-a)^2} \right]$$

For $z = 0$, i.e. along the bottom or the surface of the terrain behind the embankment, this becomes

$$v_z = -\frac{\varphi_1}{\pi} \frac{2a}{x^2 - a^2} \quad (6.9.12)$$

Fig. 6.13c shows a section of a canal formed by embankments with a very flat footing bottom $2B$ wide. After transforming the foundation into a semicircle, we obtain an embankment the semicircular foundation perimeter of which has a radius $r_0 = B/2$. The specific volume flow q per 1 m of the canal is calculated approximately, using eqn. (6.9.12):

$$q = 2 \int_0^{a-r_0} v_z dx = \frac{4a\varphi_1}{\pi} \int_0^{a-r_0} \frac{dx}{\sqrt{(a^2 - x^2)}} = -\frac{4\varphi_1}{\pi} \operatorname{argth} \left(1 - \frac{r_0}{a} \right) \quad (6.9.13)$$

6.10. Lamé's method of solving certain symmetrical three-dimensional fields

Suppose we have a parameter κ which is a function of the coordinates x, y, z :

$$\kappa = f(x, y, z)$$

Assume that the potential φ is a function of this parameter, so that in the x direction we have

$$\frac{\partial \varphi}{\partial x} = \frac{\partial \varphi}{\partial \kappa} \frac{\partial \kappa}{\partial x} \quad (6.10.1)$$

$$\frac{\partial^2 \varphi}{\partial x^2} = \frac{\partial^2 \varphi}{\partial \kappa^2} \left(\frac{\partial \kappa}{\partial x} \right)^2 + \frac{\partial \varphi}{\partial \kappa} \frac{\partial^2 \kappa}{\partial x^2} \quad (6.10.2)$$

and similarly in the y and z directions.

Upon substituting in the Laplace equation

$$\Delta \varphi = \frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} + \frac{\partial^2 \varphi}{\partial z^2} = 0$$

we obtain

$$\Delta \varphi = \frac{\partial^2 \varphi}{\partial \kappa^2} \left[\left(\frac{\partial \kappa}{\partial x} \right)^2 + \left(\frac{\partial \kappa}{\partial y} \right)^2 + \left(\frac{\partial \kappa}{\partial z} \right)^2 \right] + \frac{\partial \varphi}{\partial \kappa} \left(\frac{\partial^2 \kappa}{\partial x^2} + \frac{\partial^2 \kappa}{\partial y^2} + \frac{\partial^2 \kappa}{\partial z^2} \right) = 0 \quad (6.10.3)$$

Using the notation

$$\left(\frac{\partial \kappa}{\partial x} \right)^2 + \left(\frac{\partial \kappa}{\partial y} \right)^2 + \left(\frac{\partial \kappa}{\partial z} \right)^2 = \nabla \kappa \nabla \kappa \quad (6.10.4)$$

$$\frac{\partial^2 \kappa}{\partial x^2} + \frac{\partial^2 \kappa}{\partial y^2} + \frac{\partial^2 \kappa}{\partial z^2} = \Delta \kappa \quad (6.10.5)$$

we write, in place of eqn. (6.10.3), the equation

$$\Delta \varphi = \nabla \kappa \nabla \kappa \frac{\partial^2 \varphi}{\partial \kappa^2} + \Delta \kappa \frac{\partial \varphi}{\partial \kappa} = 0 \quad (6.10.6)$$

Eqn. (6.10.6) leads to the function g :

$$g = \frac{(\partial^2 \varphi / \partial \kappa^2)}{(\partial \varphi / \partial \kappa)} = - \frac{\Delta \kappa}{\nabla \kappa \nabla \kappa} = \frac{d}{d\kappa} \left[\ln \left(\frac{\partial \varphi}{\partial \kappa} \right) \right] \quad (6.10.7)$$

Consider now a three-dimensional field characterized by the equipotential surfaces

$$F(x, y, z, \kappa) = 0 \quad (6.10.8)$$

Differentiating with respect to x we obtain

$$\frac{dF}{dx} = \frac{\partial F}{\partial x} + \frac{\partial F}{\partial \kappa} \frac{\partial \kappa}{\partial x} = 0$$

which leads to

$$\frac{\partial \kappa}{\partial x} = - \frac{(\partial F / \partial x)}{(\partial F / \partial \kappa)} \quad (6.10.9)$$

Differentiating again we obtain

$$\frac{\partial^2 \kappa}{\partial x^2} = - \left[\frac{\partial}{\partial x} \left(\frac{\partial F / \partial x}{\partial F / \partial \kappa} \right) + \frac{\partial \kappa}{\partial x} \frac{\partial}{\partial \kappa} \left(\frac{\partial F / \partial x}{\partial F / \partial \kappa} \right) \right]$$

which leads to

$$\begin{aligned} \frac{\partial^2 \kappa}{\partial x^2} &= - \left[\frac{\partial^2 F / \partial x^2}{\partial F / \partial \kappa} - 2 \frac{(\partial F / \partial x) (\partial^2 F / \partial x \partial \kappa)}{(\partial F / \partial \kappa)^2} + \frac{(\partial^2 F / \partial \kappa^2) (\partial F / \partial x)^2}{\partial F / \partial \kappa (\partial F / \partial \kappa)^2} \right] = \\ &= - \left[\frac{\partial^2 F / \partial x^2}{\partial F / \partial \kappa} - \frac{1}{(\partial F / \partial \kappa)^2} \frac{\partial}{\partial \kappa} \left(\frac{\partial F}{\partial x} \right)^2 + \frac{(\partial^2 F / \partial \kappa) (\partial F / \partial \kappa)^2}{(\partial F / \partial \kappa) (\partial F / \partial \kappa)^2} \right] \end{aligned}$$

Similar relations can be obtained for the derivatives in the y and z directions. On writing

$$\begin{aligned} \frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial y^2} + \frac{\partial^2 F}{\partial z^2} &= \Delta F \\ \left(\frac{\partial F}{\partial x} \right)^2 + \left(\frac{\partial F}{\partial y} \right)^2 + \left(\frac{\partial F}{\partial z} \right)^2 &= \nabla F \nabla F \end{aligned}$$

and substituting the respective derivatives in eqn. (6.10.5) we obtain

$$\Delta \kappa = - \left[\frac{\Delta F}{\partial F / \partial \kappa} - \frac{1}{(\partial F / \partial \kappa)^2} \frac{\partial}{\partial \kappa} (\nabla F \nabla F) + \frac{\partial^2 F / \partial \kappa^2}{\partial F / \partial \kappa} \frac{\nabla F \nabla F}{(\partial F / \partial \kappa)^2} \right]$$

Substitution of the respective derivatives in eqn. (6.10.4) gives

$$\Delta \kappa \Delta \kappa = \frac{\nabla F \nabla F}{(\partial F / \partial \kappa)^2} \quad (6.10.10)$$

The function g defined by eqn. (6.10.7) is then expressed as follows:

$$\begin{aligned} g &= \frac{(\partial F / \partial \kappa)^2}{\nabla F \nabla F} \frac{\Delta F}{\partial F / \partial \kappa} - \frac{(\partial F / \partial \kappa)^2}{\nabla F \nabla F} \frac{1}{(\partial F / \partial \kappa)^2} \frac{\partial}{\partial \kappa} (\nabla F \nabla F) + \\ &+ \frac{(\partial F / \partial \kappa)^2}{\nabla F \nabla F} \frac{\partial^2 F / \partial \kappa^2}{\partial F / \partial \kappa} \frac{\nabla F \nabla F}{(\partial F / \partial \kappa)^2} = \end{aligned}$$

$$\begin{aligned}
&= \frac{\Delta F}{\nabla F \nabla F} \frac{\partial F}{\partial \kappa} - \frac{\partial / \partial \kappa (\nabla F \nabla F)}{\nabla F \nabla F} + \frac{\partial^2 F / \partial \kappa^2}{\partial F / \partial \kappa} \\
g &= \frac{\partial F / \partial \kappa}{\nabla F \nabla F} \Delta F - \frac{\partial}{\partial \kappa} \ln \frac{\partial F / \partial \kappa}{\nabla F \nabla F}
\end{aligned} \tag{6.10.11}$$

We now introduce another auxiliary function $\bar{g}$ defined by

$$\begin{aligned}
\bar{g} &= \frac{\nabla F \nabla F}{\partial F / \partial \kappa} \\
\nabla F \nabla F &= \bar{g} \frac{\partial F}{\partial \kappa}
\end{aligned} \tag{6.10.12}$$

With the help of this function we can write, in place of eqn. (6.10.11), the equation

$$g = \frac{\Delta F}{\bar{g}} - \frac{\partial}{\partial \kappa} \ln \frac{1}{\bar{g}}$$

So far we have generally assumed that the parameter κ depends on x, y, z . We now seek special conditions under which κ is independent of x, y, z . In the first place, it is necessary that

$$\frac{\partial^2 F}{\partial x^2} = f_1(\kappa), \quad \frac{\partial^2 F}{\partial y^2} = f_2(\kappa), \quad \frac{\partial^2 F}{\partial z^2} = f_3(\kappa) \tag{6.10.13}$$

where $f_1(\kappa), f_2(\kappa), f_3(\kappa)$ are functions of the one parameter, κ .

It is clear that in this case ΔF will be a function of a single parameter. Moreover, since we also require that $g(\kappa)$ should be a function of κ , $\bar{g}(\kappa)$ must also be a function of the single parameter κ .

The above requirements are satisfied by the function

$$F = \frac{x^2}{2} f_1(\kappa) + \frac{y^2}{2} f_2(\kappa) + \frac{z^2}{2} f_3(\kappa) - 1 \tag{6.10.14}$$

because for κ independent of x, y, z , $\Delta F = f_1(\kappa) + f_2(\kappa) + f_3(\kappa)$. At the same time we have

$$\begin{aligned}
\nabla F \nabla F &= x^2 f_1^2 + y^2 f_2^2 + z^2 f_3^2 \\
\frac{\partial F}{\partial \kappa} &= \frac{x^2}{2} \frac{\partial f_1}{\partial \kappa} + \frac{y^2}{2} \frac{\partial f_2}{\partial \kappa} + \frac{z^2}{2} \frac{\partial f_3}{\partial \kappa}
\end{aligned}$$

On substituting in eqn. (6.10.12) we obtain

$$x^2 f_1^2 + y^2 f_2^2 + z^2 f_3^2 = \bar{g}(\kappa) \left(\frac{x^2}{2} \frac{\partial f_1}{\partial \kappa} + \frac{y^2}{2} \frac{\partial f_2}{\partial \kappa} + \frac{z^2}{2} \frac{\partial f_3}{\partial \kappa} \right) \tag{6.10.15}$$

which leads to

$$\begin{aligned}\frac{\bar{g}(\kappa)}{2} \frac{\partial f_1}{\partial \kappa} &= f_1^2 \\ \frac{\bar{g}(\kappa)}{2} \frac{\partial f_2}{\partial \kappa} &= f_2^2 \\ \frac{\bar{g}(\kappa)}{2} \frac{\partial f_3}{\partial \kappa} &= f_3^2\end{aligned}\tag{6.10.16}$$

Integration of eqns. (6.10.16) yields

$$\begin{aligned}f_1(\kappa) &= -\frac{1}{2 \int \frac{d\kappa}{\bar{g}(\kappa)} + \frac{a^2}{2}} \\ f_2(\kappa) &= -\frac{1}{2 \int \frac{d\kappa}{\bar{g}(\kappa)} + \frac{b^2}{2}} \\ f_3(\kappa) &= -\frac{1}{2 \int \frac{d\kappa}{\bar{g}(\kappa)} + \frac{c^2}{2}}\end{aligned}$$

where a, b, c are constants.

According to eqn. (6.10.14) the required function is of the form

$$F(x, y, z, \kappa) = -\frac{1}{2} \left[\frac{x^2}{2 \int \frac{d\kappa}{\bar{g}(\kappa)} + \frac{a^2}{2}} + \frac{y^2}{2 \int \frac{d\kappa}{\bar{g}(\kappa)} + \frac{b^2}{2}} + \frac{z^2}{2 \int \frac{d\kappa}{\bar{g}(\kappa)} + \frac{c^2}{2}} \right] - 1\tag{6.10.17}$$

If $a = 0$ is substituted for the constant a and $\bar{g}(\kappa)$ is chosen to equal $-2/\kappa$, eqn. (6.10.17), after rearrangement, defines the following system of second-degree confocal surfaces:

$$F(x, y, z, \kappa) = \frac{x^2}{\kappa^2} + \frac{y^2}{\kappa^2 - b^2} + \frac{z^2}{\kappa^2 - c^2} - 1 = 0\tag{6.10.18}$$

The shape of the surfaces depends on the magnitude of the parameter κ . Thus, for example, for $0 \leq \kappa \leq b < c$ we obtain

$$\frac{x^2}{\kappa^2} - \frac{y^2}{b^2 - \kappa^2} - \frac{z^2}{c^2 - \kappa^2} = 1\tag{6.10.19}$$

Eqn. (6.10.19) describes a hyperboloid of two sheets. For $0 \leq b < c < \kappa < \infty$, the equation becomes

$$\frac{x^2}{\kappa^2} + \frac{y^2}{\kappa^2 - b^2} + \frac{z^2}{\kappa^2 - c^2} = 1 \quad (6.10.20)$$

i.e. the equation of an ellipsoid.

As an example of the application of Lamé's method, consider the discharge into a foundation pit which passes through a relatively impervious overburden and is

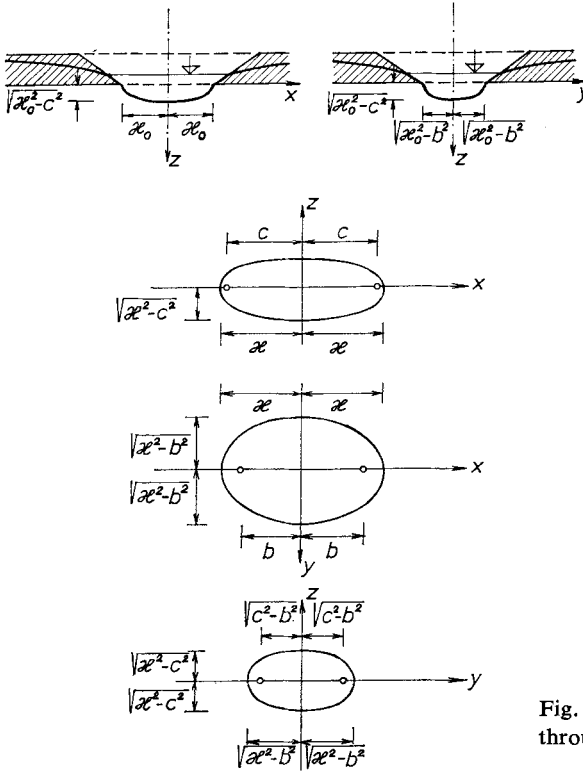


Fig. 6.14. Solution of seepage through the ellipsoidal bottom of a foundation pit.

shaped like an ellipsoid in the permeable subsoil. In the permeable half-space the equipotential surfaces have the form of confocal ellipsoids. As shown in Fig. 6.14, the semi-axis of the boundary ellipsoid, i.e. the pit bottom, in the x direction is κ_0 .

According to eqn. (6.10.20) we have

$$F = \frac{x^2}{\kappa^2} + \frac{y^2}{\kappa^2 - b^2} + \frac{z^2}{\kappa^2 - c^2} - 1$$

so that

$$\Delta F = \frac{2}{\kappa^2} + \frac{2}{\kappa^2 - b^2} + \frac{2}{\kappa^2 - c^2} \quad (6.10.21)$$

We have assumed in the derivation that $\bar{g}(\kappa) = -2/\kappa$; hence according to eqn. (6.10.12) we obtain

$$\frac{\nabla F \nabla F}{\partial F / \partial \kappa} = -\frac{2}{\kappa} \quad (6.10.22)$$

By eqns. (6.10.11) and (6.10.7)

$$g(\kappa) = \frac{\partial^2 \varphi / \partial \kappa^2}{\partial \varphi / \partial \kappa} = -\frac{\kappa}{2} \Delta F - \frac{\partial}{\partial \kappa} \ln \left(-\frac{\kappa}{2} \right) \quad (6.10.23)$$

Substitution of eqn. (6.10.21) gives

$$g(\kappa) = \frac{\partial^2 \varphi / \partial \kappa^2}{\partial \varphi / \partial \kappa} = \frac{\partial}{\partial \kappa} \left(\ln \frac{\partial \varphi}{\partial \kappa} \right) = -\frac{1}{\kappa} - \frac{\kappa}{\kappa^2 - b^2} - \frac{\kappa^2}{\kappa^2 - c^2} + \frac{1}{\kappa}$$

integrating which we obtain

$$\ln \frac{\partial \varphi}{\partial \kappa} = - \left[\frac{1}{2} \ln (\kappa^2 - b^2) + \frac{1}{2} \ln (\kappa^2 - c^2) \right] + C$$

and from this

$$\frac{\partial \varphi}{\partial \kappa} = -C_1 \frac{c}{\sqrt{[(\kappa^2 - b^2)(\kappa^2 - c^2)]}}$$

Integrating the equation again gives

$$\varphi = -C_1 \int \frac{c \, d\kappa}{\sqrt{[(\kappa^2 - b^2)(\kappa^2 - c^2)]}} \quad (6.10.24a)$$

and the substitution

$$\kappa = \frac{c}{t}, \quad d\kappa = \frac{c \, dt}{t^2}$$

leads to

$$\varphi = C_1 \int_0^t \frac{dt}{\sqrt{\{(1-t^2)[1-(b^2/c^2)t^2]\}}} + C_2 \quad (6.10.24b)$$

The integral in eqn. (6.10.24) is an incomplete elliptic integral of the first kind with modulus b/c . This can be proved readily by using yet another substitution, i.e. $t = \sin \Theta$ which leads to the form frequently found in the literature, viz

$$\varphi = C_1 \int_0^\Theta \frac{d\Theta}{\sqrt{[1-(b^2/c^2)\sin^2 \Theta]}} + C_2 \quad (6.10.25)$$

Tables of elliptic integrals were compiled, for example, by Jahnke, Emde, Lösch.

The values of the constants are obtained from the boundary conditions. On the bottom of the pit $\kappa = \kappa_0$, $\varphi = \varphi_0$, $t = t_0 = c/\kappa_0$. Substituting these values in eqn. (6.10.25) we obtain

$$\varphi_0 = C_1 \int_0^{\Theta_0} \frac{d\Theta}{\sqrt{[1 - (b^2/c^2) \sin^2 \Theta]}} + C_2 \quad (6.10.26)$$

Assume that for an ellipsoid defined by a sufficiently large $\kappa = \kappa_1$, we have $\varphi = \varphi_1$, $t = t_1 = c/\kappa_1$, $t_1 = \sin \Theta_1$. On substituting in eqn. (6.10.25), we have in this case

$$\varphi_1 = C_1 \int_0^{\Theta_1} \frac{d\Theta}{\sqrt{[1 - (b^2/c^2) \sin^2 \Theta]}} + C_2 \quad (6.10.27)$$

Expressions (6.10.26) and (6.10.27) amount essentially to a system of two equations in the two unknowns C_1 and C_2 .

The velocity of flow in the direction of the normal to the equipotential surface is calculated using eqn. (6.10.28) which reads as follows:

$$\begin{aligned} v_n &= \frac{\partial \varphi}{\partial n} = \frac{\partial \varphi}{\partial x} \cos \alpha + \frac{\partial \varphi}{\partial y} \cos \beta + \frac{\partial \varphi}{\partial z} \cos \gamma = \\ &= \frac{\partial \varphi}{\partial \kappa} \left(\frac{\partial \kappa}{\partial x} \cos \alpha + \frac{\partial \kappa}{\partial y} \cos \beta + \frac{\partial \kappa}{\partial z} \cos \gamma \right) \end{aligned} \quad (6.10.28)$$

where $\cos \alpha$, $\cos \beta$, $\cos \gamma$ are the cosines of the angles made by the normal with the directions of the x , y , z axes.

On substituting eqn. (6.10.9), eqn. (6.10.28) becomes

$$v_n = - \frac{\partial \varphi / \partial \kappa}{\partial F / \partial \kappa} \left(\cos \alpha \frac{\partial F}{\partial x} + \cos \beta \frac{\partial F}{\partial y} + \cos \gamma \frac{\partial F}{\partial z} \right) \quad (6.10.29)$$

Since we know that for a normal to the surface F we have

$$\cos \alpha = \frac{\partial F / \partial x}{\sqrt{[(\partial F / \partial x)^2 + (\partial F / \partial y)^2 + (\partial F / \partial z)^2]}}$$

substitution of the expression for the directional cosines in eqn. (6.10.29) leads to

$$v_n = - \frac{\partial \varphi / \partial \kappa}{\partial F / \partial \kappa} \sqrt{(\nabla F \nabla F)} \quad (6.10.30)$$

In the case being considered we have

$$\begin{aligned} \frac{\partial F}{\partial x} &= \frac{2x}{\kappa^2}, \quad \frac{\partial F}{\partial y} = \frac{2y}{\kappa^2 - b^2}, \quad \frac{\partial F}{\partial z} = \frac{2z}{\kappa^2 - c^2} \\ \frac{\partial F}{\partial \kappa} &= -2\kappa \left[\frac{x^2}{\kappa^4} + \frac{y^2}{(\kappa^2 - b^2)^2} + \frac{z^2}{(\kappa^2 - c^2)^2} \right] \end{aligned}$$

and further, according to eqn. (6.10.24a),

$$\frac{\partial \varphi}{\partial x} = - \frac{cC_1}{\sqrt{[(x^2 - b^2)(x^2 - c^2)]}} \quad (6.10.31)$$

Substituting the above in eqn. (6.10.30) we obtain

$$v_n = - \frac{cC_1}{x \sqrt{\left\{ (x^2 - b^2)(x^2 - c^2) \left[\frac{x^2}{x^4} + \frac{y^2}{(x^2 - b^2)^2} + \frac{z^2}{(x^2 - c^2)^2} \right] \right\}}} \quad (6.10.32)$$

The total volume flow through an equipotential surface is determined by integrating eqn. (6.10.32) along the water-discharging portion of the ellipsoid surface.

In the case being considered, we denote the total area of the ellipsoid surface by the symbol S . Since only the flow Q per second through a half of the area is of interest, we can write

$$2Q = \iint_S v_n \, dS \quad (6.10.33)$$

We omit here the procedure used to evaluate the integral appearing in eqn. (6.10.33), merely stating the result, viz.

$$Q = -2\pi cC_1 \quad (6.10.34)$$

6.11. Notes on the solution of three-dimensional motion in a non-homogeneous permeable medium

The steady-state groundwater flow in space is generally described by eqn. (1.4.58) which contains the components of the coefficient of permeability in the directions of the coordinate axes. It is assumed that in this equation is to be found the dependence of the value of k on x, y, z , viz.

$$k = k_0 f(x, y, z) \quad (6.11.1)$$

where

k — value of the coefficient of permeability at a given point of space,
 k_0 — some basic value.

The function $f(x, y, z)$ may be continuous in the whole space. If this is so, it is helpful to consider yet another function

$$F = k_0 h \sqrt{f} \quad (6.11.2)$$

where h — piezometric head at a given point of space.

Similarly as in Section 4.11 we can readily show that in space

$$\frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial y^2} + \frac{\partial^2 F}{\partial z^2} = \frac{F}{\sqrt{f}} \left[\frac{\partial^2(\sqrt{f})}{\partial x^2} + \frac{\partial^2(\sqrt{f})}{\partial y^2} + \frac{\partial^2(\sqrt{f})}{\partial z^2} \right] \quad (6.11.3)$$

Eqn. (6.11.3) is easier to solve in cases when the term in the square brackets on its right-hand side is equal to a constant or to some fairly simple function.

Since the procedure for working out practical cases is analogous to that explained in Chapter 4, we do not enter into further details here.

Whenever the continuity of the function $f(x, y, z)$ can no longer be assumed, the analysis becomes more difficult. In a number of such problems, one can resort to the application of the method of fictitious sources, or of the procedure based on the theory of Green's function. Variational methods have also been found successful in many instances. But practical experience has taught us that it is best to choose the method of calculation to suit the case at hand; the limited scope of this book hardly allows us even to mention all the versions which can possibly be applied.

CHAPTER 7

UNSTEADY FLOW OF GROUNDWATER

The character of groundwater motion in nature is usually unsteady. An exact mathematical description of unsteady flow under complicated conditions is very difficult, and to make it easier, various simplifications and schemes are employed. This strategy, however, rarely leads to the possibility of obtaining an analytical solution, and numerical and experimental methods are preferred to achieve the desired end. From a theoretical viewpoint, it serves a useful purpose to gain a rough idea of the course of an unsteady process even in the simplest of cases, for such knowledge provides fundamental information on the character and time development of the phenomenon involved.

The literature is replete with solutions of diverse problems. Some of them, however, have no parallels and are unsuitable for general application. In the sections that follow, we shall therefore concentrate on the description of such basic schemes and their solutions which, methodologically speaking, enable us to apply them to studies of practical problems specified in a similar manner.

7.1. Similarity

It has been established that a dimensional analysis based on Buckingham's theorem can serve as a helpful means for delineating the scope of similar problems, and in fact, for their solution as well.

Assume that a certain physical process is taking place in a given space. We can describe this process by means of equations containing dimensional quantities denoted by letters in square brackets. Thus, for example, the fundamental quantities which will be used in the subsequent discussion, i.e. the mass, the length and the time, are denoted by $[M]$, $[L]$, $[T]$. From these three basic dimensions we construct the remaining, more complicated, ones and study their mutual relations.

Assume that besides the object being studied there exists its geometrical reduction or enlargement, and that a physically homogeneous and kinematically similar process takes place on the model thus conceived. The physical quantities appertaining to the model are denoted by the subscript m .

The theory of physical similarity of the two processes is based on the fact that the corresponding physical quantities are in a definite, constant ratio — namely the scale.

A physical quantity is determined by the product of an absolute number which describes its magnitude and a dimension which describes its quality. Thus, for example, a length is expressed by the product $L = \bar{L}[L]$ where $\bar{L}$ is a number. The scale of the lengths, λ_L is given by the ratio

$$\lambda_L = \frac{L}{L_m} = \frac{\bar{L}[L]}{\bar{L}_m[L]} = \frac{\bar{L}}{\bar{L}_m} \quad (7.1.1)$$

Consider now a more complicated quantity A which is defined by the product

$$A = \bar{A}[A] = \bar{L}^{A_L}[L]^{A_L} \bar{M}^{A_M}[M]^{A_M} \bar{T}^{A_T}[T]^{A_T} \quad (7.1.2)$$

Its scale is given by

$$\lambda_A = \frac{\bar{L}^{A_L} \bar{M}^{A_M} \bar{T}^{A_T}}{\bar{L}_m^{A_L} \bar{M}_m^{A_M} \bar{T}_m^{A_T}} \quad (7.1.3)$$

The dimension $[A]$ of the quantity A is characterized by the product

$$[A] = [L]^{A_L} [M]^{A_M} [T]^{A_T} \quad (7.1.4)$$

Consider now the quantity π which is more complicated still, viz.

$$\pi = A^{\alpha_A} B^{\alpha_B} C^{\alpha_C} \quad (7.1.5)$$

The dimensions of the quantities $A, B, C, \dots$ are $[A], [B], [C], \dots$; they are given by

$$\begin{aligned} [A] &= [L]^{A_L} [M]^{A_M} [T]^{A_T} \\ [B] &= [L]^{B_L} [M]^{B_M} [T]^{B_T} \\ [C] &= [L]^{C_L} [M]^{C_M} [T]^{C_T} \\ &\vdots \end{aligned} \quad (7.1.6)$$

In keeping with the foregoing, we also have

$$\begin{aligned} \lambda_A &= \lambda_L^{A_L} \lambda_M^{A_M} \lambda_T^{A_T} \\ \lambda_B &= \lambda_L^{B_L} \lambda_M^{B_M} \lambda_T^{B_T} \\ \lambda_C &= \lambda_L^{C_L} \lambda_M^{C_M} \lambda_T^{C_T} \\ &\vdots \end{aligned} \quad (7.1.7)$$

The dimension of the quantity π is determined by the relation

$$[\pi] = [L]^{\beta_L} [M]^{\beta_M} [T]^{\beta_T} \quad (7.1.8)$$

The corresponding scale is calculated from the formula

$$\lambda_{\pi} = \lambda_L^{\beta_L} \lambda_M^{\beta_M} \lambda_T^{\beta_T} \quad (7.1.9)$$

where, clearly,

$$\begin{aligned} \beta_L &= \alpha_A A_L + \alpha_B B_L + \alpha_C C_L + \dots \\ \beta_M &= \alpha_A A_M + \alpha_B B_M + \alpha_C C_M + \dots \\ \beta_T &= \alpha_A A_T + \alpha_B B_T + \alpha_C C_T + \dots \end{aligned} \quad (7.1.10)$$

Imagine now that the quantity π is dimensionless. In this case the left-hand sides of the system of equations (7.1.10) must be equal to zero (an annulled system) so that, according to eqn. (7.1.8), we obtain that $[\pi] = 1$. Moreover, by eqn. (7.1.9)

$$\lambda_{\pi} = 1 \quad (7.1.11)$$

As the above implies, the values of π in reality and on the model are of equal magnitude. At the same time, we have

$$\bar{\pi} = \pi = A^{\alpha_A} B^{\alpha_B} C^{\alpha_C} \quad (7.1.12)$$

The values of the exponents $\alpha_A, \alpha_B, \alpha_C, \dots$ are obtained by solving the system (7.1.10) with the left-hand sides set equal to zero. The number of equations is the same as the number of basic dimensions. Since there is usually a greater number of unknowns, we set one of the redundant unknowns equal to an arbitrary non-zero number. In this way, we obtain the values of the unknowns for the calculation of a certain, j -th, value of π_j . Substituting non-zero values for the redundant unknowns, we determine J values of π_j .

The dimensionless quantities π_j are termed “similarity invariants”, and their mutual relation describes the process being considered in reality and on the model. The functional dependence

$$f(\pi_1, \pi_2, \dots, \pi_J) = 0 \quad (7.1.13)$$

established on the model will also be of use when evaluating the course of the homogeneous process in reality, for if it holds that

$$\pi_j = A^{(\alpha_A)_j} B^{(\alpha_B)_j} C^{(\alpha_C)_j} \dots$$

eqn. (7.1.11) yields a total of J relations necessary for the re-calculation:

$$\lambda_A^{(\alpha_A)_j} \lambda_B^{(\alpha_B)_j} \lambda_C^{(\alpha_C)_j} = 1 \quad (7.1.14)$$

The procedure just outlined enables us to analyze highly complicated processes, such as the motion of water in a medium undergoing deformation under non-isothermal conditions. However, processes taking place in groundwater hydraulics are usually assumed to be isothermal, i.e. capable of being studied with the help of three basic and five derived dimensional quantities (Table 7.1).

Table 7.1

Quantity	Code character	Symbol	Dimension
Length	<i>A</i>	<i>l</i>	<i>L</i>
Specific mass	<i>B</i>	<i>q</i>	<i>ML</i> ⁻³
Time	<i>C</i>	<i>t</i>	<i>T</i>
Velocity	<i>D</i>	<i>v</i>	<i>LT</i> ⁻¹
Pressure	<i>E</i>	<i>p</i>	<i>ML</i> ⁻¹ <i>T</i> ⁻²
Acceleration of gravity	<i>F</i>	<i>g</i>	<i>LT</i> ⁻²
Coefficient of kinematic viscosity	<i>G</i>	<i>ν</i>	<i>L</i> ² <i>T</i> ⁻¹
Surface tension	<i>H</i>	<i>c</i>	<i>MT</i> ⁻²

From these quantities we set up the following five similarity invariants (Table 7.2):

Table 7.2

Designation of the similarity of	Invariant	Name of invariant
time	<i>lv</i> ⁻¹ <i>t</i> ⁻¹	Strouhal number <i>Sh</i>
pressure forces	<i>pv</i> ⁻² <i>q</i> ⁻¹	Euler number <i>Eu</i>
gravity forces	<i>v</i> ² <i>l</i> ⁻¹ <i>g</i> ⁻¹	Froude number <i>Fr</i>
viscosity forces	<i>vlv</i> ⁻¹	Reynolds number <i>Re</i>
surface tension forces	<i>v</i> ² <i>lqc</i> ⁻¹	Weber number <i>We</i>

Simultaneous satisfaction of all the basic relations stated above is possible in specially chosen cases. In the majority of practical problems we choose those invariants which describe the predominant and determining aspects of the process being studied.

As an example, consider the flow in the post-linear regime, with a fully developed turbulence in the prototype as well as in the model.

In this case we are concerned with the similarity of the gravity forces and hence also with the similarity of the inertia forces, and we ignore the other effects, for example, that of the surface forces.

According to the Froude invariant,

$$\lambda_v = \sqrt{(\lambda_l \lambda_g)}$$

(7.1.15)

where $\lambda_v, \lambda_l, \lambda_g$ are the scales of velocity, length and acceleration of gravity, respectively.

Expressing the force *F* as the product of mass and acceleration, we obtain

$$\lambda_F = \lambda_M \lambda_A = \lambda_l^3 \lambda_q \lambda_A$$

(7.1.16)

where λ_F , λ_M , λ_A are the scales of force, mass and acceleration, respectively, $\lambda_A = \lambda_g$, and λ_ρ is the scale of specific mass.

The force action of the external pressure is expressed by the product of the pressure and the area to which the pressure is applied. Hence we also have

$$\lambda_F = \lambda_p \lambda_l^2 \quad (7.1.17)$$

where λ_p is the scale of pressure.

Comparing eqn. (7.1.17) with (7.1.16) and substituting eqn. (7.1.15), we obtain the equation

$$\lambda_p = \lambda_l \lambda_\rho \lambda_A = \lambda_l \lambda_\rho \lambda_g = \lambda_\rho \lambda_v^2 \quad (7.1.18)$$

Eqn. (7.1.18) corresponds to the Euler invariant according to which

$$\lambda_v = \sqrt{\frac{\lambda_p}{\lambda_\rho}} \quad (7.1.19)$$

In the case being considered, we shall manage with the criterion (7.1.15) to which we add – for the time-dependent processes – the following relation implied by the Strouhal invariant

$$\lambda_t' = \frac{\lambda_l}{\lambda_v} = \frac{\lambda_l}{\sqrt{(\lambda_l \lambda_\rho)}} = \sqrt{\frac{\lambda_l}{\lambda_\rho}} \quad (7.1.20)$$

Consider now the case in which the surface tension forces predominate and the effect of capillarity comes markedly into play in both the prototype and the model.

According to the Weber invariant we have

$$\lambda_v = \sqrt{\frac{\lambda_c}{\lambda_l \lambda_\rho}} \quad (7.1.21)$$

where λ_c is the scale of surface tension.

Substituting eqn. (7.1.21) in the relation implied by the Strouhal invariant we also have

$$\lambda_t = \lambda_l^{3/2} \sqrt{\frac{\lambda_\rho}{\lambda_c}} \quad (7.1.22)$$

where λ_t is the scale of time.

As a comparison of eqn. (7.1.22) with eqn. (7.1.20) suggests, the assessment of similarity in the two cases is diametrically opposite.

Studies of processes on physical models are particularly useful in cases when a mathematical description of the process involved is either not known at all, or is only partly known. A general three-dimensional motion in the post-linear regime when the assumption concerning the collinearity of the velocity vector and the vector of the gradient of the function $h(x, y, z, t)$ cannot be relied on a priori, is a case in point.

An analysis of the mathematical formulation — provided that it is known — enables us to gain a better qualitative and quantitative understanding of the problem we are dealing with. An analysis of the equation of motion which describes a developed turbulent flow through a system of grains can serve as an example. The liquid moving in the prototype as well as in the model is assumed to be water.

Referring to Section 1.5 we know that in the actual flow region

$$v = k_t \sqrt{J} = \delta \sqrt{(gdJ)} \quad (7.1.23)$$

and similarly, on the model

$$v_m = k_{tm} \sqrt{J_m} = \delta_m \sqrt{(g_m d_m J_m)} \quad (7.1.24)$$

where

k_t — coefficient of turbulent seepage,

δ — a dimensionless constant expressing the effect of material porosity and grain shape,

d — grain diameter,

J — dimensionless gradient of the function $h(x, y, z, t)$.

In the spirit of the foregoing discussion, we can write

$$\bar{v}[v] = \delta \sqrt{(\bar{g}[g] \bar{d}[d] J)} \quad (7.1.25)$$

$$\bar{v}_m[v] = \delta_m \sqrt{(\bar{g}_m[g] \bar{d}_m[d] J_m)} \quad (7.1.26)$$

The scales are defined as follows:

$$\lambda_v = \frac{\bar{v}}{\bar{v}_m}, \quad \lambda_g = \frac{\bar{g}}{\bar{g}_m}, \quad \lambda_l = \frac{\bar{d}}{\bar{d}_m}, \quad \lambda_\delta = \frac{\delta}{\delta_m}, \quad \lambda_J = \frac{J}{J_m} \quad (7.1.27)$$

$\lambda_v, \lambda_g, \lambda_l$ are the scales of velocity, acceleration and lengths, respectively,

$\lambda_\delta, \lambda_J$ are the scales of the dimensionless quantities δ, J .

Substituting eqn. (7.1.27) in eqn. (7.1.25), we find that the result is identical with eqn. (7.1.26) so long as it holds that

$$\left[\frac{\lambda_v}{\lambda_\delta \sqrt{(\lambda_g \lambda_l \lambda_J)}} \right] = 1 \quad (7.1.28)$$

The expression in the square brackets is called the unit similarity indicator, or simply the similarity indicator. It determines the rules of the process of re-calculation of the quantities obtained by model measurements to find the actual ones. It can be shown that the rules are not at variance with the similarity invariants. If we are concerned with the relation between the prototype and a geometrically similar model then $\lambda_J = 1$. If the composition grain sizes and shapes of the model soil are consistent with the assumptions concerning the geometrical similarity, then $\lambda_\delta = 1$, and we

obtain, in place of eqn. (7.1.28) the relation

$$\left[\frac{\lambda_v}{\sqrt{(\lambda_g \lambda_l)}} \right] = 1 \quad (7.1.29)$$

The indicator (7.1.29) corresponds directly to the Froude invariant; the indicator (7.1.28), on the other hand, is more general. It describes, for example, the effect of the difference between the grains of the model and of the actual soil. It can also be obtained via a dimensional analysis but the procedure is somewhat more complicated.

In other cases, the advantage of an analysis of the mathematical formulations lies in the fact that such an analysis obliterates (or conceals) the effect of a number of additional phenomena which are likely to complicate the dimensional analysis.

Consider, for example, the flow of water with a free surface. In a saturated medium below the surface, the motion obeys Darcy's law whose formulation is based on the principle of continualization and pays no special regard to the structure of the porous medium. In Section 1.4 we obtained the following equation for the free surface:

$$-\frac{\mu}{k} \frac{\partial h}{\partial t} + \left(\frac{\partial h}{\partial x} \right)^2 + \left(\frac{\partial h}{\partial y} \right)^2 + \left(\frac{\partial h}{\partial z} \right)^2 - \frac{\partial h}{\partial z} + \frac{v_0}{k} \frac{\partial h}{\partial z} - \frac{v_0}{k} = 0 \quad (7.1.30)$$

where

μ — active porosity,

k — coefficient of permeability,

v_0 — velocity of water discharge into the free surface.

Below the free surface the situation is adequately described by the Laplace equation

$$\frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2} + \frac{\partial^2 h}{\partial z^2} = 0 \quad (7.1.31)$$

Suppose that eqns. (7.1.30) and (7.1.31) describe certain actual conditions which are being studied with the help of a model. The process which is taking place in the model is described by the equations

$$-\frac{\mu_m}{k_m} \frac{\partial h_m}{\partial t_m} + \left(\frac{\partial h_m}{\partial x_m} \right)^2 + \left(\frac{\partial h_m}{\partial y_m} \right)^2 + \left(\frac{\partial h_m}{\partial z_m} \right)^2 - \frac{\partial h_m}{\partial z_m} + \frac{v_{0m}}{k_m} \frac{\partial h_m}{\partial z_m} - \frac{v_{0m}}{k_m} = 0 \quad (7.1.32)$$

$$\frac{\partial^2 h_m}{\partial x_m^2} + \frac{\partial^2 h_m}{\partial y_m^2} + \frac{\partial^2 h_m}{\partial z_m^2} = 0 \quad (7.1.33)$$

Choose the scales of lengths, active porosity, coefficient of permeability and discharge velocity, respectively, as follows:

$$\lambda_l = \frac{\bar{x}}{\bar{x}_m} = \frac{\bar{y}}{\bar{y}_m} = \frac{\bar{z}}{\bar{z}_m} = \frac{\bar{h}}{\bar{h}_m}, \quad \lambda_\mu = \frac{\mu}{\mu_m}, \quad \lambda_k = \frac{\bar{k}}{\bar{k}_m}, \quad \lambda_v = \frac{\bar{v}_0}{\bar{v}_{0m}} \quad (7.1.34)$$

On substituting the scales in eqns. (7.1.30) and (7.1.31) we find that the resulting equations are identical with eqns. (7.1.32) and (7.1.33), provided that

$$\left[\frac{\lambda_\mu \lambda_l}{\lambda_k \lambda_t} \right] = 1 \quad (7.1.35)$$

$$\left[\frac{\lambda_v}{\lambda_k} \right] = 1 \quad (7.1.36)$$

The indicators (7.1.35) and (7.1.36) enable us to compare qualitatively the process which is taking place in the prototype with that in the model. A quantitative comparison is not possible because the initial equations fail to provide a unique definition of the process. Additional information which is needed for a quantitative appraisal, is obtained by an analysis of the boundary and initial conditions which endow the basic partial differential equations with a uniqueness property.

At first sight, the indicators (7.1.35) and (7.1.36) do not seem to follow from the similarity invariants. They do not contain, for example, the Reynolds invariant which prescribes identical Reynolds numbers in the prototype and on the model. The indicators encompass an extensive domain of processes whose physical nature a priori assumes the existence of a linear law of flow. We speak of a self-similarity domain and require that the similarity of boundary conditions should be maintained within its extent.

The definition of the extent and volume of the self-similarity domain is of considerable theoretical significance. Obviously, the proposed principles also apply to the mathematical description, i.e. to the initial partial differential equations (disregarding the actual reality). In an analysis of these equations we can completely omit the concepts of prototype and model which are nothing more than an aid for theoretical considerations.

7.2. Boussinesq's equation and its linearization

In our explanation of the application of auto-modelling we turn to eqn. (1.4.66) which, in its somewhat simplified form,

$$\frac{\partial h}{\partial t} = \frac{k}{\mu} \left[\frac{\partial}{\partial x} \left(H \frac{\partial h}{\partial x} \right) + \frac{\partial}{\partial y} \left(H \frac{\partial h}{\partial y} \right) \right] + \frac{v_0}{\mu} \quad (7.2.1)$$

is known as Boussinesq's equation. It describes the plane flow which is shown schematically in Fig. 7.1a. The letter H denotes the thickness of the saturated medium, which is a function of both position and time. The magnitude of H is co-determined by the magnitude of the dependent variable, the piezometric head h . Hence the partial differential equation is non-linear.

We can assume that eqn. (7.2.1) has two solutions of two similar cases. In the interest of clarity, we denote by the subscript m the quantities which will be used in the second solution. The scales are defined as follows:

$$\begin{aligned}\lambda_h &= \frac{h}{h_m}, \quad \lambda_t = \frac{t}{t_m}, \quad \lambda_\mu = \frac{\mu}{\mu_m} \\ \lambda_H &= \frac{H}{H_m}, \quad \lambda_l = \frac{x}{x_m} = \frac{y}{y_m}, \quad \lambda_v = \frac{v_0}{v_m}\end{aligned}\quad (7.2.2)$$

$\lambda_h, \lambda_t, \lambda_\mu, \lambda_H, \lambda_l, \lambda_v$ — scales of piezometric heads, time, active porosity, thickness of saturated permeable region, horizontal lengths and velocities, respectively.

Applying the procedure outlined in the previous section, we obtain the similarity indicators

$$\left[\frac{\lambda_\mu \lambda_l^2}{\lambda_H \lambda_k \lambda_t} \right] = 1 \quad (7.2.3)$$

$$\left[\frac{\lambda_v \lambda_l^2}{\lambda_k \lambda_H \lambda_h} \right] = 1 \quad (7.2.4)$$

The indicators (7.2.3) and (7.2.4) are somewhat more complicated than the indicators (7.1.35) and (7.1.36). Their applicability is confined to cases which can be described by means of the theory of plane seepage (assuming a linear law of flow in the saturated medium).

The following dimensionless similarity invariants can be set up, by a reverse procedure, from the indicators (7.2.3) and (7.2.4):

$$\pi_1 = \left\{ \frac{\mu l^2}{H k t} \right\} \quad \text{or} \quad \pi_1 = \left\{ \frac{l}{\sqrt{[(kH/\mu) t]}} \right\} \quad (7.2.5)$$

$$\pi_2 = \left\{ \frac{v_0 l^2}{k H h} \right\} \quad (7.2.6)$$

We can assume that the solution to the partial differential equation has the form of a functional dependence of the dimensionless similarity invariants. The necessary operations are made extremely difficult by the non-linearity of Boussinesq's equation. The determination of the function of the dimensionless arguments becomes incomparably easier if eqn. (7.2.1) is linearized. One of the possible ways to achieve this is to replace the function $H(x, y, t)$ by a mean value H_s regarded as constant throughout the entire flow region. To simplify, assume that $v_0 = 0$. Then

$$\frac{k H_s}{\mu} = a = \text{const.} \quad (7.2.7)$$

so that

$$\frac{\partial h}{\partial t} = a \left(\frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2} \right) \quad (7.2.8)$$

The similarity of the second process which is assumed to be the image of the first one, is ensured under the condition that

$$\left[\frac{\lambda_l^2}{\lambda_a \lambda_r} \right] = 1 \quad (7.2.9)$$

with

$$a = \lambda_a a_m \quad (7.2.10)$$

The similarity of the two processes is also ensured if the functions of the dimensionless parameter

$$\left\{ \frac{l}{\sqrt{at}} \right\} \quad (7.2.11)$$

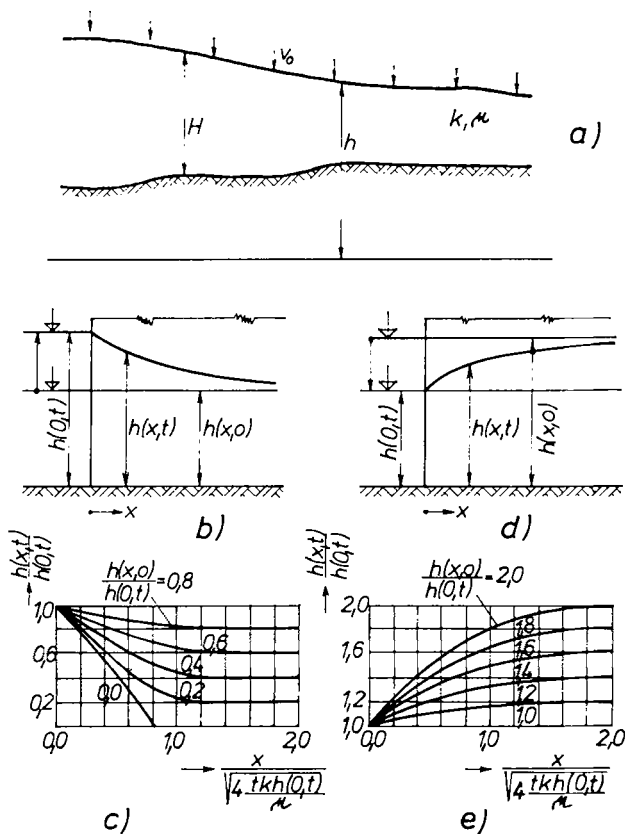


Fig. 7.1. Solution of one-dimensional unsteady flow by means of the method of small parameter.

are the same in both cases. The required functional dependence is obtained numerically, experimentally or analytically.

The solution will become unique after the introduction of the boundary and initial conditions. Suppose, for example, that we are studying a one-dimensional flow in the region $(0 \leq x \leq \infty)$ on the assumption that at the time $t = 0$, h was equal to h_0 throughout the entire region. At the time $t = 0$, a sudden rise or fall of the surface occurred, i.e. a sudden increase or decrease of h from the value h_0 to the value $h(0, t) = \text{const}$. After the change the water surface on the boundary remains in its position for an infinitely long time.

The above essentially represents two auto-modelling problems. One refers to all cases involving a rise of the water surface on the boundary of a geometrically delineated region of groundwater flow (assuming a linear law of plane flow through a saturated soil). The other refers to all cases involving a fall of the surface.

From the point of view of the definition of the boundary conditions, linearized equations are simpler because they do not a priori assume an exact knowledge of the shape and surface of the relatively impervious subsoil (seeing that the free surface is experiencing variations). Boussinesq's equation does not possess this advantage but offers a more complete physical description.

It can be shown that a solution of a linearized equation represents the first approximation to the solution of the non-linear equation. Consider the one-dimensional flow referred to above, and choose the assumption of the coincidence of the surface of the relatively impervious subsoil and the reference plane, i.e. $h = H$, as a supplementary condition. Moreover, let $v_0 = 0$. Under these conditions we obtain, in place of eqn (7.2.1), the equation

$$\frac{\partial h}{\partial t} = \frac{k}{\mu} \frac{\partial}{\partial x} \left(h \frac{\partial h}{\partial x} \right) \quad (7.2.12)$$

According to Polubarinova-Kochina the solution of eqn. (7.2.12) can be taken to be in the form of a series containing a small parameter κ , viz.

$$h(x, t) = h_0 + \kappa h_1(x, t) + \kappa^2 h_2(x, t) + \dots \quad (7.2.13)$$

where h_0 is a certain constant value.

It clearly holds that

$$\begin{aligned} \frac{\partial h}{\partial t} &= \kappa \frac{\partial h_1}{\partial t} + \kappa^2 \frac{\partial h_2}{\partial t} + \dots, \quad \frac{\partial h}{\partial x} = \kappa \frac{\partial h_1}{\partial x} + \kappa^2 \frac{\partial h_2}{\partial x} + \dots \\ \frac{\partial}{\partial x} \left(h \frac{\partial h}{\partial x} \right) &= \frac{\partial}{\partial x} \left[(h_0 + \kappa h_1 + \kappa^2 h_2 + \dots) \left(\kappa \frac{\partial h_1}{\partial x} + \kappa^2 \frac{\partial h_2}{\partial x} + \dots \right) \right] = \\ &= \frac{\partial}{\partial x} \left[\kappa h_0 \frac{\partial h_1}{\partial x} + \kappa^2 \left(h_0 \frac{\partial h_2}{\partial x} + h_1 \frac{\partial h_1}{\partial x} \right) + \dots \right] \end{aligned}$$

Eqn. (7.2.12) can be replaced by a system of partial differential equations which are obtained by comparing the terms containing like powers of the parameter κ :

$$\frac{\partial h_1}{\partial t} = \frac{kh_0}{\mu} \frac{\partial^2 h_1}{\partial x^2} = a \frac{\partial^2 h_1}{\partial x^2} \quad (7.2.14)$$

$$\frac{\partial h_2}{\partial t} = \frac{kh_0}{\mu} \frac{\partial^2 h_2}{\partial x^2} + \frac{k}{\mu} \frac{\partial}{\partial x} \left(h_1 \frac{\partial h_1}{\partial x} \right) = a \frac{\partial^2 h_2}{\partial x^2} + \frac{k}{\mu} \frac{\partial}{\partial x} \left(h_1 \frac{\partial h_1}{\partial x} \right) \quad (7.2.15)$$

It is obvious from the above that the non-linear equation (7.2.12) can be replaced by a system of linear equations. The non-linear term on the right-hand side of eqn. (7.2.15) can be disposed of by continuing the operation as indicated.

Polubarinova-Kochina presents the results of the solution of the problem being considered in the form of a function of the parameter

$$\eta = \frac{x}{\sqrt{\frac{4kt}{\mu} h(0, t)}} \quad (7.2.16)$$

which corresponds to the second of the invariants (7.2.5).

For the two self-similarity domains the author shows diagrams representing the results obtained by calculation of eqn. (7.2.12). The diagram shown in Fig. 7.1c relates to the case of a rise of the surface on the boundary (Fig. 7.1b), that shown in Fig. 7.1e to the case of a fall of the surface on the boundary (Fig. 7.1d).

Concerning the solution of the linear equation (7.2.14) taken separately: we assume it to be in the form of a function of the argument

$$\xi = \frac{x}{\sqrt{(4at)}} \quad (7.2.17)$$

which corresponds to the invariant (7.2.11), and obtain

$$h_1(x, t) = - \frac{x}{2t^{3/2} \sqrt{(4a)}} \frac{dh_1}{d\xi} = - \frac{\xi}{2t} \frac{dh_1}{d\xi} \quad (7.2.18)$$

$$\frac{\partial h_1}{\partial x^2} = \frac{1}{4at} \frac{d^2 h_1}{d\xi^2} \quad (7.2.18)$$

so that

$$\frac{d^2 h_1}{d\xi^2} + 2\xi \frac{dh_1}{d\xi} = 0 \quad (7.2.19)$$

A general solution of the ordinary differential equation (7.2.19) is of the form

$$h_1(\xi) = A + B_1 \int_0^\xi e^{-\xi^2} d\xi$$

Substituting $\xi = x/\sqrt{(4at)}$ for the upper limit we obtain

$$h_1(x, t) = A + B_1 \int_0^{x/\sqrt{(4at)}} e^{-\xi^2} d\xi \quad (7.2.20)$$

where A, B_1 are constants.

The second term on the right-hand side of eqn. (7.2.20) resembles the Gauss integral which is tabulated in almost every publication dealing with special functions. Hence, using a form more convenient for practical calculations, we can write

$$h_1(x, t) = A + B \frac{2}{\sqrt{\pi}} \int_0^{x/\sqrt{(4at)}} e^{-\xi^2} d\xi = A + B \operatorname{Erf} \left(\frac{x}{\sqrt{(4at)}} \right) \quad (7.2.21)$$

Since eqn. (7.2.21) is to approximate the exact solution to a sufficient degree of accuracy, we assume that $h_1 = h$, $h_1(x, 0) = h(x, 0)$, $h_1(0, t) = h(0, t)$. The initial and boundary conditions thus defined imply that

$$h(x, 0) = A + B$$

$$h(0, t) = A$$

and therefore

$$B = h(x, 0) - h(0, t)$$

$$h(x, t) = h(0, t) + [h(x, 0) - h(0, t)] \operatorname{Erf} \left(\frac{x}{\sqrt{(4at)}} \right) \quad (7.2.22)$$

Formally, the solution (7.2.22) corresponds to the case of a sudden rise, as well as to that of a sudden fall, of the surface on the boundary. Its practical applicability, however, depends on the choice of $H_s = h_s$ in eqn. (7.2.7). If we choose $h_s = h(x, 0)$, a single self-similarity domain (7.2.22) will be sufficient for our purpose. This fact can best be shown to be true by using the substitution

$$h_r(x, t) = \frac{h(0, t) - h(x, t)}{h(0, t) - h(x, 0)} \quad (7.2.23)$$

which changes the solution (7.2.22) to

$$h_r(x, t) = \operatorname{Erf} \left(\frac{x}{\sqrt{(4at)}} \right) \quad (7.2.24)$$

If we choose the same a for the case of a sudden rise as for that of a sudden fall of the surface on the boundary, the numerical value of eqn. (7.2.24) becomes independent of which case is being studied. This is, of course, at variance with the results of the more exact solution shown in Figs. 7.1c, e.

A closer approximation is obtained by choosing

$$h_s = h(x, 0) + 9[h(0, t) - h(x, 0)]$$

$$9 = \frac{1}{2} \sim \frac{1}{3} \quad (7.2.25)$$

In this case, the value of a is naturally different for each of the examined alternatives. The results closely approach the values obtained by the use of the descriptive initial equation.

Eqn. (7.2.12) can be linearized in another way. Consider, for example, the function (Shestakov [38])

$$u = 2 \int_0^x h \frac{\partial h}{\partial x} dx \quad (7.2.26)$$

and carry out the integration on the assumption that the thickness of the saturated medium experiences only small variations, i.e. that

$$u = \int_0^x \frac{\partial}{\partial x} \left(\frac{h^2}{2} \right) dx = h^2 \quad (7.2.27)$$

From there we obtain

$$\frac{\partial u}{\partial t} = 2h \frac{\partial h}{\partial t} \quad (7.2.28)$$

Differentiating the integral with respect to the limits gives

$$\frac{\partial^2 u}{\partial x^2} = 2 \frac{\partial}{\partial x} \left(h \frac{\partial h}{\partial x} \right) \quad (7.2.29)$$

We can write, in place of eqn. (7.2.12), the equation

$$\frac{\partial u}{\partial t} = \frac{kh}{\mu} \frac{\partial^2 u}{\partial x^2} \quad (7.2.30)$$

and linearize the latter by setting h equal to some mean value h_s . For this value,

$$a = \frac{kh_s}{\mu} \quad (7.2.31)$$

and the solution is again a function of the parameter ξ defined by eqn. (7.2.17). For $u(x, 0) = \text{const.}$ and $u(0, t) = \text{const.}$, we have

$$u(x, t) = u(0, t) + [u(x, 0) - u(0, t)] \text{Erf} \left[\frac{x}{\sqrt{(4at)}} \right] \quad (7.2.32)$$

Since eqn. (7.2.27) applies, we obtain

$$h(x, t) = \sqrt{\left\{ h^2(0, t) + [h^2(x, 0) - h^2(0, t)] \text{Erf} \left[\frac{x}{\sqrt{(4at)}} \right] \right\}} \quad (7.2.33)$$

The form of eqn. (7.2.33) is different from that of eqn. (7.2.22). A comparison with the exact solution will reveal which of the two equations is more suitable. Alternatively, this can be decided on the basis of a simple logical deduction based

on the easiest of the particular solutions to the initial equation (7.2.14) for $h = h_1$, and of eqn. (7.2.30) for $u = h^2$. In the former case the satisfactory value is

$$h(x) = A + Bx \quad (7.2.34)$$

in the latter

$$h(x) = \sqrt{(A + Bx)} \quad (7.2.35)$$

where A, B are constants.

Eqns. (7.2.34) and (7.2.35) describe a special case of nonstationarity, i.e. the steady state. The quadratic form of eqn. (7.2.35) corresponds better to the conclusions which were drawn in Chapter 2 in the analysis of one-dimensional steady flow. It seems, therefore, that the second method of linearization is more appropriate. In many cases the uncertainty as to which of the methods should be chosen, is removed by basing the theory of plane seepage on the concept of the G potential.

7.3. Unsteady flow described by means of the G potential theory

The theory of the G potential is used to describe plane flow in layered soils. Assuming the absence of zones with markedly different seepage regimes, we can use eqn. (1.4.67) which, after substituting eqn. (1.4.84), gives

$$\frac{\partial h}{\partial t} = \frac{1}{\mu_{n+1}} \left(-\frac{\partial^2 G}{\partial x^2} - \frac{\partial^2 G}{\partial y^2} + v_0 \right) \quad (7.3.1)$$

where μ_{n+1} is the active porosity in the $(n + 1)$ th layer which contains the free surface. Since eqn. (1.4.83) also applies, we have

$$\frac{\partial G}{\partial t} = -\frac{\partial h}{\partial t} \left[k_{n+1}h + \sum_{i=1}^n T_i(k_i - k_{i+1}) \right] \quad (7.3.2)$$

and on substituting the above in eqn. (7.3.1),

$$\frac{\partial G}{\partial t} = \frac{k_{n+1}h + \sum_{i=1}^n T_i(k_i - k_{i+1})}{\mu_{n+1}} \left(\frac{\partial^2 G}{\partial x^2} + \frac{\partial^2 G}{\partial y^2} - v_0 \right) \quad (7.3.3)$$

Eqn. (7.3.3) is non-linear but we are frequently satisfied with the solution of a simpler relation obtained by linearization. In the process, the value of h is replaced by a suitably chosen mean value, h_s . This leads to the equations

$$\frac{\partial G}{\partial t} = a \left(\frac{\partial^2 G}{\partial x^2} + \frac{\partial^2 G}{\partial y^2} - v_0 \right) \quad (7.3.4)$$

$$a = \frac{k_{n+1}h_s + \sum_{i=1}^n T_i(k_i - k_{i+1})}{\mu_{n+1}} \quad (7.3.5)$$

The error caused by linearization is not too large, provided that

$$\sum_{i=1}^n T_i(k_i - k_{i+1}) \gg k_{n+1}h$$

The effect of infiltration into the free surface is neglected in many cases of engineering practice.

On setting $v_0 = 0$ in eqn. (7.3.4) we obtain

$$\frac{\partial G}{\partial t} = a \left(\frac{\partial^2 G}{\partial x^2} + \frac{\partial^2 G}{\partial y^2} \right) \quad (7.3.6)$$

7.4. Application of the G potential to the solution of one-dimensional problems

The initial equation of one-dimensional flow follows directly from eqn. (7.3.6):

$$\frac{\partial G}{\partial t} = a \frac{\partial^2 G}{\partial x^2} \quad (7.4.1)$$

Of the various methods which the theory of differential equations offers for the solution of eqn. (7.4.1), we can use that of substitution, i.e. assume that the solution is the product of two factors, viz.

$$G(x, t) = U(x) V(t)$$

each a function of one variable. After substituting eqn. (7.4.1) in the above we obtain

$$\frac{1}{U} \frac{d^2 U}{dx^2} = \frac{1}{aV} \frac{dV}{dt} \quad (7.4.3)$$

Each side of the ordinary differential equation (7.4.3) is a function of one variable and equal to a constant. Let us write, for example,

$$\frac{1}{U} \frac{d^2 U}{dx^2} = -\delta^2, \quad \frac{1}{aV} \frac{dV}{dt} = -\delta^2 \quad (7.4.4)$$

From the integrals of eqn. (7.4.4)

$$U = C \cos \delta x + D \sin \delta x$$

$$V = \text{Exp}(-a\delta^2 t)$$

where C, D are constants, we set up a particular solution of eqn. (7.4.2) which we write in the form

$$G(x, t) = (C \cos \delta x + D \sin \delta x) \text{Exp}(-a\delta^2 t) \quad (7.4.5)$$

In Section 7.2 we referred to another solution of the Fourier equation, in the form of the function (7.2.32). In the present notation this solution takes the form

$$G(x, t) = A + B \operatorname{Erf} \frac{x}{\sqrt{(4at)}} \quad (7.4.6)$$

Differentiating eqn. (7.4.6) with respect to x yields another particular solution, viz.

$$G(x, t) = \frac{c}{\sqrt{(4at)}} \operatorname{Exp} \left(-\frac{x^2}{4at} \right) \quad (7.4.7)$$

where c is a constant.

By mentioning the various alternatives and their combinations, we intend to convey the idea that the solution of practical problems is essentially a matter of a judicious selection of the applicable form of the particular integral, i.e. of a form which corresponds to the physical nature of the problem being studied while permitting easy introduction of the specified boundary and initial conditions. We shall devote the next few subsections to examples of practical applications of the proposed method.

7.4.1. A sudden change of the water surface on the boundary of a very wide earth massif

The problem is essentially the same as that discussed in the previous section except that in the range $0 \leq x \leq \infty$ we now consider, instead of the levels of the surface $h(x, t)$, the values of $G(x, t)$ corresponding to them.

For the case of a sudden rise of the surface on the boundary (Fig. 7.1b) we have

$$G(x, 0) = \text{const.}, \quad G(0, t) = \text{const.} \quad (7.4.8)$$

For $x = 0$, the particular solution (7.4.6) gives $G(0, t) = A$; for $t = 0$, $G(x, 0) = G(0, t) + B$. Hence

$$\begin{aligned} G(x, t) &= G(0, t) + [G(x, 0) - G(0, t)] \operatorname{Erf} \left[\frac{x}{\sqrt{(4at)}} \right] = \\ &= G(x, 0) + [G(0, t) - G(x, 0)] \operatorname{Erfc} \left[\frac{x}{\sqrt{(4at)}} \right] \end{aligned} \quad (7.4.9)$$

The volume flow q at the point x is obtained from the formula

$$q = \frac{\partial G}{\partial x} = -\frac{G(0, t) - G(x, 0)}{\sqrt{(\pi at)}} \operatorname{Exp} \left(-\frac{x^2}{4at} \right) \quad (7.4.10)$$

In the inlet cross-section, this clearly becomes

$$q = -\frac{G(0, t) - G(x, 0)}{\sqrt{(\pi at)}} \quad (7.4.11)$$

Under the conditions (7.4.8), the equation applicable to a sudden fall of the surface on the boundary is

$$G(x, t) = G(0, t) + [G(x, 0) - G(0, t)] \operatorname{Erf} \left[\frac{x}{\sqrt{4at}} \right] \quad (7.4.12)$$

Just as in the former case, the volume flow q at the point x is

$$q = \frac{\partial G}{\partial x} = \frac{G(x, 0) - G(0, t)}{\sqrt{(\pi at)}} \operatorname{Exp} \left(-\frac{x^2}{4at} \right) \quad (7.4.13)$$

and in the inlet cross-section, where $x = 0$, it is

$$q = \frac{G(x, 0) - G(0, t)}{\sqrt{(\pi at)}} \quad (7.4.14)$$

7.4.2. A sudden change of the water surface on the boundary of an earth massif of finite length

Fig. 7.2a shows the scheme of an earth massif of width L bounded by two reservoirs. At the beginning of the unsteady process the surfaces in the two reservoirs were at the same level. Since the free surface in the soil was horizontal, the coordinates of its points were $h(x, 0) = \text{const}$. At the time $t = 0$, the surface in the left-hand reservoir experienced a sudden rise to the level $h(0, t) = \text{const}$. The unsteady process which develops inside the massif as a result, is described by eqn. (7.4.1).

If we suppose the solution to be of the form

$$G(x, t) = \Delta G(x, t) + G(x, 0) \quad (7.4.15)$$

$$\Delta G(x, t) = G(x, t) - G(x, 0) \quad (7.4.16)$$

we obtain, after substituting in eqn. (7.4.1),

$$\frac{\partial(\Delta G)}{\partial t} = a \frac{\partial^2(\Delta G)}{\partial x^2} \quad (7.4.17)$$

To the coordinates of the surface, $h(x, t)$, correspond certain values of $G(x, t)$ as well as certain values of $\Delta G(x, t)$. The boundary and the initial conditions are then defined as follows:

$$\begin{aligned} \Delta G(x, 0) &= 0 \\ \Delta G(0, t) &= \text{const.} \\ \Delta G(L, t) &= 0 \end{aligned} \quad (7.4.18)$$

In the solution proper we make use of the physical notions concerning the course of the process. We start from the particular integral (7.4.5), which we write in the form

$$(C_1 \cos \delta x + D \sin \delta x) \text{Exp}(-a\delta^2 t) \quad (7.4.19)$$

We also know that the solution will be a function of two dimensionless arguments, one implied by the requirement of geometrical similarity of the flow regions, the other by the requirement of similarity between the mathematical descriptions

$$\left\{ \frac{x}{L} \right\}, \quad \left\{ \frac{at}{L^2} \right\} \quad (7.4.20)$$

Furthermore, at the point $x = L$, we must clearly have $\Delta G(L, t) = 0$ at any time. This condition can be satisfied by choosing $\delta = n\pi/L$, $C_1 = 0$. From the physical idea which takes the above facts into account, follows the form of the particular integral

$$\sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{L} \text{Exp} \left[- \left(\frac{n\pi}{L} \right)^2 at \right] \quad (7.4.21)$$

We note, however, that for $t = \infty$, eqn. (7.4.21) yields zero, and this disagrees with the specification of the problem. In the steady state, the function $G(x)$ should have the form of a straight line. Therefore, we combine eqn. (7.4.21) linearly with the particular integral $\Delta G = A + Bx$ and obtain the new particular integral

$$\Delta G(x, t) = C \left(1 - \frac{x}{L} \right) + \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{L} \text{Exp} \left[- \left(\frac{n\pi}{L} \right)^2 at \right] \quad (7.4.22)$$

The values of the constants C , B_n are obtained from the boundary and the initial conditions. At the point $x = 0$, $\Delta G = \Delta G(0, t)$ and hence

$$C = \Delta G(0, t) \quad (7.4.23)$$

At the same time, $\Delta G(x, 0) = 0$ at $t = 0$. According to eqn. (7.4.8)

$$-C \left(1 - \frac{x}{L} \right) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{L} \quad (7.4.24)$$

The expression on the right-hand side of eqn. (7.4.24) is a Fourier series. On the left-hand side, we obtain $-C$ for $x = 0$, and zero for $x = L$.

Hence the constant of the series

$$B_n = - \frac{2C}{n\pi} \quad (7.4.25)$$

The determination of this constant completes the solution and we can write the equa-

tion in its final form:

$$\Delta G(x, t) = \Delta G(0, t) \left\{ 1 - \frac{x}{L} - \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \sin \frac{n\pi x}{L} \exp \left[- \left(\frac{n\pi}{L} \right)^2 at \right] \right\} \quad (7.4.26)$$

The function (7.4.19) enables us to calculate the volume flow q at any point x . The appropriate equation is

$$q = \frac{\partial G}{\partial x} = \frac{\partial(\Delta G)}{\partial x} = - \frac{\Delta G(0, t)}{L} \left\{ 1 + 2 \sum_{n=1}^{\infty} \cos \frac{n\pi x}{L} \exp \left[- \left(\frac{n\pi}{L} \right)^2 at \right] \right\} \quad (7.4.27)$$

In practical calculations we are usually satisfied with the first few terms of the series (7.4.26) and (7.4.27).

7.4.3. Emptying of an infiltration field after the interruption of seepage with backwater

We use this example to show how the Fourier equation of unsteady groundwater flow is solved under non-zero initial conditions. The scheme in Fig. 7.2c represents a section of one of several identical fields. Both infiltration and collection take place through a narrow, hydraulically perfect cut. Before the beginning of infiltration, the level was at the height $H = \text{const.}$ above the surface of a relatively impervious subsoil. To this height corresponds a certain value $G = \text{const.}$ The steady state which the system had reached after a prolonged period of operation of the infiltration, existed until the instant $t = 0$ when both the supply and removal of water from the cuts were interrupted. Assume that the water surface corresponding to the period of operation is defined by the function $G(x, 0)$, and that it was formed symmetrical about the original level of G . The axes of the infiltration and collection cuts are also the axes of symmetry of the flow in the entire system.

The values of $G(x, t)$ which are characteristic of the levels of the surfaces after the interruption of the operation, are determined from the relation

$$G(x, t) = G + \Delta G(x, t), \quad \Delta G(x, t) = G(x, t) - G \quad (7.4.28)$$

Eqn. (7.4.17) is solved under the conditions that

$$G(x, 0) = \Delta G(0, 0) \left(1 - \frac{2x}{L} \right) \quad (7.4.29)$$

$$\left. \frac{\partial(\Delta G)}{\partial x} \right|_{x=0} = 0, \quad \left. \frac{\partial(\Delta G)}{\partial x} \right|_{x=L} = 0 \quad (7.4.30)$$

Writing the condition (7.4.29) in the form

$$\Delta G(x, 0) = \frac{8 \Delta G(0, 0)}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} \cos \frac{(2n-1)\pi x}{L} \quad (7.4.31)$$

we see that the function (7.4.31) contains cosine terms only.

Choosing the particular integral (7.4.5) for $D = 0$ for the determination of the function $\Delta G(x, t)$, we obtain, after rearrangement,

$$\Delta G(x, t) = \frac{8 \Delta G(0, 0)}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} \cos \frac{(2n-1) \pi x}{L} \text{Exp} \left\{ - \left[\frac{(2n-1) \pi}{L} \right]^2 at \right\} \quad (7.4.32)$$

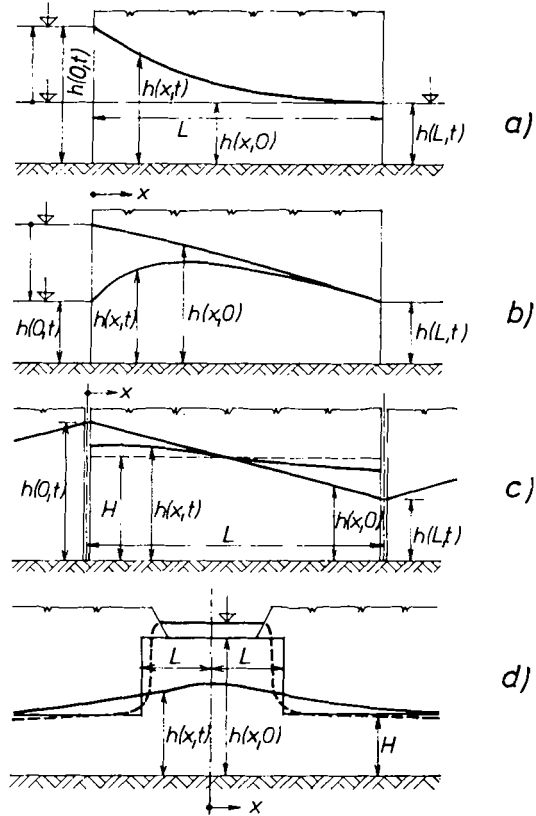


Fig. 7.2. Solution of the motion on a horizontal base.

For $t = 0$, the above function $\Delta G(x, t)$ satisfies the condition (7.4.29). Differentiation with respect to x yields

$$\begin{aligned} \frac{\partial}{\partial x} [\Delta G(x, t)] &= \\ &= - \frac{8 \Delta G(0, 0)}{\pi L} \sum_{n=1}^{\infty} \frac{1}{2n-1} \sin \frac{(2n-1) \pi x}{L} \text{Exp} \left\{ - \left[\frac{(2n-1) \pi}{L} \right]^2 at \right\} \end{aligned} \quad (7.4.33)$$

The derivative will be equal to zero for $x = 0$ as well as for $x = L$. Hence the boundary conditions (7.4.30) are also satisfied.

7.4.4. Flow through an infiltration field after the interruption of seepage without backwater

The solution of the case when the water from the infiltration reservoir seeps into the subsoil without backwater, is somewhat different from the solution just discussed. The dashed line in the scheme in Fig. 7.2d shows the character of the depression curve. At the time $t = 0$, the infiltration is interrupted and for the sake of simplification, the water surface is assumed to become immediately detached from the bottom of the infiltration reservoir, and to take the form of a right-angle polygon. To the value of H at infinity corresponds a certain value of G . The positions of the surfaces are determined with the help of the relation $h = f(G)$. Choosing

$$G(x, t) = G + \Delta G(x, t)$$

$$\Delta G(x, t) = G(x, t) - G$$

and again using eqn. (7.4.17) for the solution, we establish the boundary and the initial conditions as follows:

$$\begin{aligned} \Delta G(x, 0) &= 0 & \text{for } |x| > L \\ \Delta G(x, 0) &= \text{const.} & \text{for } |x| < L \end{aligned} \quad (7.4.34)$$

On the axis of symmetry, i.e. at the point $x = 0$, we have

$$\frac{\partial(\Delta G)}{\partial x} = 0$$

Starting from the solution (7.4.5) and integrating it for positive values of δ between the limits 0 and ∞ , we obtain the following new particular solution:

$$\Delta G(x, t) = \int_0^\infty [C(\delta) \cos \delta x + D(\delta) \sin \delta x] \text{Exp}(-a\delta^2 t) d\delta \quad (7.4.35)$$

For $t = 0$, we must have

$$f(x) = \int_0^\infty [C(\delta) \cos \delta x + D(\delta) \sin \delta x] d\delta \quad (7.4.36)$$

Writing eqn. (7.4.36) in terms of the Fourier integral

$$\begin{aligned} f(x) &= \frac{1}{\pi} \int_0^\infty d\delta \int_{-\infty}^\infty f(\eta) \cos \delta(\eta - x) d\eta = \\ &= \frac{1}{\pi} \int_0^\infty \left[\cos \delta x \int_{-\infty}^\infty f(\eta) \cos \delta \eta d\eta + \sin \delta x \int_{-\infty}^\infty f(\eta) \sin \delta \eta d\eta \right] d\delta \end{aligned} \quad (7.4.37)$$

and comparing eqn. (7.4.36) with eqn. (7.4.37) we obtain

$$C(\delta) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(\eta) \cos \delta \eta \, d\eta$$

$$D(\delta) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(\eta) \sin \delta \eta \, d\eta$$

Then

$$\Delta G(x, t) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(\eta) \left[\int_0^{\infty} \text{Exp}(-a\delta^2 t) \cos \delta(\eta - x) \, d\delta \right] d\eta$$

We can show that the expression in the square brackets of the above integral is equal to

$$\sqrt{\frac{\pi}{4at}} \text{Exp} \left[-\frac{(\eta - x)^2}{4at} \right]$$

On the strength of this fact, we can write the solution in the simplified form

$$\Delta G(x, t) = \frac{1}{\sqrt{(4\pi at)}} \int_{-\infty}^{\infty} f(\eta) \text{Exp} \left[-\frac{(\eta - x)^2}{4at} \right] d\eta \quad (7.4.38)$$

According to the conditions (7.4.34), we have, at the time $t = 0$, the values $\Delta G = 0$ at the points $|x| > L$, and $\Delta G(x, 0) = \text{const.}$ at the points $|x| < L$. It is, therefore, sufficient to write the integral in eqn. (7.4.38) as follows:

$$\Delta G(x, t) = \frac{\Delta G(x, 0)}{\sqrt{(4\pi at)}} \int_{-L}^L \text{Exp} \left[-\frac{(\eta - x)^2}{4at} \right] d\eta \quad (7.4.39)$$

Using the substitution

$$\frac{\eta - x}{\sqrt{(4at)}} = \zeta$$

$$\begin{aligned} \Delta G(x, t) &= \frac{\Delta G(x, 0)}{\sqrt{\pi}} \int_{-(L+x)/\sqrt{(4at)}}^{(L-x)/\sqrt{(4at)}} e^{-\zeta^2} d\zeta = \\ &= \frac{\Delta G(x, 0)}{\sqrt{\pi}} \left\{ \int_0^{(L-x)/\sqrt{(4at)}} e^{-\zeta^2} d\zeta - \int_0^{-(L+x)/\sqrt{(4at)}} e^{-\zeta^2} d\zeta \right\} \end{aligned}$$

we obtain

$$\Delta G(x, t) = \frac{\Delta G(x, 0)}{2} \left\{ \text{Erf} \left[\frac{L-x}{\sqrt{(4at)}} \right] + \text{Erf} \left[\frac{L+x}{\sqrt{(4at)}} \right] \right\} \quad (7.4.40)$$

7.4.5. Problems without the initial condition

The advantages of superposition can be exploited in a number of interesting problems. Assume, for example, that a horizontal initial water surface exists in a very wide earth massif, $0 \leq x \leq \infty$. A sudden change of the surface on the boundary, which occurs at the time t_i , corresponds to the value ΔG_i . After an interval of time, the surface returns to its original position. After superposition, the solution takes the form

$$\Delta G_i[f(x, t - t_i) - f(x, t - t_{i+1})]$$

Assume that a sequence of sudden rises and falls (pulses) developed on the boundary. After superposing all these changes, we have, inside the field

$$\Delta G(x, t) = \sum_{i=1}^n \Delta G_i[f(x, t - t_i) - f(x, t - t_{i+1})]$$

Assume further that the sequence contains equally long pulses of duration Δt ,

$$\Delta G(x, t) = \sum_{i=1}^n \Delta G_i \frac{f(x, t - t_i) - f(x, t - t_{i+1})}{\Delta t} \Delta t$$

Changing to the elementary time intervals we obtain

$$\Delta G(x, t) = \int_0^t [\Delta G(0, \tau) - \Delta G(x, 0)] \frac{\partial}{\partial t} [f(x, t - \tau)] d\tau \quad (7.4.41)$$

Suppose that the envelope of the elementary pulses on the boundary has the form of a sine curve:

$$\Delta G(0, t) = \Delta G_0 \sin \omega t \quad (7.4.42)$$

In the determination of the elementary sudden changes we make use of the particular solution obtained by rearrangement of eqn. (7.4.6), viz.

$$f(x, t - \tau) = \text{Erf} \frac{x}{\sqrt{[4a(t - \tau)]}}$$

Differentiation with respect to t gives

$$\frac{\partial}{\partial t} f(x, t - \tau) = - \frac{x \text{Exp} [-x^2/4a(t - \tau)]}{(t - \tau)^{3/2} \sqrt{(4\pi a)}}$$

By eqn. (7.4.41), we obtain for $\Delta G(x, 0) = 0$

$$\Delta G(x, t) = \frac{x \Delta G_0}{\sqrt{(4\pi a)}} \int_0^t \sin \omega \tau \frac{\text{Exp} [-x^2/4a(t - \tau)]}{(t - \tau)^{3/2}} d\tau \quad (7.4.43)$$

Using the substitution

$$\eta = \frac{x}{\sqrt{[4a(t - \tau)]}}$$

from which it follows that

$$\tau = t - \frac{x^2}{4a\eta^2}, \quad d\tau = \frac{4a(t - \tau)^{3/2}}{x} d\eta$$

in eqn. (7.4.43), we obtain

$$\Delta G(x, t) = \frac{2 \Delta G_0}{\sqrt{\pi}} \int_{x/\sqrt{4at}}^{\infty} \sin \omega \left(t - \frac{x^2}{4a\eta^2} \right) \text{Exp}(-\eta^2) d\eta \quad (7.4.44)$$

The lower limit of the integral describes the effect of the initial stage of fluctuations of the surfaces on the boundary of the earth massif. Assume that the subject of our study is the situation which arises after the periodic fluctuations on the boundary have been going on for a considerable time. In this case, the lower limit of the integral in eqn. (7.4.44) is close to zero:

$$\Delta G(x, t) = \frac{2 \Delta G_0}{\sqrt{\pi}} \int_0^{\infty} \sin \omega \left(t - \frac{x^2}{4a\eta^2} \right) \text{Exp}(-\eta^2) d\eta \quad (7.4.45)$$

On evaluating the integral in eqn. (7.4.45), we obtain the resulting relation

$$\Delta G(x, t) = \Delta G_0 \text{Exp} \left(-x \sqrt{\frac{\omega}{2a}} \right) \sin \left(\omega t - x \sqrt{\frac{\omega}{2a}} \right) \quad (7.4.46)$$

The solution (7.4.46) assumes a substantial weakening of the effect of the initial stage of the process. In the literature, such solutions are called solutions without the initial condition.

7.4.6. Motion of water along an inclined surface of an impervious layer

So far, we have assumed that the surface of the impervious layer which forms the base of the permeable material, is horizontal. Such a configuration very seldom occurs in nature, however, and this is why attempts have been and are still made to obtain an analytical solution of the case when the base is of a curvilinear shape or, at least, resembles an inclined plane.

The existing solutions which are based on linearization, are correct from the mathematical point of view. Their practical inadequacy lies in the fact that we do not know how to choose the quantities h_s or H_s , i.e. the mean values of the piezometric heads. In our opinion, it is sufficient in many cases to use an approximate procedure based on the fundamental equation (1.4.67) which for a one-dimensional flow and

$v_0 = 0$ takes the form

$$\frac{\partial h}{\partial t} = \frac{1}{\mu} \left(-\frac{\partial q}{\partial x} \right) \quad (7.4.47)$$

where q is the quantity of water which flows per unit time through a vertical.

Assume that the base of the permeable medium is of the shape shown in Fig. 7.3a. The coordinates of its points relative to a horizontal reference plane, are

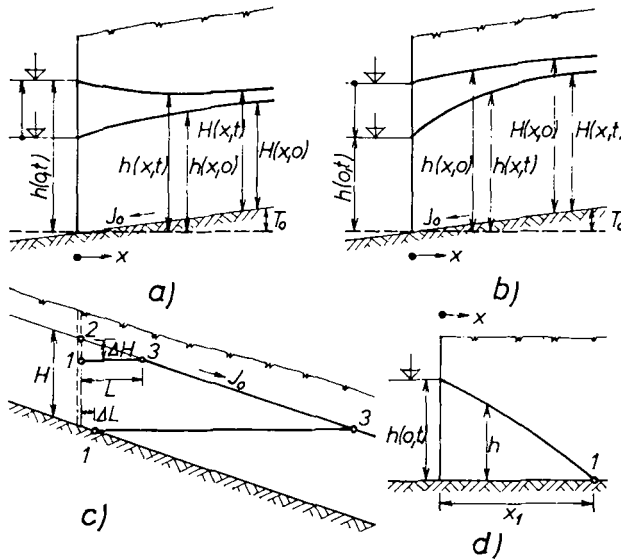


Fig. 7.3. Solution of the motion on an inclined base.

$T_0(x) = T_0$, the thickness of the saturated layer is $H(x, t)$, the coordinates of the points of the free surface are $h(x, t)$, and the initial water surface is characterized by the coordinates $H(x, 0) = H_0$ or $h(x, 0) = h_0$.

It is clear, from a simple consideration of the flow in a homogeneous soil, that

$$q = -kH \frac{\partial h}{\partial x} \quad (7.4.48)$$

$$q_0 = -kH_0 \frac{\partial h_0}{\partial x} \quad (7.4.49)$$

where q_0 is the initial quantity of flow (quantity of water which has seeped through in unit time). Denote the difference of the two quantities by Δq , and write

$$\Delta q = q - q_0 = -k \left(H \frac{\partial h}{\partial x} - H_0 \frac{\partial h_0}{\partial x} \right) \quad (7.4.50)$$

Assume that in the location where the water surface on the boundary has experienced a rise, we have

$$h(0, t) = \text{const.} \quad (7.4.51)$$

Assume also that a change in h produced by the change Δq took place inside the earth massif, and write the equation

$$\frac{\partial h}{\partial t} = \frac{1}{\mu} \left[- \frac{\partial}{\partial x} (\Delta q) \right] = \frac{k}{\mu} \frac{\partial}{\partial x} \left(H \frac{\partial h}{\partial x} - H_0 \frac{\partial h_0}{\partial x} \right) \quad (7.4.52)$$

in place of eqn. (7.4.8).

Consider now the possibility of using the G potential by setting up a different function $G = f(h)$ for each vertical line. By eqn. (1.4.83), we have for $k_{n+1} = k$, $k_n = 0$,

$$G = -k \left(\frac{h^2}{2} - T_0 h + \frac{T_0^2}{2} \right)$$

$$G_0 = -k \left(\frac{h_0^2}{2} - T_0 h_0 + \frac{T_0^2}{2} \right)$$

where G_0 corresponds to the initial water surface.

Taking the difference

$$\Delta G = G - G_0 = -k \left[\frac{h^2 - h_0^2}{2} - T_0(h - h_0) \right] \quad (7.4.53)$$

and differentiating it with respect to x we obtain

$$\begin{aligned} \frac{\partial}{\partial x} (\Delta G) &= -k \left[(h - T_0) \frac{\partial h}{\partial x} - (h_0 - T_0) \frac{\partial h_0}{\partial x} - (h - h_0) \frac{\partial T_0}{\partial x} \right] = \\ &= -k \left[H \frac{\partial h}{\partial x} - H_0 \frac{\partial h_0}{\partial x} - (h - h_0) \frac{\partial T_0}{\partial x} \right] \end{aligned}$$

Suppose now that during the unsteady flow the value of h experiences small variations and the inclination of the base is small. Alternatively assume that $(h - h_0)$ can be replaced by a constant value. According to eqn. (7.4.50), we then have

$$\frac{\partial}{\partial x} (\Delta q) \doteq \frac{\partial^2 (\Delta G)}{\partial x^2} \quad (7.4.54)$$

Differentiating eqn. (7.4.53) with respect to t , we obtain

$$\frac{\partial (\Delta G)}{\partial t} = -k \left[h \frac{\partial h}{\partial t} - T_0 \frac{\partial h}{\partial t} \right] = -k(h - T_0) \frac{\partial h}{\partial t} = -kH \frac{\partial h}{\partial t} \quad (7.4.55)$$

and on substituting eqns. (7.4.55), (7.4.54) in eqn. (7.4.52)

$$\frac{\partial(\Delta G)}{\partial t} = \frac{kH}{\mu} \frac{\partial^2(\Delta G)}{\partial x^2} \quad (7.4.56)$$

Consider also the mean value $H = H_s$, and simultaneously

$$a = \frac{kH_s}{\mu} \quad (7.4.57)$$

Then

$$\frac{\partial(\Delta G)}{\partial t} = a \frac{\partial^2(\Delta G)}{\partial x^2} \quad (7.4.58)$$

On the boundary, we have

$$\Delta G(0, t) = \text{const.} \quad (7.4.59)$$

and at the beginning of the unsteady process

$$\Delta G(x, 0) = 0 \quad (7.4.60)$$

Since we have solved eqn. (7.4.56) under the specified boundary and initial conditions several times, we just write

$$\Delta G(x, t) = \Delta G(0, t) \text{Erfc} \left[\frac{x}{\sqrt{(4at)}} \right] \quad (7.4.61)$$

$$\frac{\partial}{\partial x} [\Delta G(x, t)] = -\Delta G(0, t) \frac{1}{\sqrt{(\pi at)}} \text{Exp} \left(-\frac{x^2}{4at} \right) \quad (7.4.62)$$

We assumed that, approximately,

$$\Delta q \doteq \frac{\partial}{\partial x} [\Delta G(x, t)] \quad (7.4.63)$$

$$q = q_0 + \Delta q \quad (7.4.64)$$

In keeping with eqn. (7.4.53), we have

$$\Delta G = -k(h - h_0) \left(\frac{h + h_0}{2} - T_0 \right) \quad (7.4.65)$$

$$\Delta G(0, t) = -\frac{k}{2} [h^2(0, t) - h_0^2(0, 0)] \quad (7.4.66)$$

$$\Delta G(0, t) = -\frac{k}{2} [H^2(0, t) - H_0^2(0, 0)] \quad (7.4.67)$$

$$h(x, t) = T_0 + \sqrt{\left[(h_0 - T_0)^2 - \frac{2\Delta G}{k} \right]} = T_0 + \sqrt{\left(H_0^2 - \frac{2\Delta G}{k} \right)} \quad (7.4.68)$$

The inclination of the base is denoted by the letter J_0 , as usual, and its sign is chosen to correspond to the sense of the flow. If the base is linear, then

$$T_0 = J_0 x \quad (7.4.69)$$

As the above analysis suggests, the approximate procedure outlined in Section 2.3 can also be applied under the conditions of unsteady flow. Assuming first a horizontal base and determining h for this simplified scheme, we can regard the resulting values as the coordinates H of the points of the free surface and measure them from the actual linear surface of the base.

The approximate method of calculation is not universal, and its application calls for a certain amount of care on the part of the designer. An important point is the requirement that the assumptions on which the theory is based, should be acceptable in every particular case. The following example will show that this is not always the case. Fig. 7.3c shows the scheme of a uniform flow of water along a slope, which was suddenly interrupted by the erection of an imaginary impervious curtain. The discharge into the space underneath the curtain just stopped suddenly (Fig. 7.3c). To simplify, we assume the flow in the lower parts to continue and the wedge 1, 2, 3 to be emptying gradually. From the equation of continuity we have

$$V = kHJ_0 t = \mu L \frac{\Delta H}{2} \quad (7.4.70)$$

where

- k — the coefficient of permeability,
- μ — the active porosity,
- L — the length of the segment $\overline{I3}$,
- J_0 — the inclination of the water surface as well as of the base,
- ΔH — the length of the segment $\overline{I2}$,
- t — the time,
- V — the volume of water which has flown away in time t .

As eqn. (7.4.70) implies, the point 1 moves along the curtain so that for

$$L = \frac{\Delta H}{J_0} \quad (7.4.70)$$

we obtain

$$\Delta H = J_0 \sqrt{\frac{2kHt}{\mu}} \quad (7.4.71)$$

Eqn. (7.4.71) applies until the instant when the point 1 reaches the toe of the curtain.

This happens at the time $t = t_H$,

$$t_H = \frac{\mu H}{2kJ_0^2} \quad (7.4.72)$$

For times $t > t_H$, the equation of continuity gives

$$V = kH(t - t_H)J_0 = \mu(\Delta H - H)(L - \Delta L) \quad (7.4.73)$$

In addition to eqn. (7.4.70), the equation

$$\Delta L = \frac{\Delta H - H}{J_0} \quad (7.4.74)$$

also applies. Hence we obtain from eqn. (7.4.73) that

$$t = \frac{\mu \Delta H}{kJ_0^2} - t_H, \quad t > t_H \quad (7.4.75)$$

What is important is the fact that in the time $t > t_H$, the specified range in which the flow is being studied, experiences a change. The point of contact between the base and the free surface moves down the slope.

A change in the specified range is also examined in the next example. Fig. 7.3d shows the case when a sudden rise of the water surface occurs on the boundary of a dry earth massif underlain by a horizontal impervious base. The free surface which arises in the process touches the base at the point *I*. We assume that the flow can be described using the principles of the theory of plane seepage, i.e. we expect the equipotential lines to be vertical so that for each streamline and its point lying on the free surface, we have

$$v = \mu \frac{dx}{dt} = k \frac{h(0, t) - h(x, t)}{x} \quad (7.4.76)$$

where

v is the fictitious flow velocity,

μ is the active porosity.

On integrating eqn. (7.4.45) we obtain for $h(x, 0) = 0$, $h(0, t) = \text{const.}$ that

$$h(x, t) = h(0, t) - \frac{\mu x^2}{2kt}$$

At the point *I*, $h(x, t) = 0$, so that

$$x_1 = \sqrt{\frac{2kth(0, t)}{\mu}}$$

The free surface is calculated in the range $0 \leq x \leq x_1$.

7.5. Quasi-linear equations for one-dimensional flow

The solutions of seepage into a dry medium revealed a possible way of studying unsteady flows by examining horizontal flows isolated from one another, and adding up the results so as to gain a comprehensive view of the development of the free surface. Although the solution also applies to layered media, we must generally acknowledge that the free surface varies from one layer to another. The interface between the layers is intersected by the surface at transition points whose position generally varies with time. It is assumed that the motion inside the layers can be described by linear equations. On the whole, however, the problem is nonlinear.

Suppose that the permeable medium is divided into a set of infinitely thin layers. In view of this assumption we can use a linear description for the flow in each layer, and on carrying out a synthesis of the individual partial flows, obtain a solution of the non-linear problem discussed in Section 7.2 (application of the method of small parameter).

Quasi-linear equations essentially constitute an intermediate stage between a fully linearized description and a non-linear formulation. We shall demonstrate the application of this technique by solving the case of a stratified earth massif on the boundary of which there occurred a rise of the water surface in such a way as to cause the free surface of groundwater to move from one layer to another.

7.5.1. A quasi-linear description of the effect of a sudden rise of the water surface on the boundary of a very wide earth massif

Fig. 7.4 shows the situation in an earth massif on whose boundary a sudden rise of the surface occurred at the time $t = 0$. The initial water surface was horizontal. Suppose that at the time $t = 0$, the surface inside the massif formed the interface between the zero and the first layers which have identical properties. On the boundary, the surface rose to a certain level. We can assume that this level forms the interface between the n -th and the $(n + 1)$ th layers. Under these conditions, we can define the G potential in the n -th layer as follows:

$$G_n = -h \left[k_n \frac{h}{2} + \sum_{j=0}^{n-1} T_j (k_j - k_{j+1}) \right] + \frac{1}{2} \sum_{j=0}^{n-1} T_j (k_j - k_{j+1}) \quad (7.5.1)$$

On taking the difference between the values of the G potential in the n -th layer and the G potential at the level of the lower edge of the n -th layer, we obtain a new function, the so-called ΔG potential whose value in the n -th layer is

$$\begin{aligned} \Delta G_n &= -h_n \left[k_n \frac{h_n}{2} + \sum_{j=0}^{n-1} T_j (k_j - k_{j+1}) \right] + T_{n-1} \left[k_n \frac{T_{n-1}}{2} + \sum_{j=0}^{n-1} T_j (k_j - k_{j+1}) \right] = \\ &= -k_n (h_n - T_{n-1}) \left[\frac{h_n - T_{n-1}}{2} + \frac{1}{k_n} \sum_{j=-1}^{n-1} k_j (T_j - T_{j-1}) \right] \end{aligned} \quad (7.5.2)$$

Writing

$$\Delta h_n = h_n - T_{n-1} \quad (7.5.3)$$

eqn. (7.5.2), after a simple rearrangement, gives

$$\Delta G_n = -k_n \Delta h_n \left[\frac{\Delta h_n}{2} + \frac{1}{k_n} \sum_{j=-1}^{n-1} k_j (T_j - T_{j-1}) \right] \quad (7.5.4)$$

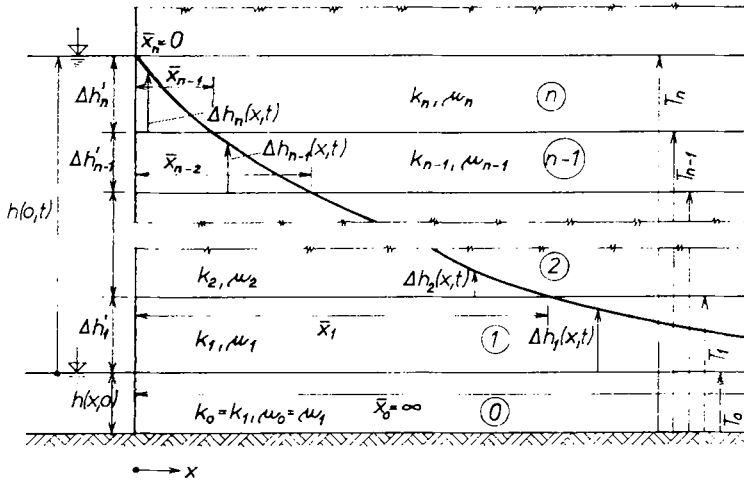


Fig. 7.4. Two-dimensional flow through an n -layer medium at a sudden rise of the surface on the boundary.

In the limiting case, the free surface can touch the upper edge of the n -th layer, and we have

$$\Delta G_n = \Delta G'_n = -k_n (T_n - T_{n-1}) \left[\frac{T_n - T_{n-1}}{2} + \frac{1}{k_n} \sum_{j=-1}^{n-1} k_j (T_j - T_{j-1}) \right] \quad (7.5.5)$$

Denoting the expression in the square brackets of eqn. (7.5.5) by the symbol T_{sn}

$$T_{sn} = \frac{T_n - T_{n-1}}{2} + \frac{1}{k_n} \sum_{j=-1}^{n-1} k_j (T_j - T_{j-1}) \quad (7.5.6)$$

eqn. (7.5.5) becomes

$$\Delta G'_n = -k_n T_{sn} (T_n - T_{n-1}) \quad (7.5.7)$$

It is clear that ΔG_n is a function of Δh_n , and that

$$\Delta h_n = -\frac{1}{k_n} \sum_{j=-1}^{n-1} k_j (T_j - T_{j-1}) \left\{ 1 - \sqrt{1 - \frac{2 \Delta G_n k_n}{\left[\sum_{j=-1}^{n-1} k_j (T_j - T_{j-1}) \right]^2}} \right\} \quad (7.5.8)$$

In the limiting case when the free surface touches the upper edge of the n -th layer we have

$$\Delta h_n = \Delta h'_n = - \frac{\Delta G'_n}{k_n T_{sn}} \quad (7.5.9)$$

It can readily be shown that the derivatives of the function ΔG in the directions of the coordinate axes give the components of the volume flow in the directions of these axes and have, therefore, the properties of the potential. Hence the equation

$$\frac{\partial(\Delta G_n)}{\partial t} = a_n \frac{\partial^2(\Delta G_n)}{\partial x^2} \quad (7.5.10)$$

which follows directly from eqn. (7.4.1) applies to each (generally to the n -th) layer. The value of a_n is, of course, different for each layer since, after linearization,

$$a_n = \frac{k_n(h_n)_s + \sum_{j=0}^{n-1} T_j(k_k - k_{j+1})}{\mu_n} \quad (7.5.11)$$

where $(h_n)_s$ is a certain chosen mean value of h which applies within the range of the n -th layer, and μ_n is the active porosity of the n -th layer. Denoting the various layers by the numbers $N = 1, 2, \dots, n$ we obtain the following system of equations which solve the quasi-linear problem

$$\frac{\partial(\Delta G_N)}{\partial t} = a_N \frac{\partial^2(\Delta G_N)}{\partial x^2} \quad (7.5.12)$$

with

$$0 \leq x \leq \infty$$

in the case being considered.

We shall call the points of intersection of the depression curve and the edges of the layers "the transition points", and denote their abscissae by the symbol $\bar{x}_n$. We then have

$$\bar{x}_N \leq x \leq \bar{x}_{N-1}, \quad \Delta G_N(\bar{x}_N, t) = \Delta G'_N, \quad \Delta G_N(\bar{x}_{N-1}, t) = 0 \quad (7.5.13)$$

and at the same time

$$\bar{x}_n = 0, \quad \bar{x}_0 = \infty \quad (7.5.14)$$

According to the similarity theory, the solution (7.5.12) is a function of the dimensionless argument $x/\sqrt{(4a_N t)}$. It is clear that the transition points move along the interfaces between the layers where the value of ΔG is constant independently of time. Hence the values of the dimensionless arguments for the transition points must also be constants, independent of time:

$$\bar{x}_N = c_N \sqrt{t} \quad (7.5.15)$$

where c_N are certain constants.

The solution which pays heed to the given boundary conditions is written as follows:

$$\Delta G_N(x, t) = \Delta G'_N \frac{\text{Erf} [x/\sqrt{(4a_N t)}] - \text{Erf} [c_{N-1}/\sqrt{(4a_N)}]}{\text{Erf} [c_N/\sqrt{(4a_N)}] - \text{Erf} [c_{N-1}/\sqrt{(4a_N)}]} \quad (7.5.16)$$

In the segment in which the free surface is developing within the range of the N -th layer, the volume flow q_N turns out to be

$$q_N(x, t) = \frac{\partial(\Delta G_N)}{\partial x} = \frac{\Delta G'_N}{\sqrt{(\pi a_N t)}} \frac{\text{Exp}(-x^2/4a_N t)}{\text{Erf} [c_N/\sqrt{(4a_N)}] - \text{Erf} [c_{N-1}/\sqrt{(4a_N)}]} \quad (7.5.17)$$

At the transition points at which the free surface moves along the interfaces and where no storage takes place, we can write the equation of continuity in the form

$$\begin{aligned} & \frac{\Delta G'_N}{\sqrt{(\pi a_N t)}} \frac{\text{Exp}(-c_{N-1}^2/4a_N)}{\text{Erf} [c_N/\sqrt{(4a_N)}] - \text{Erf} [c_{N-1}/\sqrt{(4a_N)}]} = \\ & = \frac{\Delta G'_{N-1}}{\sqrt{(\pi a_{N-1} t)}} \frac{\text{Exp}(-c_{N-1}^2/4a_{N-1})}{\text{Erf} [c_{N-1}/\sqrt{(4a_{N-1})}] - \text{Erf} [c_{N-2}/\sqrt{(4a_{N-1})}]} \end{aligned}$$

and from there obtain

$$\frac{\Delta G'_N}{\Delta G'_{N-1}} = \sqrt{\frac{a_N}{a_{N-1}}} \frac{\text{Exp}(-c_{N-1}^2/4a_{N-1})}{\text{Exp}(-c_{N-1}^2/4a_N)} \frac{\text{Erf} [c_N/\sqrt{(4a_N)}] - \text{Erf} [c_{N-1}/\sqrt{(4a_N)}]}{\text{Erf} [c_{N-1}/\sqrt{(4a_{N-1})}] - \text{Erf} [c_{N-2}/\sqrt{(4a_{N-1})}]} \quad (7.5.18)$$

The expression (7.5.18) is essentially a system of transcendental equations for the unknown constants c_N . The solution becomes simpler in the case where the development of the depression is considered within the range of two layers of soil:

$$\frac{\Delta G'_2}{\Delta G'_1} = \sqrt{\frac{a_2}{a_1}} \frac{\text{Exp}(-c_1^2/4a_1)}{\text{Exp}(-c_1^2/4a_2)} \frac{\text{Erf} [c_2/\sqrt{(4a_2)}] - \text{Erf} [c_1/\sqrt{(4a_2)}]}{\text{Erf} [c_1/\sqrt{(4a_1)}] - \text{Erf} [c_0/\sqrt{(4a_1)}]}$$

Under the given conditions, $c_2 = 0$, $c_0 = \infty$, so that

$$\frac{\Delta G'_2}{\Delta G'_1} = \sqrt{\frac{a_2}{a_1}} \frac{\text{Exp}(-c_1^2/4a_1)}{\text{Exp}(-c_1^2/4a_2)} \frac{\text{Erf} [c_1/\sqrt{(4a_2)}]}{\text{Erf} [c_1/\sqrt{(4a_1)}]} \quad (7.5.19)$$

The best way of solving eqn. (7.5.19) consists of selecting successively c_1 for prescribed a_1, a_2 , calculating $\Delta G'_2/\Delta G'_1$ from eqn. (7.5.19), and on the basis of the results obtained, drawing a diagram of the function

$$\frac{\Delta G'_2}{\Delta G'_1} = f(c_1)$$

from which we read off the sought values of c_1 for known $\Delta G'_2/\Delta G'_1$.

7.5.2. A quasi-linear description of the effect of a sudden fall of the water surface on the boundary of a very wide earth massif

Fig. 7.5 shows the scheme of a stratified earth massif. The initial surface of groundwater is horizontal, and at the time $t = 0$, a sudden fall of the level in the reservoir of surface water occurred on the boundary. Suppose that the initial surface lay just on the interface between the n -th and the $(n + 1)$ th layer. The surface on the boundary fell to the level of the interface between the first and the zero layers.

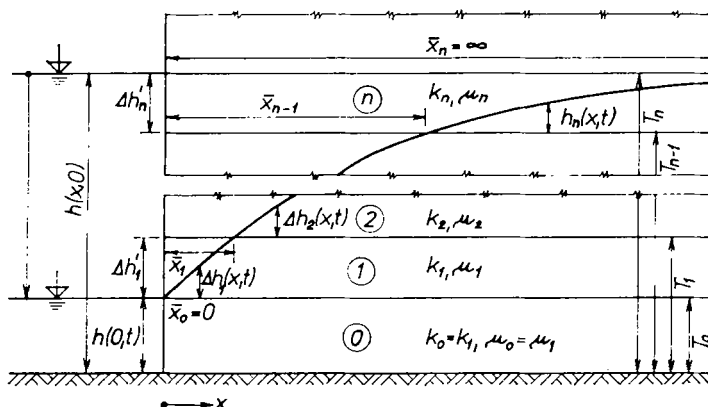


Fig. 7.5. Two-dimensional flow through an n -layer medium at a sudden fall of the surface on the boundary.

The time development of the free surface is again described by means of the ΔG potential and with the help of the system of quasi-linear equations (7.5.12). In the specified range $0 \leq x \leq \infty$, we study the course of the free surface which crosses the interfaces between the layers at the transition points $\bar{x}_N$. We have

$$\bar{x}_N \leq x \leq \bar{x}_{N-1}, \quad \Delta G_N(\bar{x}_N, t) = \Delta G'_N, \quad \Delta G_N(\bar{x}_{N-1}, t) = 0 \quad (7.5.20)$$

and, at the same time,

$$\bar{x}_n = \infty, \quad \bar{x}_0 = 0 \quad (7.5.21)$$

Just as in the case discussed in the previous subsection, a physical interpretation of the situation implies that

$$\bar{x}_N = c_N \sqrt{t} \quad (7.5.22)$$

the value of c_N being different for the case of a sudden fall from that for a sudden rise of the surface on the boundary.

Although the solution which pays heed to the conditions (7.5.20) is again of the form of eqns. (7.5.16), (7.5.17) and (7.5.18), the conditions (7.5.21) lead to the solution of a different system of transcendental equations. That this is indeed the case is best

seen when studying the development of the depression curve in two layers. Since clearly $c_2 = \infty$, $c_0 = 0$, eqn. (7.5.18) will become

$$\frac{\Delta G'_2}{\Delta G'_1} = \sqrt{\frac{a_2}{a_1}} \frac{\text{Exp}(-c_1^2/4a_1)}{\text{Exp}(-c_1^2/4a_2)} \frac{\text{Erfc}[c_1/\sqrt{(4a_2)}]}{\text{Erf}[c_1/\sqrt{(4a_1)}]} \quad (7.5.23)$$

The differences in the form as well as in the content of the two solutions confirm the fact established by practical experience, namely, that there is no symmetry between charging and emptying of stratified sediments in river shore zones. This circumstance affects the function of water conservation objects, for example, systems of wells supplied by shoreline infiltration, operating in such locations.

7.6. Application of the G potential to the solution of axi-symmetrical problems

Extensive water conservation systems are connected to the natural regime of groundwater flow. By taking an active part in it, they give rise to new conditions. A motion of a somewhat more regular character is, however, apt to form in certain parts of the flow field.

Consider, for example, an isolated well with an axially-symmetrical inflow of water. In this axi-symmetrical flow, an unsteady motion develops, and we assume that it can be described in terms of the theory of plane seepage, as its special case. The corresponding partial differential equations are deduced from the more general form (7.3.4). Since the magnitude of the polar angle is of no importance in an axi-symmetrical flow, it will be enough to use the independent variable $r = \sqrt{(x^2 + y^2)}$ and write

$$\begin{aligned} \frac{\partial G}{\partial x} &= \frac{\partial G}{\partial r} \frac{\partial r}{\partial x} = \frac{\partial G}{\partial r} \cdot \frac{x}{\sqrt{(x^2 + y^2)}} = \frac{\partial G}{\partial r} \frac{x}{r} \\ \frac{\partial^2 G}{\partial x^2} &= \frac{\partial^2 G}{\partial r^2} \frac{\partial r}{\partial x} \frac{x}{r} + \frac{\partial G}{\partial r} \frac{\partial}{\partial x} \left(\frac{x}{r} \right) = \frac{\partial^2 G}{\partial r^2} \frac{x^2}{r^2} + \frac{\partial G}{\partial r} \frac{r - x(\partial r/\partial x)}{r^2} = \\ &= \frac{\partial^2 G}{\partial r^2} \frac{x^2}{r^2} + \frac{\partial G}{\partial r} \frac{r^2 - x^2}{r^3} = \frac{\partial^2 G}{\partial r^2} \frac{x^2}{r^2} + \frac{\partial G}{\partial r} \frac{y^2}{r^3} \end{aligned}$$

and similarly for the y direction:

$$\frac{\partial^2 G}{\partial y^2} = \frac{\partial^2 G}{\partial r^2} \frac{y^2}{r^2} + \frac{\partial G}{\partial r} \frac{x^2}{r^3}$$

On substituting in eqn. (7.3.4), we have

$$\frac{\partial G}{\partial t} = a \left(\frac{\partial^2 G}{\partial r^2} + \frac{1}{r} \frac{\partial G}{\partial r} - v_0 \right) \quad (7.6.1)$$

If we consider only the case when no infiltration takes place from the terrain surface, i.e. $v_0 = 0$, eqn. (7.6.1) becomes

$$\frac{\partial G}{\partial t} = a \left(\frac{\partial^2 G}{\partial r^2} + \frac{1}{r} \frac{\partial G}{\partial r} \right) \quad (7.6.2)$$

The above equation is then solved for the prescribed boundary and initial conditions. We shall next discuss several cases which are of theoretical as well as practical importance.

7.6.1. Flow into a well operating under the assumption of a constant fall of the water surface

For the case of a sudden fall of the water surface in a well of radius $r = r_0$, we choose the following boundary and initial conditions:

$$G(r_0, t) = \text{const.}, \quad G(r, 0) = \text{const.}, \quad r_0 \leq r \leq \infty$$

The problem is made somewhat simpler by using the substitution

$$\begin{aligned} G(r, t) &= G(r, 0) + \Delta G(r, t) \\ \Delta G(r, t) &= G(r, t) - G(r, 0) \\ \frac{\partial(\Delta G)}{\partial t} &= a \left[\frac{\partial^2(\Delta G)}{\partial r^2} + \frac{1}{r} \frac{\partial(\Delta G)}{\partial r} \right] \end{aligned} \quad (7.6.3)$$

$$\Delta G(r_0, 0) = 0, \quad \Delta G(r_0, t) = \text{const.} \quad (7.6.4)$$

In the literature, equations of the type of (7.6.4) are successfully solved by the application of the Laplace transformation.

Using a transformation function of the type

$$L[f(t)] = F(p) = \int_0^\infty \text{Exp}(-pt) f(t) dt \quad (7.6.5)$$

the partial differential equation is converted to an ordinary one.

All the terms of eqn. (7.6.3) are multiplied by $\text{Exp}(-pt)$ and integrated with respect to t between the limits zero and infinity:

$$\int_0^\infty e^{-pt} \frac{\partial(\Delta G)}{\partial t} dt = a \int_0^\infty e^{-pt} \frac{\partial^2(\Delta G)}{\partial r^2} dt + \frac{a}{r} \int_0^\infty e^{-pt} \frac{\partial(\Delta G)}{\partial r} dt \quad (7.6.6)$$

The integral on the left-hand side of eqn. (7.6.6) is integrated by parts:

$$\int_0^\infty e^{-pt} \frac{\partial(\Delta G)}{\partial t} dt = [e^{-pt} \Delta G(r, t)]_0^\infty + p \int_0^\infty e^{-pt} \Delta G(r, t) dt$$

Assume now that

$$\lim_{t \rightarrow \infty} e^{-pt} \Delta G(r, t) = 0$$

and that according to eqn. (7.6.4), $\Delta G(r, 0) = 0$, so that

$$\int_0^\infty e^{-pt} \frac{\partial[\Delta G(r, t)]}{\partial t} dt = p \int_0^\infty e^{-pt} \Delta G(r, t) dt = p \overline{\Delta G}(r, p) \quad (7.6.7)$$

where $\overline{\Delta G}(r, p)$ is a new function introduced in the solution.

After rearranging the two integrals on the right-hand side of eqn. (7.6.6) as follows:

$$\begin{aligned} a \int_0^\infty e^{-pt} \frac{\partial^2(\Delta G)}{\partial r^2} dt + \frac{a}{r} \int_0^\infty e^{-pt} \frac{\partial(\Delta G)}{\partial r} dt &= a \frac{\partial^2}{\partial r^2} \int_0^\infty e^{-pt} \Delta G dt + \\ &+ \frac{a}{r} \frac{\partial}{\partial r} \int_0^\infty e^{-pt} \Delta G dt = a \frac{\partial^2(\overline{\Delta G})}{\partial r^2} + \frac{a}{r} \frac{\partial(\overline{\Delta G})}{\partial r} \end{aligned} \quad (7.6.8)$$

we can write, in place of eqn. (7.6.3), the equation

$$p \overline{\Delta G}(r, p) = a \frac{\partial^2[\overline{\Delta G}(r, p)]}{\partial r^2} + \frac{a}{r} \frac{\partial[\overline{\Delta G}(r, p)]}{\partial r} \quad (7.6.9)$$

Eqn. (7.6.9) contains the derivatives with respect to only one independent variable, r , and can, therefore, be treated as an ordinary differential equation, viz.

$$r \frac{d^2(\overline{\Delta G})}{dr^2} + \frac{d(\overline{\Delta G})}{dr} - \frac{pr}{a} \overline{\Delta G} = 0 \quad (7.6.10)$$

In this way, we converted the problem to the solution of Bessel's equation whose general integral is of the form

$$\overline{\Delta G} = C_1 I_0 \left(r \sqrt{\frac{p}{a}} \right) + C_2 K_0 \left(r \sqrt{\frac{p}{a}} \right) \quad (7.6.11)$$

where

I_0 is the modified zero order Bessel function of the first kind (of an imaginary argument),

K_0 is the modified zero order Bessel function of the second kind (of an imaginary argument),

C_1, C_2 are constants.

Noting that eqn. (7.6.11) has a finite value at the point $r = \infty$, we must assume that $C_1 = 0$ and therefore

$$\overline{\Delta G} = C_2 K_0 \left(r \sqrt{\frac{p}{a}} \right) \quad (7.6.12)$$

According to the specification of the problem, ΔG has a certain constant value at the point $r = r_0$,

$$\int_0^\infty e^{-pt} \Delta G(r_0, t) dt = \Delta G(r_0, t) \left(0 + \frac{1}{p}\right) = \frac{\Delta G(r_0, t)}{p}$$

$$\overline{\Delta G}(r_0, p) = \frac{\Delta G(r_0, t)}{p} = C_2 K_0 \left(r_0 \sqrt{\frac{p}{a}}\right)$$

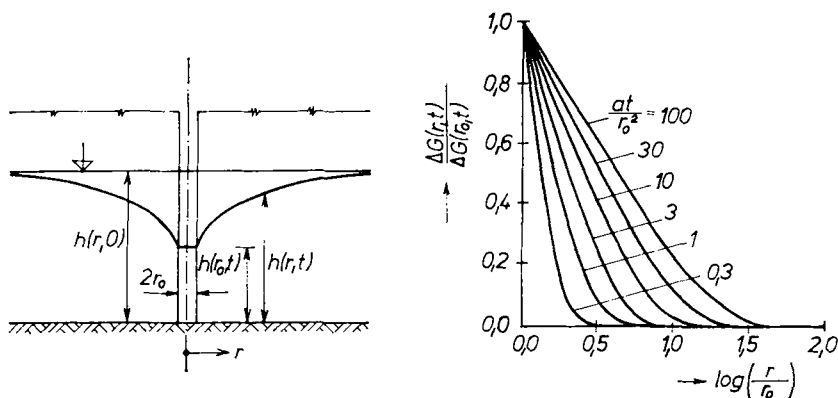


Fig. 7.6. Unsteady flow into a well operating under a constantly falling water surface.

from which

$$C_2 = \frac{\Delta G(r_0, t)}{p K_0[r_0 \sqrt{(p/a)}]}$$

Hence the resulting function $\overline{\Delta G}$ is of the form

$$\overline{\Delta G} = \frac{\Delta G(r_0, t) K_0[r \sqrt{(p/a)}]}{p K_0[r_0 \sqrt{(p/a)}]} \quad (7.6.13)$$

In the next phase of the solution, we are required to find the original $f(t)$ in the expression (7.6.5)

$$\Delta G(r, t) = f(t) = L^{-1}[F(p)]$$

This is usually done by referring to comprehensive tables. Untabulated cases are solved by special procedures which are frequently fairly difficult to apply. In our solution we can adopt an expression fit for immediate use, which was obtained by a modification of the formulae proposed in a book by Kristea [22], viz.

$$\Delta G(r, t) = \Delta G(r_0, t) \left[1 + \frac{2}{\pi} \int_0^\infty \text{Exp}(-au^2 t) \frac{J_0(ru) Y_0(r_0 u) - Y_0(ru) J_0(r_0 u)}{J_0^2(r_0 u) + Y_0^2(r_0 u)} \frac{du}{u} \right] \quad (7.6.14)$$

The flow into a well with $r = r_0$ is calculated from the equation

$$Q = 2\pi r_0 \left. \frac{\partial(\Delta G)}{\partial r} \right|_{r=r_0} = -\frac{8}{\pi} \Delta G(r_0, t) \int_0^\infty \frac{\text{Exp}(-au^2t)}{J_0^2(r_0u) + Y_0^2(r_0u)} \frac{du}{u} \quad (7.6.15)$$

where

I_0 is the zero order Bessel function of the first kind,

Y_0 — the zero order Bessel function of the second kind.

Since the practical calculations using eqn. (7.6.14) are time consuming and laborious, it may serve a useful purpose to show the resulting solution (Fig. 7.6) in the form of a diagram which enables us to gain an idea of the course of the unsteady process.

A more suitable analytical expression of the function $\Delta G(r, t)$ is obtained for comparatively short times t . On expressing the values of $K_v(z)$ in eqn. (7.6.13) by means of the series

$$K_v(z) = \sqrt{\frac{\pi}{2z}} \text{Exp}(-z) \left\{ 1 + \frac{4v^2 - 1}{1! 8z} + \frac{(4v^2 - 1)(4v^2 - 3^2)}{2! (8z)^2} + \dots \right\}$$

we have

$$\begin{aligned} \Delta \bar{G} &= \frac{\Delta G(r_0, t)}{p} \sqrt{\frac{r_0}{r}} \text{Exp} \left[-(r - r_0) \sqrt{\frac{p}{a}} \frac{1 - 1/8r \sqrt{(p/a)} + 9/128r^2 (p/a) + \dots}{1 - 1/8r_0 \sqrt{(p/a)} + 9/128r_0^2 (p/a) + \dots} \right] = \\ &= \frac{\Delta G(r_0, t)}{p} \sqrt{\frac{r_0}{r}} \text{Exp} \left[-(r - r_0) \sqrt{\frac{p}{a}} \right] \left[1 + \frac{r - r_0}{8rr_0 \sqrt{(p/a)}} + \right. \\ &\quad \left. + \frac{9r_0^2 - 2rr_0 - 7r^2}{128r^2 r_0^2 (p/a)} + \dots \right] \end{aligned}$$

Referring to the tables of the Laplace transformations, we obtain for the transform

$$\Delta \bar{G}(p) = \frac{\text{Exp}[-x \sqrt{(p/a)}]}{p^{1+n/2}}, \quad n = 1, 2, \dots$$

the original

$$\Delta G(t) = (4t)^{n/2} (i)^n \text{Erfc} \left(\frac{x}{\sqrt{(4at)}} \right)$$

Hence in the case being considered, we have

$$\begin{aligned} \Delta G(r, t) &= \Delta G(r_0, t) \left[\sqrt{\frac{r_0}{r}} \text{Erfc} \left(\frac{r - r_0}{\sqrt{(4at)}} \right) + \frac{(r - r_0) \sqrt{(at)}}{4r \sqrt{(rr_0)}} i \text{Erfc} \left(\frac{(r - r_0)}{\sqrt{(4at)}} \right) + \right. \\ &\quad \left. + \frac{(9r_0^2 - 2rr_0 - 7r^2) at}{32r_0^{3/2} r^{5/2}} i^2 \text{Erfc} \left(\frac{(r - r_0)}{\sqrt{(4at)}} \right) + \dots \right] \quad (7.6.16) \end{aligned}$$

For $n = 1, 2, 3, 4, 5, 6$, the function $(i)^n \operatorname{Erfc}(x)$ which appears in the resulting series, is tabulated. Since the first one or two terms of the series are enough for practical calculations, we can write, accurately enough, that

$$\Delta G(r, t) = \Delta G(r_0, t) \sqrt{\frac{r_0}{r}} \operatorname{Erfc}\left(\frac{r - r_0}{\sqrt{(4at)}}\right) \quad (7.6.17)$$

The equation applying to the well contour for small t is, approximately,

$$Q|_{r=r_0} = -\pi \Delta G(r_0, t) \left(1 + \frac{2}{\sqrt{(\pi at/r_0^2)}}\right) \quad (7.6.18)$$

7.6.2. Flow into a well operating under the assumption of constant charging or discharging

Essentially, we have to solve eqn. (7.6.3) under the assumption that

$$\Delta G(r, 0) = 0, \quad 2\pi r \frac{\partial(\Delta G)}{\partial r} \Big|_{r=r_s} = Q_s = \text{const.} \quad (7.6.19)$$

where Q_s is taken to be positive or negative depending on whether charging or discharging is involved.

An exact solution performed by means of the operational calculus leads essentially to eqn. (7.6.11), i.e. to expressions which are hard to cope with in practical calculations. Relaxing somewhat the requirement of an accurate solution, we assume, in place of eqns. (7.6.19), that

$$\Delta G(r, 0) = 0, \quad \lim_{r \rightarrow 0} \left[2\pi r \frac{\partial(\Delta G)}{\partial r} \right] = Q_s = \text{const.} \quad (7.6.20)$$

Take note of the fact that, in keeping with the similarity theory, the solution (7.6.3) should be a function of the dimensionless number

$$\left[\frac{r^2}{4at} \right]$$

On the strength of this fact, choose the substitution

$$\zeta = \frac{r^2}{4at}$$

which leads to the ordinary differential equation

$$\frac{d^2(\Delta G)}{d\zeta^2} + \frac{1 + \zeta}{\zeta} \frac{d(\Delta G)}{d\zeta} = 0$$

The form of the general solution of this equation is

$$\Delta G(r, t) = c_1 + c_2 \int \frac{\text{Exp}(-\zeta)}{\zeta} d\zeta$$

or

$$\Delta G(r, t) = C_1 + C \int_{r^2/4at}^{\infty} \frac{\text{Exp}(-\zeta)}{\zeta} d\zeta \quad (7.6.21)$$

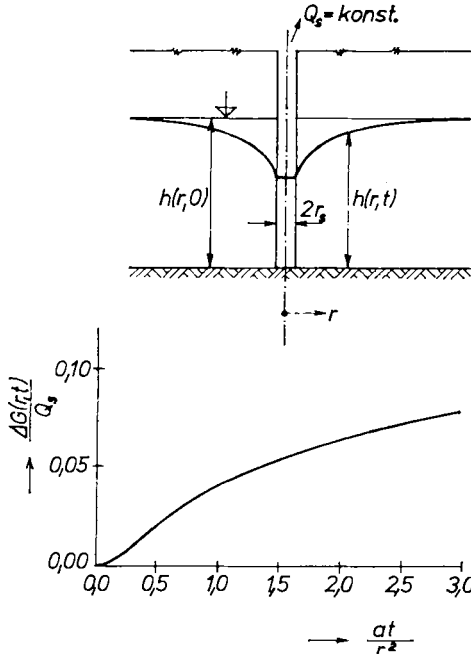


Fig. 7.7. Unsteady flow into a well operating under a periodic take-off.

Since ΔG is to equal zero for $t = 0$, $C_1 = 0$ and

$$\frac{\partial(\Delta G)}{\partial r} = -\frac{2C}{r} \text{Exp}\left(-\frac{r^2}{4at}\right)$$

The condition (7.6.20) implies that

$$\lim_{r \rightarrow 0} \left[2\pi r \frac{\partial(\Delta G)}{\partial r} \right] = -4\pi C = Q_s, \quad C = -\frac{Q_s}{4\pi}$$

hence

$$\Delta G(r, t) = -\frac{Q_s}{4\pi} \int_{r^2/4at}^{\infty} \frac{e^{-\zeta}}{\zeta} d\zeta = -\frac{Q_s}{4\pi} \left[-\text{Ei}\left(-\frac{r^2}{4at}\right) \right] \quad (7.6.22)$$

In the literature, eqn. (7.6.22) goes under the name of the Theis equation [41]. The

function $-\text{Ei}(-x)$ is tabulated, for example, in [18]. The diagram of the function

$$\left| \frac{\Delta G(r, t)}{Q_s} \right| = f\left(\frac{at}{r^2}\right)$$

is shown in Fig. 7.7. The function $-\text{Ei}(-x)$ can be written as the infinite series

$$-\text{Ei}(-x) = -0.57721 - \ln x + \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n! n}, \quad n = 1, 2, 3, \dots \quad (7.6.23a)$$

For small values of the argument, it is usually sufficient to take the first terms of the series. Substituting the argument $r^2/4at$, we obtain

$$-\text{Ei}\left(-\frac{r^2}{4at}\right) \approx \ln \frac{2.25at}{r^2} \quad (7.6.23b)$$

from which it follows that for $at/r^2 > 8.0$, the simple formula

$$\Delta G(r, t) = -\frac{Q_s}{4\pi} \ln \frac{2.25at}{r^2} \quad (7.6.24)$$

leads to a result that is less than 1% in error.

The volume flow through a cylinder of radius r is obtained by differentiating eqn. (7.6.21):

$$Q = 2\pi r \frac{\partial(\Delta G)}{\partial r} = Q_s \text{Exp}\left(-\frac{r^2}{4at}\right) \quad (7.6.25)$$

For small values of the argument, we use the derivative of eqn. (7.6.23b) which — within appropriate limits — gives a constant and time independent volume flow. This means that the ensuing flow is close to a steady flow. On the other hand, since

$$\frac{\partial(\Delta G)}{\partial t} = -\frac{Q_s}{4\pi t} \quad (7.6.26)$$

the water surface as an entity is in motion independently of r .

Within certain limits, the flow can be regarded as steady at any time and subjected to a corresponding analysis. What we have to know in order to be in a position to undertake it, is not only the extent of the region of steady flow but the values of ΔG on the boundaries as well. One of the values used in the analysis is ΔG on the well lining where $r = r_s$, the level of the water surface in the well corresponding approximately to the respective value of $\Delta G(r_s, t)$. The other boundary is taken to lie inside the region in which the simplified eqn. (7.6.23b) still applies.

7.6.3. Flow into a well with a periodic take-off

It was shown in Section 7.1 that the application of the similarity principle demands that due regard be paid not only to the partial differential equation which describes the process being studied but also to the boundary conditions. If we are dealing with periodic fluctuations which can be described by continuous trigonometric functions, we start from the equation

$$\frac{\partial u}{\partial t} = a \left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} \right) \quad (7.6.27)$$

and assume that in the specified range $r_s < r < \infty$, the solution is of the form

$$u = u(r) \text{Exp}(i\omega t) \quad (7.6.28)$$

where ω is the angular frequency.

Eqn. (7.6.28) implies the boundary condition for $r = r_s$. If we apply the method whose principle was explained in Section 7.1, we find that the solution is a function of the parameters

$$\left\{ r \sqrt{\frac{\omega}{a}} \right\}, \quad \left\{ \frac{r^2}{at} \right\} \quad (7.6.29)$$

Writing

$$\xi = r \sqrt{\frac{\omega}{a}} \quad (7.6.30)$$

and substituting in eqn. (7.6.27) we obtain

$$\frac{\partial u}{\partial t} = \omega \left(\frac{\partial^2 u}{\partial \xi^2} + \frac{1}{\xi} \frac{\partial u}{\partial \xi} \right) \quad (7.6.31)$$

According to eqn. (7.6.28), we can assume the solution to have the form of the product

$$u = \zeta(t) \eta(\xi) \quad (7.6.32)$$

where $\zeta(t)$ is a function of time.

The function

$$\eta(\xi) = \frac{\partial w}{\partial \xi} \quad (7.6.33)$$

is a function of ξ , w , and w is the integral of the equation

$$\frac{d^2 w}{d\xi^2} + \frac{1}{\xi} \frac{dw}{d\xi} - iw = 0, \quad w = w(\xi) \quad (7.6.34)$$

On substituting eqn. (7.6.32) in eqn. (7.6.31) we have

$$\frac{1}{\zeta} \frac{d\zeta}{dt} = \omega \left(\frac{d^2\eta}{d\xi^2} + \frac{1}{\xi} \frac{d\eta}{d\xi} \right) \frac{1}{\eta} \quad (7.6.35)$$

Because of the equality sign in eqn. (7.6.35), both sides of the equation must be equal to the same constant, for example, $i\omega$. Hence, for the left-hand side, we obtain the equation

$$\frac{1}{\zeta} \frac{d\zeta}{dt} = i\omega \quad (7.6.36)$$

whose integral is

$$\zeta = c_1 \text{Exp}(i\omega t) \quad (7.6.37)$$

Similarly, for the right-hand side of eqn. (7.6.35), we have

$$\frac{d^2\eta}{d\xi^2} + \frac{1}{\xi} \frac{d\eta}{d\xi} - i\eta = 0 \quad (7.6.38)$$

Note that eqn. (7.6.38) is formally identical with eqn. (7.6.34) whose integral is $w(\xi)$. On differentiating eqn. (7.6.34) with respect to ξ and rearranging, we therefore obtain

$$\frac{d}{d\xi} \left(\frac{d^2w}{d\xi^2} \right) + \frac{1}{\xi} \frac{d}{d\xi} \left(\frac{dw}{d\xi} \right) - \frac{1}{\xi^2} (1 + i\xi^2) \frac{dw}{d\xi} = 0 \quad (7.6.39)$$

and on referring to eqn. (7.6.33)

$$\xi^2 \frac{d^2\eta}{d\xi^2} + \xi \frac{d\eta}{d\xi} - (1 + i\xi^2) \eta = 0 \quad (7.6.40)$$

The solution of eqn. (7.6.40), which is of the Bessel type, is

$$\eta = c_2 [\text{ker}_0(\xi) + i \text{kei}_0(\xi)] + c_3$$

where

c_2, c_3 are constants (c_3 being assumed to be equal to zero),

$\text{ker}_0(\xi), \text{kei}_0(\xi)$ are the zero order Kelvin functions of the second kind.

Hence, in keeping with eqn. (7.6.32), the solution of eqn. (7.6.32) is of the form

$$u = c [\text{ker}_0(\xi) + i \text{kei}_0(\xi)] \text{Exp}(i\omega t) \quad (7.6.41)$$

where c is a constant.

Now write the Kelvin functions in terms of the zero order amplitude function of the second order, $N_0(\xi)$, and the zero order phase function of the second order,

$\Phi_0(\xi)$ (the diagrams of these functions are shown in Fig. 7.8):

$$u = c e^{i\omega t} N_0(\xi) e^{i\Phi_0(\xi)} = c N_0(\xi) \text{Exp } i[\omega t + \Phi_0(\xi)] \quad (7.6.42)$$

$$u = c N_0 \left(r \sqrt{\frac{\omega}{a}} \right) \left\{ \cos \left[\omega t + \Phi_0 \left(r \sqrt{\frac{\omega}{a}} \right) \right] + i \sin \left[\omega t + \Phi_0 \left(r \sqrt{\frac{\omega}{a}} \right) \right] \right\} \quad (7.6.43)$$

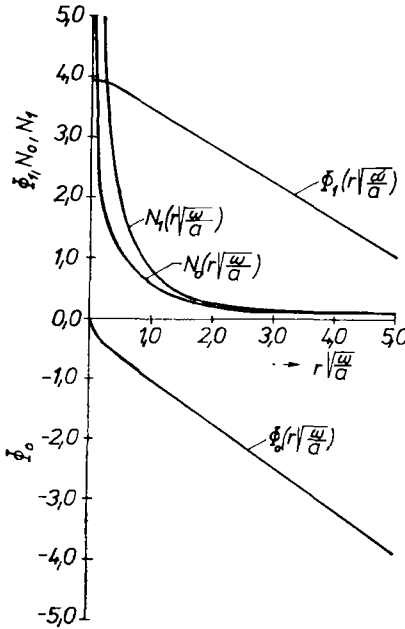


Fig. 7.8. Amplitude and phase functions corresponding to the Kelvin functions.

The equation (7.6.43) has a real and an imaginary part, each of which may be a solution of the initial equation. Let us start, for example, from the real part and assume that

$$\Delta G(r, t) = G(r, t) - G(r, 0) \quad (7.6.44)$$

The difference between the G potentials satisfies eqn. (7.6.3) which is formally identical with eqn. (7.6.27). Hence the solution based on the use of the real part of eqn. (7.6.43), is

$$\Delta G(r, t) = c N_0 \left(r \sqrt{\frac{\omega}{a}} \right) \cos \left[\omega t + \Phi_0 \left(r \sqrt{\frac{\omega}{a}} \right) \right] \quad (7.6.45)$$

Assume further that the periodic take-off has been going on for a long time and that the effect of the initial condition has already vanished (problem without the initial condition). The flow which passes at a given instant through a cylinder of radius r is obtained from the equation

$$Q = 2\pi r \frac{\partial(\Delta G)}{\partial r} = 2\pi r c \sqrt{\frac{\omega}{a}} N_1 \left(r \sqrt{\frac{\omega}{a}} \right) \cos \left[\omega t + \Phi_1 \left(r \sqrt{\frac{\omega}{a}} \right) - \frac{\pi}{4} \right] \quad (7.6.46)$$

where $N_1(\xi)$, $\Phi_1(\xi)$ are the first order amplitude and phase functions of the second kind. They correspond to the first order Kelvin functions of the second kind, and their diagrams are shown in Fig. 7.8.

Assume that on the well lining, $r = r_s$ and $Q = Q_s$, where

$$Q_s = Q_0 \cos \left[\omega t - \Phi_1 \left(r_s \sqrt{\frac{\omega}{a}} \right) - \frac{\pi}{4} \right] \quad (7.6.47)$$

Q_0 is the amplitude of the take-off from the well (the maximum take-off).

Eqn. (7.6.47) implies that

$$c = \frac{Q_0}{2\pi r_s \sqrt{(\omega/a)} N_1[r_s \sqrt{(\omega/a)}]}$$

We can consider

$$Q = Q_0 \frac{r N_1[r \sqrt{(\omega/a)}]}{r_s N_1[r_s \sqrt{(\omega/a)}]} \cos \left[\omega t + \Phi_1 \left(r \sqrt{\frac{\omega}{a}} \right) - \frac{\pi}{4} \right] \quad (7.6.48)$$

in place of eqn. (7.6.45), and write

$$\Delta G(r, t) = \frac{Q_0 N_0[r \sqrt{(\omega/a)}] \cos \{ \omega t + \Phi_0[r \sqrt{(\omega/a)}] \}}{2\pi r_s \sqrt{(\omega/a)} N_1[r_s \sqrt{(\omega/a)}]} \quad (7.6.49)$$

in place of eqn. (7.6.45).

On the well lining or, more exactly, inside the well,

$$\Delta G(r_s, t) = \frac{Q_0 N_0[r_s \sqrt{(\omega/a)}] \cos \{ \omega t + \Phi_0[r_s \sqrt{(\omega/a)}] \}}{2\pi r_s \sqrt{(\omega/a)} N_1[r_s \sqrt{(\omega/a)}]} \quad (7.6.50)$$

Since for $r \sqrt{(\omega/a)} < 0.05$, we have, approximately, $\Phi_1[r \sqrt{(\omega/a)}] \doteq 5\pi/4$, and the take-off from the well is characterized by the equation

$$Q_s = Q_0 \cos(\omega t) \quad (7.6.51)$$

which follows from eqn. (7.6.48). As a comparison of eqn. (7.6.51) with eqn. (7.6.50) suggests, the fluctuations of the water surface in the well are out of phase with the take-off.

The amplitude and the phase of the fluctuations of the water surface inside an earth massif vary with r . The time increment of $\Delta G(r, t)$ is obtained by differentiating eqn. (7.6.49) with respect to t :

$$\frac{\partial}{\partial t} [\Delta G(r, t)] = \frac{Q_0 \omega N_0[r \sqrt{(\omega/a)}] \sin \{ \omega t + \Phi_0[r \sqrt{(\omega/a)}] \}}{2\pi r_s \sqrt{(\omega/a)} N_1[r_s \sqrt{(\omega/a)}]}$$

At the instant when the wave has a maximum, we have

$$\omega t + \Phi_0 \left(r \sqrt{\frac{\omega}{a}} \right) = \frac{\pi}{2}$$

From the above, we calculate the velocity of propagation of the wave from the contour of the borehole towards the inside of the massif, viz.

$$-\frac{dr}{dt} = \sqrt{a\omega} \frac{N_0[r \sqrt{(\omega/a)}]}{N_1[r \sqrt{(\omega/a)}] \sin \{ \Phi_1[r \sqrt{(\omega/a)}] - \Phi_0[r \sqrt{(\omega/a)}] - \pi/4 \}}$$

For large values of the arguments, we can write simply that

$$\frac{dr}{dt} \doteq \sqrt{(2\omega a)} = \text{const.}$$

The surface waves propagate into the surroundings with a nearly constant velocity.

Analogous results can be obtained by considering the imaginary part of eqn. (7.6.43). In this case, we have in the well

$$Q_s = Q_0 \sin(\omega t) \quad (7.6.52)$$

In the problem without the initial condition, eqn. (7.6.52) agrees in fact with eqn. (7.6.51), since the phase displacement by $\pi/2$ is clearly of no practical significance. The results of the solutions obtained under the condition (7.6.52) and under the condition (7.5.51) are, however, important from a theoretical point of view because their synthesis enables us to describe the effect of take-offs which are apt to follow any periodic relation whatever.

Applying the Fourier analysis, we can resolve any periodic function $f(t)$ into elementary trigonometric functions, viz.

$$f(t) = A_0 + \sum_{n=1}^{\infty} A_n \cos n\omega t + B_n \sin n\omega t \quad (7.6.53)$$

$$A_0 = \frac{1}{t_0} \int_0^T f(t) dt \quad (7.6.54)$$

$$A_n = \frac{2}{t_0} \int_0^T f(t) \cos n\omega t \quad (7.6.55)$$

$$B_n = \frac{2}{t_0} \int_0^T f(t) \sin n\omega t \quad (7.6.56)$$

$$t_0 = \frac{2\pi}{\omega} \quad (7.6.57)$$

where ω is the angular frequency, and t_0 is the time for a period of the function $f(t)$.

Instead of dwelling on details of the harmonic analysis which are fully covered in mathematics textbooks, we shall present some complete solutions which we believe will be found helpful in practical calculations.

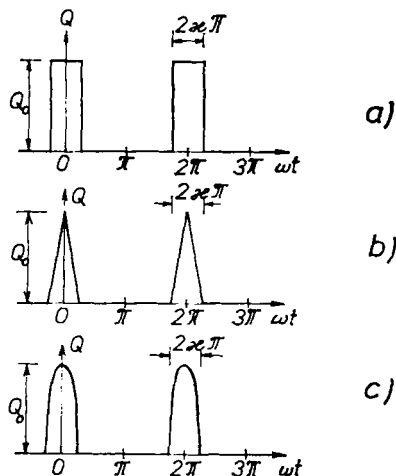


Fig. 7.9. Typical periodic wave forms.

Fig. 7.9a shows a sequence of rectangular wave forms for which

$$Q = Q_0 \left(\kappa + \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \sin n\kappa\pi \cos n\omega t \right) \quad (7.6.58)$$

and Fig. 7.9b a sequence of sawtooth wave forms for which

$$Q = Q_0 \left[\kappa + \frac{1}{\kappa\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2} (1 - \cos 2n\kappa\pi) \cos n\omega t \right] \quad (7.6.59)$$

Fig. 7.9c shows a sequence of round-top wave forms whose curvilinear parts derive from a sine curve. In this case, the analysis leads to the conclusion that

$$Q = \frac{Q_0}{\pi(1 - \cos \kappa\pi)} \left\{ \sin \kappa\pi - \kappa\pi \cos \kappa\pi + (\kappa\pi - \sin \kappa\pi \cos \kappa\pi) \cos \omega t + \right. \\ \left. + \sum_{n=2}^{\infty} \left[\frac{\sin(n+1)\kappa\pi}{n+1} + \frac{\sin(n-1)\kappa\pi}{n-1} - \frac{2 \sin n\kappa\pi \cos \kappa\pi}{n} \right] \cos n\omega t \right\} \quad (7.6.60)$$

In practical calculations, it is usually sufficient to take a few first terms of the series. An idea of the accuracy of the corresponding approximations can be gained by referring to Fig. 7.10, the bottom diagram in which shows a sequence of rectangular

wave forms that can be described by the equation

$$Q = Q_0 \left[\frac{1}{2} + \frac{2}{\pi} (\sin \omega t + \frac{1}{3} \sin 3\omega t + \frac{1}{5} \sin 5\omega t + \dots) \right] \quad (7.6.61)$$

Starting from the top of Fig. 7.10, the diagrams show the two-term, the three-term, the four-term and the five-term approximation, respectively. It is seen that the first five terms of the series give a very good approximation indeed.

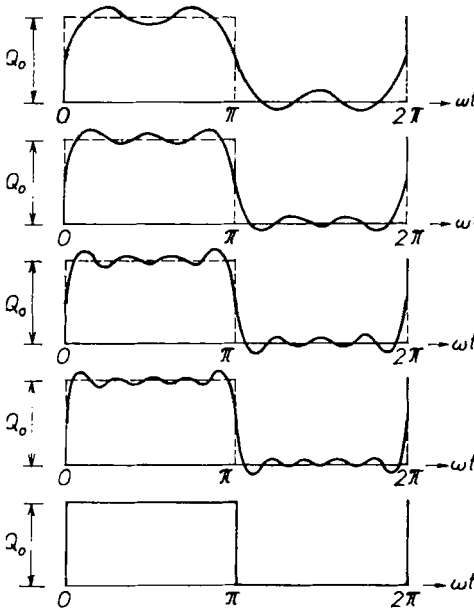


Fig. 7.10. Accuracy of the process of approximating a rectangular wave form.

7.6.4. The action radius of wells

The effect of groundwater surface fluctuations, induced by a periodic take-off from a well, is weaker in locations which are at some distance from the object. In an illustrative quantitative evaluation, we shall use one of the elementary periodic functions, denoting by the number m the position of this function in the frequency spectrum.

In keeping with the previous section, we have

$$\Delta G_m(r, t) = \frac{Q_0}{2\pi r_s} \frac{N_0[r \sqrt{(m\omega/a)}] \cos \{m\omega t + \Phi_0[r \sqrt{(m\omega/a)}]\}}{\sqrt{(m\omega/a)} N_1[r_s \sqrt{(m\omega/a)}]} \quad (7.6.62)$$

The angular frequency is expressed in terms of the time for one basic period, viz.

$$\omega = \frac{2\pi}{t_1}$$

The effect of the amplitude functions is evaluated using the diagrams shown in Fig. 7.8. It is found that the ratio between the maximum values of the function $\Delta G_m(r, t)$ and of the function $\Delta G_m(r_s, t)$, calculated for the water surface in the well, decreases with increasing ratio r/r_s as follows:

$$\frac{[\Delta G_m(r, t)]_{\max}}{[\Delta G_m(r_s, t)]_{\max}} = \frac{N_0[r \sqrt{(m\omega/a)}]}{N_0[r_s \sqrt{(m\omega/a)}]} \quad (7.6.63)$$

If the ratio r/r_s relative to a given frequency is chosen large enough, the effect of fluctuations of the water surface in the well cannot be expected to be too clearly defined on a cylinder of radius r . The corresponding $r = R$ is considered to be the action radius of the well. It is clear that during a periodic take-off of prolonged duration, the action radius R is not a function of time, but depends on the frequency (i.e. the time for one period), permeability and thickness of the saturated medium, and on active porosity.

Referring to the diagrams in Fig. 7.8, we assume that for small values of the argument $r_s \sqrt{(m\omega/a)}$, the ratio (7.6.63) is close to zero for

$$r \sqrt{\frac{m\omega}{a}} \geq (3 \sim 4)$$

Hence

$$R \doteq (3 \sim 4) \sqrt{\frac{a}{m\omega}}$$

We find, on the other hand, that if we theoretically choose $t_1 \rightarrow \infty$, we automatically obtain $\omega \rightarrow \infty$ and also $R/r_s \rightarrow \infty$, since the fluctuation of the surface in the well is not a true periodic function.

In such cases, the practical action radius is defined as the radius of a cylinder on which the motion of water at a given time is not very noticeable. Assume, for example, that pumping from a well was started at the time $t = 0$, and that the amount of take-off $Q_s = \text{const}$. In accordance with eqn. (7.6.25), we can write, for the ratio between the volume flows through a cylinder of radius $r = R_n$ and through the well lining, that

$$\frac{Q(r_s, t)}{Q(R_n, t)} = \frac{\text{Exp}[-(r_s^2/4at)]}{\text{Exp}[-(R_n^2/4at)]} = \text{Exp}\left(\frac{R_n^2 - r_s^2}{4at}\right) \quad (7.6.64)$$

Hence

$$R_n = \sqrt{\left[r_s^2 + 4at \ln \frac{Q(r_s, t)}{Q(R_n, t)}\right]} \quad (7.6.65)$$

For $R_n \gg r_s$, the flow through a cylinder of radius r_s is many times larger than that through a cylinder of radius R_n . After a comparatively long time, the value of the second term under the square root sign on the right-hand side of eqn. (7.6.65) is much larger than the value of the first term. Assume that R_n , determined under such con-

ditions, is the action radius in unsteady flow. In this calculation, we choose the ratio $Q(r, t)/Q(R_n, t)$ reasonably large, assuming, for example, that the volume flow through the well lining is larger by the factor α than that through the cylinder of radius R_n . Eqn. (7.6.65) then gives

$$R_n = \sqrt{(at \ln \alpha)} \quad (7.6.66)$$

In a homogeneous medium, we have

$$a = \frac{kh_s}{\mu} \quad (7.6.67)$$

where

k is the coefficient of permeability,
 h_s — the mean thickness of the saturated medium,
 μ — the active porosity.

Substituting the above in eqn. (7.6.66) we obtain

$$R_n = \sqrt{\left(\frac{kh_s t}{\mu} \ln \alpha\right)} \quad (7.6.68)$$

The literature contains a number of formulae and recommended expressions of the type of eqn. (7.6.8), differing, for the most part, by the value of α . Supposing, for example, that the volume flow through the well lining is larger by a factor of 100 than that through the cylinder of radius R_n , then

$$R_n \doteq 2.1 \sqrt{(at)}$$

For $\alpha = 1000$ we obtain, on the other hand,

$$R_n = 2.6 \sqrt{(at)}$$

Assume now that inside a large enough R_n , there exists a region $r \leq R_s$ to which we can apply the simplification (7.6.23b), and that the limiting case of this situation occurs for

$$\frac{r^2}{at} \leq 0.05 \quad (7.6.69)$$

Inside the region R_s , Dupuit's equation

$$\Delta G(r, t) - \Delta G(R_s, t) = -\frac{Q_s}{4\pi} \left[\ln \frac{2.25at}{r^2} - \ln \frac{2.25at}{R_s^2} \right] = -\frac{Q_s}{2\pi} \ln \frac{R_s}{r} \quad (7.6.70)$$

then holds good with an accuracy better than 10%.

By eqn. (7.6.69), the action radius of the steady state, conceived in this way, is

$$R_s \doteq 0.7 \sqrt{(at)} \quad (7.6.71)$$

and in an homogeneous medium where eqn. (7.6.67) is valid,

$$R_s \doteq 0.7 \sqrt{\frac{kh_s t}{\mu}} \quad (7.6.72)$$

Both the action radius of unsteady flow, R_n , and the action radius of steady flow, R_s , are time-dependent quantities. Inside a region of a gradually varying steady flow, eqn. (7.6.70) or simply the equation

$$\Delta G(r, t) = -\frac{Q_s}{2\pi} \ln \frac{3\sqrt{(at)}}{2r} \quad (7.6.73)$$

can be used for each definitely specified t .

In an homogeneous medium, we have

$$h = \sqrt{\left[h^2(r, 0) + \frac{Q_s}{\pi k} \ln \left(\frac{3}{2r} \sqrt{\frac{kh_s t}{\mu}} \right) \right]} \quad (7.6.74)$$

where $h(r, 0) = h_0$ is the piezometric head of the initial, un-lowered water surface.

Q_s is regarded as negative whenever the flow of water is towards the inside of the well.

7.6.5. Collocated wells

The principle of superposition of the boundary conditions, which was used in Section 7.6.3, suggests the possibility that a given problem can be decomposed into a number of simpler solutions. We suppose that several partial processes whose effects are superposed upon one another, are taking place in a well at the same time. The same results are obtained by supposing that several wells operating under simplified conditions exist at one place. Such wells are collocated wells.

Assume, for example, that there are $j = 1, 2, \dots, J$ collocated wells, from each of which the take-off $Q_{sj} = \text{const.}$ was started at the time t_j . Under the assumption that $t_1 = 0$, we have, according to the principle of superposition,

$$\Delta G(r, t) = -\frac{1}{4\pi} \sum_{j=1}^J Q_{sj} \left\{ -\text{Ei} \left[-\frac{r^2}{4a(t-t_j)} \right] \right\} \quad (7.6.75)$$

As an illustrative example, consider the operation of a well from which the take-off $Q_s = \text{const.}$ was started at the time $t_1 = 0$. A fall of the water surface was observed on a cylinder of radius r . At the time $t = t_2$, the take-off was stopped. This process

can be interpreted as one during which the original take-off continues while an in flow of water, $-Q_s = \text{const.}$ starts in a collocated well. The time variations of the water surface fluctuations are described by the equation

$$\Delta G(r, t) = -\frac{Q_s}{4\pi} \left[\left[-\text{Ei} \left(-\frac{r^2}{4at} \right) \right] - \left\{ -\text{Ei} \left[-\frac{r^2}{4a(t-t_i)} \right] \right\} \right] \quad (7.6.76)$$

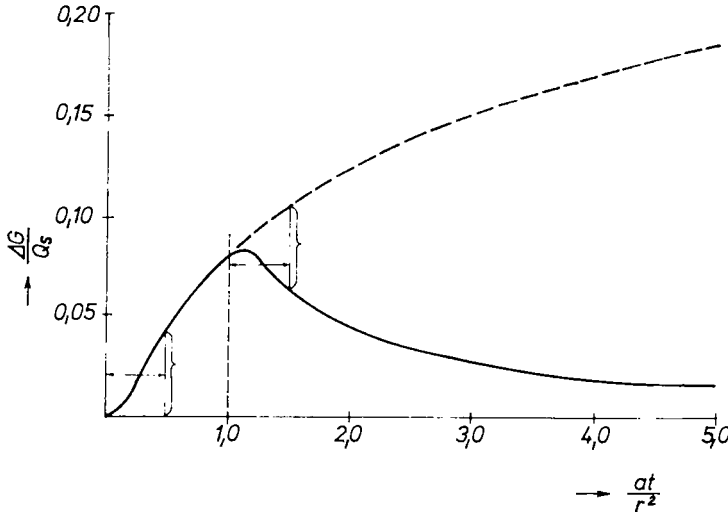


Fig. 7.11. Collocated wells. Graphical superposition.

The result of calculations made for $|\Delta G/Q_s|$ and

$$\frac{r^2}{at_2} = 1$$

is plotted in Fig. 7.11, which also shows the graphical interpretation of the superposition procedure.

It is seen in the figure that a sudden interruption of the take-off does not produce an instantaneous rise of the water surface. The fall of the water surface continues for a time and does not change into a rise until after several moments. This process is very clearly defined at the time at which the basic function has the steepest slope, or in other words, where the basic function has a point of inflection. The position of this point is established by differentiating eqn. (7.6.22) twice with respect to time t , viz.

$$\frac{\partial^2 [\Delta G(r, t)]}{\partial t^2} = -\frac{Q_s}{4\pi t^2} \left(\frac{r^2}{4at} - 1 \right) \text{Exp} \left(-\frac{r^2}{4at} \right) \quad (7.6.77)$$

Since

$$\frac{\partial^2 [\Delta G(r, t)]}{\partial t^2} = 0$$

at the point of inflection, eqn. (7.6.77) implies that

$$\frac{r^2}{at} = 4.0 \quad (7.6.78)$$

The fastest fall of the water surface on a cylinder of radius r will occur at the time

$$t = \frac{r^2}{4a} \quad (7.6.79)$$

The corresponding time for the water surface in the well is given by the equation

$$t = \frac{r_s^2}{4a} \quad (7.6.80)$$

7.6.6. Effect of inertia forces inside a well

The theory explained in the previous subsection is not always applicable in practice; this fact was proved to be true in a field test of a well passing through coarse tertiary sand and gravel (with a k of the order of 1×10^{-2} m/s) overlain by materials of a relatively poor permeability (Fig. 7.12a).

A constant amount of water has been pumped from the well for a fairly long period of time by an oil engine-pump set. After the source of power was shut down, the effect of the inertia of the set came into play and the water surface increased continuously for some time. After a while, it started to fluctuate. The time diagram of the motion of the water surface in the well is shown in Fig. 7.12b.

The motion has a character of damped vibration. To gain further insight into the matter, we inserted a piston in the borehole and repeated the experiment. After shutting-down the source of power, we attempted to maintain the undamped amplitudes of the water surface by the application of a periodic external excitation force. We found that it was possible to do this and that the excitation force was not large, so long as its period approximately corresponded to the period of free vibration.

The phenomenon can be explained by the effect of inertia forces inside the well. In the course of a continuous take-off, the water inside the casing moves with a certain velocity (in the case being considered, the velocity at the borehole entrance was 0.5 m/s). After the pump has been stopped, the water surface tends to remain in motion, initiated by the inflow of groundwater and inertia. The water surface in the well overshoots the original, unlowered level, thus producing extra pressure which forces the water from the well into the underground. The water surface in the well then falls below the original, un-lowered level, and the process is repeated. The motion is damped by the resistance of the soil in the vicinity of the borehole, in the filling and on the entry to the well, and by pipe wall friction. These losses are characterized by the magnitude of the external excitation force which was applied to overcome the damping.

In attempting a theoretical analysis of the phenomenon we faced many problems associated, for example, with the definition of the damping force in the underground, inlet losses, etc., nonlinearity of the resulting partial differential equation for the water surface motion, difficulties with the solution, etc. We believe, however, that it may serve a useful purpose to show at least the fundamental properties of a simplified equation obtained by linearization, i.e. of the equation

$$\frac{\partial^2(\Delta h)}{\partial t^2} + 2\omega_G \frac{\partial(\Delta h)}{\partial t} + \omega_0^2 \Delta h = 0 \quad (7.6.81)$$

$$\Delta h(t) = h_s(t) - h_s(0) \quad (7.6.82)$$

where

- $h_s(t)$ is the ordinate of the water surface in the well at time t ,
- $h_s(0)$ — the ordinate of the un-lowered water surface in the well,
- ω_0 — the natural frequency of vibration,
- ω_G — the frequency of damping.

We can write, approximately, that

$$\omega_0 = \sqrt{\frac{g}{S + \frac{1}{2}T}} \quad (7.6.83)$$

$$\omega_G = \frac{1}{2(2S + T)} \left[\left(\frac{\lambda_T T + 3\lambda S}{3D} + \sum \xi + 1 \right) \frac{4Q_0}{\pi D^2} + \frac{g D^2 \ln(2R/D_0)}{4kT} \right] \quad (7.6.84)$$

The meaning of the quantities S , T is clearly seen in Fig. 7.12a,

- $\sum \xi$ — the sum of loss coefficients,
- λ_T, λ — the coefficients of friction in the perforated and solid part of the pipe in the borehole,
- D — the diameter of the pipe in the borehole,
- g — the acceleration of gravity,
- D_0 — the outside diameter of the borehole,
- k — the coefficient of permeability,
- Q_0 — the continuous take-off before pumping from the well was interrupted, (positive, whenever the flow is towards the inside of the well),
- R — the action radius.

Evaluation of the formulae is usually made difficult by poor knowledge of the values which appear in them. In this respect, the action radius R which may be a function of time, is one of the hardest to describe. It is, therefore, more convenient to determine ω_0 and ω_G experimentally.

The solution of eqn. (7.7.81) can be found in textbooks on vibration theory (e.g. Den Hartog [6]). We shall, therefore, present here only the resulting relation-

ships which can be derived for three special cases. All are based on the assumption that at the instant $t = 0$, when the take-off is interrupted, the mean profile velocity in the pipe is

$$v_0 = \frac{4Q_0}{\pi D^2} \quad (7.6.85)$$

Moreover, at the same instant, $t = 0$, the water surface in the well was lowered by $\Delta h = \Delta h_0$.

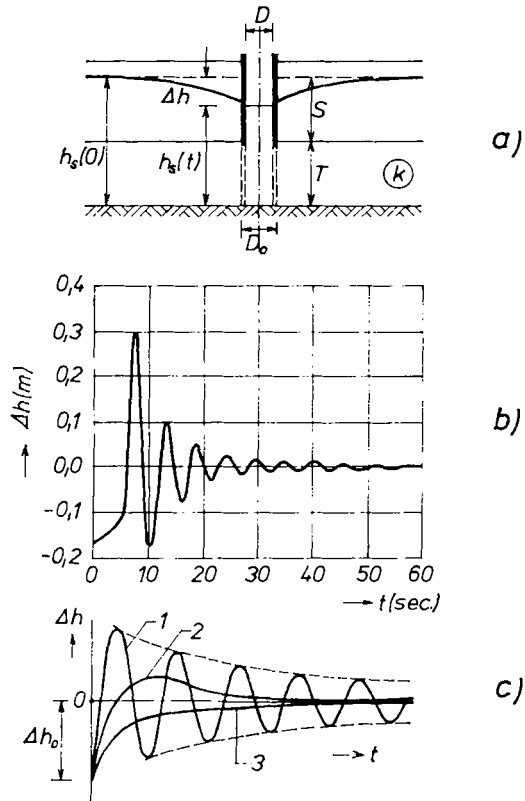


Fig. 7.12. Oscillations of the water surface in a well.

a) Case 1 – The motion of the water surface under subcritical damping (curve 1 in Fig. 7.12c) is characterized by the inequality

$$\omega_G < \omega_0$$

The vibration has a periodic character. Its frequency ω'_0 is obtained from the equation

$$\omega'_0 = \sqrt{(\omega_0^2 - \omega_G^2)} \quad (7.6.86)$$

The time interval for one period is given by

$$\Delta t = \frac{2\pi}{\omega'_0} \quad (7.6.87)$$

The fluctuations of the water surface in the well are described by the equation

$$\Delta h = \Delta h_0 \left[\frac{1}{\omega'_0} \left(\frac{v_0}{\Delta h_0} + \omega_G \right) \sin \omega'_0 t + \cos \omega'_0 t \right] \text{Exp}(-\omega_G t) \quad (7.6.88)$$

The water surface reaches the maximum level at the time

$$t = \frac{1}{\omega'_0} \text{arctg} \frac{v_0 \omega'_0}{\omega_G v_0 + \Delta h_0 \omega_0^2} \quad (7.6.89)$$

The elevation of the water surface above the original level is obtained from eqn. (7.6.88) after the substitution of eqn. (7.6.89).

The envelope of the maxima has a logarithmic form (Fig. 7.12c) and is characterized by the so-called logarithmic decrement of damping, defined by

$$\vartheta = \frac{2\pi\omega_G}{\omega'_0} \quad (7.6.90)$$

It is the logarithm of the ratio of the maximum deflections in two successive periods.

The water surface overshoots the initial, un-lowered level at the times given by

$$t = \frac{1}{\omega'_0} \text{arctg} \frac{-\omega'_0 \Delta h_0}{v_0 + \omega_G \Delta h_0} \quad (7.6.91)$$

The quantity of water which flows into the borehole, is determined by differentiating eqn. (7.6.88) with respect to time, viz.

$$Q = Q_0 \left(\cos \omega'_0 t - \frac{\omega_G v_0 + \Delta h_0 \omega_0^2}{v_0 \omega'_0} \sin \omega'_0 t \right) \text{Exp}(-\omega_G t) \quad (7.6.92)$$

b) Case 2 – The motion of the water surface under critical damping is characterized by the equality

$$\omega_G = \omega_0$$

The vibration has an a periodic character (curve 2 in Fig. 7.12c). The water surface quickly becomes stabilized on the original, un-lowered level. The motion is defined by the equation

$$\Delta h = [\Delta h_0 + (v_0 + \omega_0 \Delta h_0) t] \text{Exp}(-\omega_0 t) \quad (7.6.93)$$

During stabilization, the water surface is apt to overshoot the original, un-lowered level whenever the inequality

$$\frac{\Delta h_0}{v_0 + \Delta h_0 \omega_0} < 0 \quad (7.6.94)$$

is satisfied. The overshooting occurs at the time given by

$$t = - \frac{\Delta h_0}{v_0 + \Delta h_0 \omega_0} \quad (7.6.95)$$

and the maximum level is reached at the time defined by

$$t = \frac{v_0}{\omega_0(v_0 + \omega_0 \Delta h_0)} \quad (7.6.96)$$

The maximum value of Δh is obtained from eqn. (7.5.93) after the substitution of eqn. (7.6.96). The quantity of water Q which flows through the borehole, is determined from the equation

$$Q = Q_0 \left[1 - \left(1 + \frac{\omega_0 \Delta h_0}{v_0} \right) \omega_0 t \right] \text{Exp}(-\omega_0 t) \quad (7.6.97)$$

c) Case 3 — The supercritical damping is also aperiodic (curve 3 in Fig. 7.12c), characterized by the inequality

$$\omega_G > \omega_0$$

In the calculations, we make use of the auxiliary frequency

$$\bar{\omega}_0 = \sqrt{(\omega_G^2 - \omega_0^2)} \quad (7.6.98)$$

for which we have

$$\Delta h = \Delta h_0 \left[\cosh \bar{\omega}_0 t + \frac{1}{\bar{\omega}_0} \left(\frac{v_0}{\Delta h_0} + \omega_G \right) \sinh \bar{\omega}_0 t \right] \text{Exp}(-\omega_G t) \quad (7.6.99)$$

During stabilization, the water surface can overshoot the original, un-lowered level whenever the inequality

$$-1 < \frac{\bar{\omega}_0 \Delta h_0}{v_0 + \omega_G \Delta h_0} < 0 \quad (7.6.100)$$

is satisfied.

The time t during which the overshooting takes place is obtained from the equation

$$t = \frac{1}{\bar{\omega}_0} \text{argtgh} \frac{-\bar{\omega}_0 \Delta h_0}{v_0 + \omega_G \Delta h_0} \quad (7.6.101)$$

The time variation of the water surface motion has a maximum at a finite time so long as there is no overshooting of the water surface. The maximum level is reached at the time

$$t = \frac{1}{\bar{\omega}_0} \text{argtgh} \frac{\bar{\omega}_0 v_0}{\omega_0^2 \Delta h_0 - \omega_G v_0} \quad (7.6.102)$$

The corresponding maximum ordinate is calculated from eqn. (7.6.99) after the substitution of eqn. (7.6.102). The volume flow Q through the borehole, which varies with time, is determined by the equation

$$Q = Q_0 \left(\cosh \bar{\omega}_0 t - \frac{\omega_0^2 \Delta h_0 + \omega_G v_0}{\bar{\omega}_0 v_0} \sinh \bar{\omega}_0 t \right) \text{Exp}(-\omega_G t) \quad (7.6.103)$$

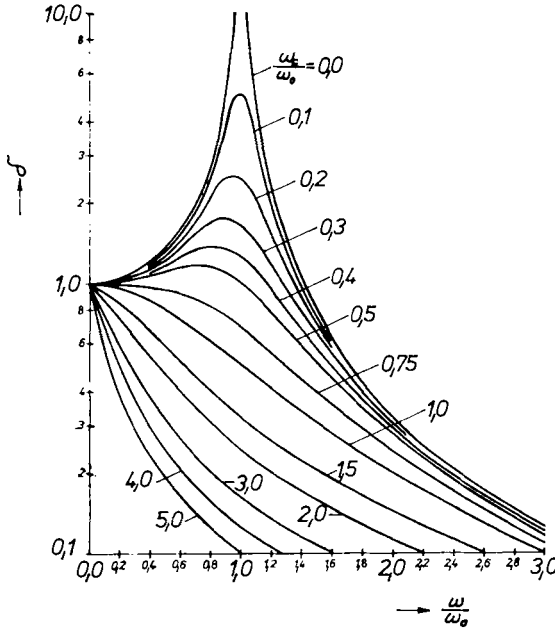


Fig. 7.13. Resonance curves.

The three fundamental solutions presented above enable us to analyze the dynamical character of the motion of the water surface in the well under the effects of an external periodic force. Assume first that a stationary load forces the water surface to a depth Δh_p below the original level. This action excites forced vibrations of the water surface, and the amplitude of these vibrations varies at first with time. Later, the vibrations become almost regular and their frequency is identical with the frequency ω of the excitation force. Denote the amplitude of these vibrations by Δh_{pm} .

The ratio

$$\delta = \frac{\Delta h_{pm}}{\Delta h_p}$$

is called the dynamic coefficient and expresses the ratio of the amplitude of the forced motion to the decrease of the water surface under the same but stationary force:

$$\delta = \frac{1}{\sqrt{\{[1 - (\omega/\omega_0)^2]^2 - 4(\omega/\omega_0)^2 (\omega_G/\omega_0)^2\}}}$$

The dependence of the dynamic coefficient on the natural frequency ω_0 and the damping frequency is shown in Fig. 7.13. The curves have a maximum at subcritical damping.

The corresponding frequency

$$\omega = \sqrt{(\omega_0^2 - 2\omega_G^2)}$$

is called the resonance frequency. For light damping, i.e. for $\omega_G \rightarrow 0$, the value of $\delta \rightarrow \infty$, and $\omega \rightarrow \omega_0$.

No resonance arises under supercritical damping. In this case, the amplitude of forced vibrations is always smaller than the fall of the water surface produced by the stationary force. The resonance curves drawn in Fig. 7.13 have no maxima.

Under actual conditions, the character of the water surface vibrations is not entirely the same as that predicted by the theory. Theory, on the other hand, explains the nature of the effect of inertia forces and gives reasons for the occurrence of resonance during our experiments. Resonance can only be achieved if the material in the neighbourhood of the well has a relatively high permeability. In fine materials the damping increases and, whenever $\omega_G \gg \omega_0$, we must adopt the assumptions which underlie the description of flow in terms of the G potential.

7.6.7. The balance method

In Chapter 5 we analyzed the conditions of plane steady flow into a large foundation pit or sand working shaft on the assumption that these objects can be regarded as wells of a relatively large diameter r_s . In an unsteady take-off, we make use of the underground inflow as well as of the quantities of water already contained in the well.

When water is pumped from large-diameter wells and the water surface moves at a comparatively slow rate, the effect of inertia forces can, therefore, be ignored.

For the sake of simplification suppose that the groundwater surface falls within the range of an overburden layer with a low permeability so that the effect of the term $k_{n+1}h_s$ on the coefficient a defined by eqn. (7.3.6) can be neglected. Since the initial water surface in the well and its neighbourhood is horizontal, we have, according to eqn. (1.4.83)

$$\Delta G(r, t) = -[h(r, t) - h(r, 0)] \sum_{i=1}^n T_i(k_i - k_{i+1}) = \mu_{n+1} a \Delta h(r, t)$$

or, more simply,

$$\Delta G(r, t) = \mu a \Delta h(r, t), \quad \Delta G(r, 0) = \Delta h(r, 0) = 0 \quad (7.6.104)$$

The groundwater inflow to the well, denoted by the letter Q_p is obtained from eqn.

(7.6.22) as

$$Q_p = - \frac{4\pi a \mu \Delta h(r_s, t)}{[-\text{Ei}(-r_s^2/4at)]} \quad (7.6.105)$$

Assume now that an average value can be used in calculations relating to the interval between the beginning of the take-off and the time Δt .

Hence

$$Q_p = - \frac{4\pi a \mu \Delta h(r_s, \Delta t)}{[-\text{Ei}(-r_s^2/4a \Delta t)]} \quad (7.6.106)$$

If the water surface falls by $\Delta h(r_s, t)$, the average take-off of water accumulated in the well is given by

$$Q_a = - \frac{\pi r_s^2 \Delta h(r_s, t)}{\Delta t} \quad (7.6.107)$$

From the beginning of the take-off to the instant t the well was supplied with rain-water. The average supply of such water per unit time is

$$Q_d = \frac{\pi r_s^2}{\Delta t} \int_0^{\Delta t} v_d dt \quad (7.6.108)$$

where v_d (a negative value) is the inflow of rainwater per unit time and unit area.

In the same interval of time, water evaporates from the well, and the average losses per unit time are described by the equation

$$Q_v = \frac{\pi r_s^2}{\Delta t} \int_0^{\Delta t} v_v dt \quad (7.6.109)$$

where v_v (a positive value) are the losses by evaporation per unit time and unit area.

The quantity of water pumped from the well per unit time (a negative value) is

$$Q = Q_a + Q_p + Q_d + Q_v \quad (7.6.110)$$

Hence for

$$|Q_a + Q_p| > |Q_d + Q_v|$$

$$Q = - \pi \Delta h(r_s, \Delta t) \frac{r_s^2 [-\text{Ei}(-r_s^2/4a \Delta t)] + 4a\mu\Delta t}{\Delta t [-\text{Ei}(-r_s^2/4a \Delta t)]} + Q_d + Q_v \quad (7.6.111)$$

$$\Delta h(r_s, \Delta t) = (Q_d + Q_v - Q) \frac{\Delta t [-\text{Ei}(-r_s^2/4a \Delta t)]}{r_s^2 [-\text{Ei}(-r_s^2/4a \Delta t)] + 4a\mu\Delta t} \quad (7.6.112)$$

The calculations illustrate the application of a method based on the balance equation (7.6.110), which enables us to study the conditions of quantitative balance and interpret physically the various terms of this equation in some detail.

7.7. Quasi-linear equations for axi-symmetrical flow through a saturated medium

We have assumed so far that in unsteady flow the groundwater surface moves within the range of a single layer. The solution of problems in which this is not the case, is based on the principles which were explained in Section 7.5. However, the nature of the problem compels us to consider, in addition to the layers in which we

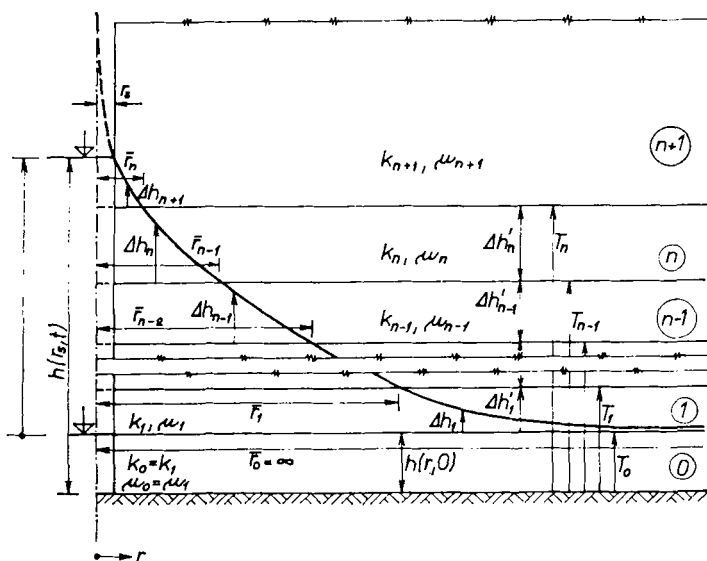


Fig. 7.14. Axi-symmetrical flow through an n -layer soil under constant inflow into the underground.

were concerned in that section, also the $(n + 1)$ th layer shown in Fig. 7.14. The ΔG potential for the free surface of this new configuration is described by the equation

$$\Delta G_{n+1} = -k_{n+1}(h_{n+1} - T_n) \left[\frac{h_{n+1} - T_n}{2} + \frac{1}{k_{n+1}} \sum_{j=0}^n k_j (T_j - T_{j-1}) \right] \quad (7.7.1)$$

Assuming that

$$\Delta h_{n+1} = h_{n+1} - T_n$$

this gives

$$\Delta G_{n+1} = -k_{n+1} \Delta h_{n+1} \left[\frac{\Delta h_{n+1}}{2} + \frac{1}{k_{n+1}} \sum_{j=0}^n k_j (T_j - T_{j-1}) \right] \quad (7.7.2)$$

The above equations lead to a reciprocal relation which makes it possible to deter-

mine the values of Δh_{n+1} for known values of ΔG_{n+1} , viz.

$$\Delta h_{n+1} = -\frac{1}{k_{n+1}} \sum_{j=0}^n k_j (T_j - T_{j-1}) - \left\{ \sqrt{1 - \frac{2 \Delta G_{n+1} k_{n+1}}{\left[\sum_{j=0}^n k_j (T_j - T_{j-1}) \right]^2}} \right\} \quad (7.7.3)$$

In the solution of the axi-symmetrical flow in a layer whose number is $N = 1, 2, 3, \dots, n$, we employ a system of nonlinear equations obtained by transforming the basic non-linear equations for plane flow into cylindrical coordinates

$$\frac{\partial(\Delta G_N)}{\partial t} = a_N \left[\frac{\partial^2(\Delta G_N)}{\partial r^2} + \frac{1}{r} \frac{\partial(\Delta G_N)}{\partial r} \right] \quad (7.7.4)$$

In addition, we also use the non-linear equations for the $(n+1)$ th and zero layers, i.e. the equations

$$\frac{\partial(\Delta G_{n+1})}{\partial t} = a_{n+1} \left[\frac{\partial^2(\Delta G_{n+1})}{\partial r^2} + \frac{1}{r} \frac{\partial(\Delta G_{n+1})}{\partial r} \right] \quad (7.7.5)$$

$$\frac{\partial(\Delta G_0)}{\partial t} = a_0 \left[\frac{\partial^2(\Delta G_0)}{\partial r^2} + \frac{1}{r} \frac{\partial(\Delta G_0)}{\partial r} \right] \quad (7.7.6)$$

A suitable choice of the average value of h_s enables us to assign to the coefficients a_N, a_{n+1}, a_0 the meaning of constants and thus convert the system of non-linear equations to a system of quasi-linear equations.

We know from the similarity theory that the solution of the equations can have the form of functions of the argument $r^2/a_N t$. Assume that the depression curve intersects all the interfaces between the layers so that the transition point $r = \bar{r}$ moves away from the well to the surroundings. Hence two arguments, i.e. $\bar{r}_N^2/a_N t$ and $\bar{r}_{N-1}^2/a_{N-1} t$ must be applicable to the transition points. The results of the solution, independently of time, must yield a zero value on all the interfaces. Hence we expect the values of the arguments to be time independent at the transition points. This can be achieved by setting, for example,

$$\bar{r}_N = c_N \sqrt{t} \quad (7.7.7)$$

where c_N is a constant.

Assume that the axis of symmetry of the flow region is identical with the axis of an hydraulically perfect well. The initial water surface lies on the level of the interface between the zero and the first layers whose hydraulic properties are the same (i.e. $k_1 = k_0, \mu_1 = \mu_0$). At the time $t = 0$, we start seepage into the underground and, at the same time, bring to the well the quantity $Q = Q_s = \text{const.}$

Within the range of the N -th layer, where $N = 1, 2, 3, 4, \dots, n$, we assume that

$$\bar{r}_N \leq r \leq \bar{r}_{N-1}, \quad \Delta G_N(\bar{r}_N, t) = \Delta G'_N, \quad \Delta G_N(\bar{r}_{N-1}, t) = 0 \quad (7.7.8)$$

Within the range of the $(n + 1)$ th layer we have

$$\lim_{r \rightarrow 0} 2\pi r \frac{\partial(\Delta G_{n+1})}{\partial r} = Q_s \quad (7.7.9)$$

Q_s is considered to be positive when water from the well seeps into the surroundings. On the interface between the zero and the first layer, $\bar{r}_0 = \infty$ and hence also $c_0 = \infty$.

The solution of eqn. (7.7.4) for the N -th layer, i.e. for the interval $\bar{r}_N \leq r \leq \bar{r}_{N-1}$ is found to be of the form

$$\Delta G_N(r, t) = \Delta G'_N \frac{F(r^2/4a_N t) - F(c_{N-1}^2/4a_N)}{F(c_N^2/4a_N) - F(c_{N-1}^2/4a_N)} \quad (7.7.10)$$

where

$$F(\zeta) = -\text{Ei}(-\zeta) = \int_{\zeta}^{\infty} \frac{e^{-\tau}}{\tau} d\tau \quad (7.7.11)$$

The volume flow which passes through a cylindrical surface of radius r belonging to the interval $\bar{r}_N \leq r \leq \bar{r}_{N-1}$ is defined by the equation

$$Q = -4\pi \Delta G'_N \frac{\text{Exp}(-r^2/4a_N t)}{F(c_N^2/4a_N) - F(c_{N-1}^2/4a_N)} \quad (7.7.12)$$

Solving eqn. (7.7.5) within the range of the $(n + 1)$ th layer, i.e. in the interval $0 \leq r \leq \bar{r}_n$, we obtain

$$\Delta G_{n+1}(r, t) = -\frac{Q_s}{4\pi} \left[F\left(\frac{r^2}{4a_{n+1}t}\right) - F\left(\frac{c_n^2}{4a_{n+1}}\right) \right] \quad (7.7.13)$$

The volume flow through a cylindrical surface of radius r belonging to that interval is described by the equation

$$Q = 2\pi r \frac{\partial(\Delta G)}{\partial r} = Q_s \text{Exp}\left(-\frac{r^2}{4a_{n+1}t}\right) \quad (7.7.14)$$

The values of c_N are determined from the equation of continuity at the transition points. On the interface between the n -th and the $(n + 1)$ th layer we have

$$-\frac{Q_s}{\Delta G'_n} = 4\pi \frac{\text{Exp}(-c_n^2/4a_n)}{\text{Exp}(-c_n^2/4a_{n+1}) [F(c_n^2/4a_n) - F(c_{n-1}^2/4a_n)]} \quad (7.7.15)$$

and at the other transition points

$$\frac{\Delta G'_N}{\Delta G'_{N-1}} = \frac{\text{Exp}(-c_{N-1}^2/4a_{N-1})}{\text{Exp}(-c_{N-1}^2/4a_N)} \frac{F(c_N^2/4a_N) - F(c_{N-1}^2/4a_N)}{F(c_{N-1}^2/4a_{N-1}) - F(c_{N-2}^2/4a_{N-1})} \quad (7.7.16)$$

The required values of c_N are obtained by solving a system of transcendental equations. On the assumption that $c_0 = \infty$ for a rise of the water surface within the range of two layers, we find it sufficient to use a single transcendental equation, viz.

$$-\frac{Q_s}{\Delta G'_1} = 4\pi \frac{\text{Exp}(-c_1^2/4a_1)}{\text{Exp}(-c_1^2/4a_2) F(c_1^2/4a_1)} \quad (7.7.17)$$

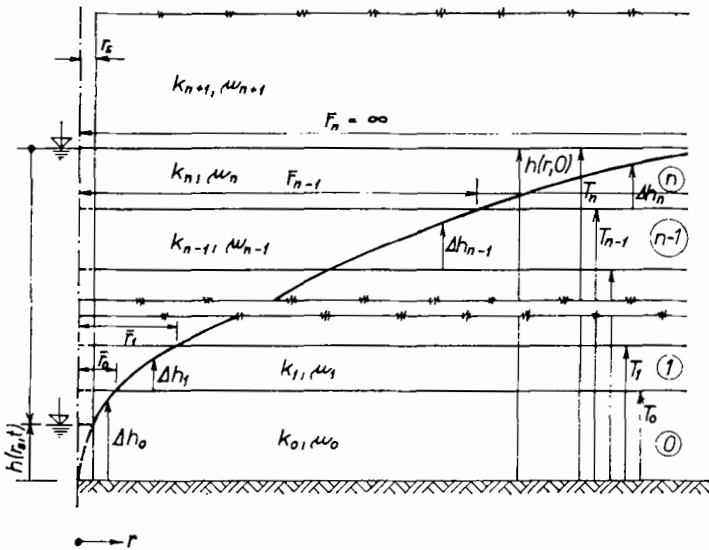


Fig. 7.15. Axi-symmetrical flow through an n -layer soil under a constant take-off from a well.

The solution consists of choosing successively various values of c_1 for given values of a_1, a_2 , calculating the ratio $Q_s/\Delta G'_1$, plotting the results, and reading off from the diagram the value of c_1 which corresponds to the required ratio $Q_s/\Delta G'_1$.

The equations derived above describe the discharge (seepage into the underground) from an infinitely narrow well. The position of the water surface in an actual well of a certain radius is obtained, approximately, by substituting $r = r_s$ in the equations. In the range of the N -th layer we clearly have

$$\Delta G_N(r_s, t) = \Delta G'_N \frac{F(r_s^2/4a_N t) - F(c_{N-1}^2/4a_N)}{F(c_N^2/4a_N) - F(c_{N-1}^2/4a_N)} \quad (7.7.18)$$

Eqn. (7.7.18) is applicable whenever

$$\frac{r_s^2}{c_{N-1}^2} \leq t \leq \frac{r_s^2}{c_N^2}$$

At the time $t > r_s^2/c_n^2$ the water surface in the well will reach up to the interface of the $(n + 1)$ th layer where

$$\Delta G_{n+1}(r_s, t) = -\frac{Q_s}{4\pi} \left[F\left(\frac{r_s^2}{4a_{n+1}t}\right) - F\left(\frac{c_n^2}{4a_{n+1}}\right) \right] \quad (7.7.19)$$

The solution of the case when a constant take-off Q_s from the well starts to take place at the time $t = 0$, is similar to that just discussed. The initial water surface is made identical with the upper boundary of the n -th layer, and it is assumed that $c_n = \infty$ there. In the N -th layer we have (Fig. 7-15)

$$\bar{r}_N \leq r \leq \bar{r}_{N-1}, \quad \Delta G(\bar{r}_N, t) = \Delta G'_N, \quad \Delta G_N(\bar{r}_{N-1}, t) = 0 \quad (7.7.20)$$

and within the range of the zero layer,

$$\lim_{r \rightarrow \infty} 2\pi r \frac{\partial(\Delta G_0)}{\partial r} = Q_s \quad (7.7.21)$$

The pumped quantity Q_s is regarded as negative when water from the underground seeps into the well.

For the N -th layer, i.e. for the interval $\bar{r}_{N-1} \leq r \leq \bar{r}_N$ we use eqn. (7.7.4) in the form (7.7.10), and find the corresponding volume flow from eqn. (7.7.12).

Within the range of the zero layer, i.e. in the interval $0 \leq r \leq \bar{r}_0$, we use eqn. (7.7.6) which yields, assuming the validity of eqn. (7.7.21),

$$\Delta G_0(r, t) = -\frac{Q_s}{4\pi} \left[F\left(\frac{r^2}{4a_0t}\right) - F\left(\frac{c_0^2}{4a_0}\right) \right] \quad (7.7.22)$$

$$Q = 2\pi r \frac{\partial(\Delta G_0)}{\partial r} = Q_s \text{Exp}\left(-\frac{r^2}{4a_0t}\right) \quad (7.7.23)$$

The equation of continuity at the transition points is as follows:

$$\frac{\Delta G'_N}{\Delta G'_{N-1}} = \frac{\text{Exp}(-c_{N-1}^2/4a_{N-1})}{\text{Exp}(-c_{N-1}^2/4a_N)} \frac{F(c_N^2/4a_N) - F(c_{N-1}^2/4a_N)}{F(c_{N-1}^2/4a_{N-1}) - F(c_{N-2}^2/4a_{N-1})} \quad (7.7.24)$$

$$-\frac{Q_s}{\Delta G'_1} = 4\pi \frac{\text{Exp}(-c_0^2/4a_1)}{\text{Exp}(-c_0^2/4a_0) [F(c_1^2/4a_1) - F(c_0^2/4a_1)]} \quad (7.7.25)$$

Eqns. (7.7.24) and (7.7.25) essentially represent a system of transcendental equations whose solution leads to the required values of the constants c_0, c_N .

In the simpler case when the water surface moves within the range of two layers, i.e. within the range of the first and the zero layers, the following single transcendental equation suffices for the determination of the constant c_0 :

$$\frac{Q_s}{\Delta G'_1} = 4\pi \frac{\text{Exp}(-c_0^2/4a_1)}{\text{Exp}(-c_0^2/4a_0) F(c_0^2/4a_1)} \quad (7.7.26)$$

The equations put forward above describe the effect of pumping from an infinitely narrow well on the development of the groundwater surface in the neighbourhood of the well. The conditions in an actual well having a certain finite radius are approximately determined by substituting $r = r_s$ in the equations. Within the range of the N -th layer we make use of eqn. (7.7.18) which holds for

$$\frac{r_s^2}{c_N^2} \leq t \leq \frac{r_s^2}{c_{N-1}^2} \quad (7.7.27)$$

At the time $t > r_s^2/c_0^2$ the water surface in the well will reach the interface of the zero layer where

$$\Delta G(r_s, t) = -\frac{Q_s}{4\pi} \left[F\left(\frac{r_s^2}{4a_0 t}\right) - F\left(\frac{c_0^2}{4a_0}\right) \right] \quad (7.7.28)$$

Eqn. (7.7.28) continues to apply until the instant when the water surface in the well reaches the surface of the relatively impervious subsoil. The applicability of the theory is restricted by imposing the condition

$$\Delta G_0(r_s, t) \leq 0 \quad (7.7.29)$$

In conclusion, let us take note of the fact that quasi-linear equations are essentially a special case of non-linear equations. By choosing a nearly infinitesimal thickness of layers having identical properties we could obtain a solution of the non-linear Boussinesq equation describing axi-symmetrical flow.

7.8. Non-linear problems of one-dimensional and axi-symmetrical flow through a saturated medium

The non-linearity of Boussinesq's equation results from the fact that the parameters are dependent on a dependent variable. Yet the law of flow is intrinsically linear. Solutions of one-dimensional and axi-symmetrical problems become complicated as soon as this assumption can no longer be used.

In flows with a free surface which can be described by means of the theory of plane seepage, different layers of the permeable medium behave differently. Definite laws apply to each layer. Moreover, the free surface can pass from layer to layer. Along a given vertical line, the character of flow is subject to changes which depend not only on the magnitude of the gradient, but on the overall thickness of the saturated medium as well.

The quality of flow is sometimes determined by one or several layers contained in the permeable stratum, while the effect of the remaining layers is only secondary. In such cases a description of water motion is somewhat simpler.

Consider, for example, the case of one-dimensional flow with the free surface lying within the range of a layer with a relatively poor permeability, of active porosity μ . One or several layers of decisive importance lie inside the saturated stratum. Denote the ordinates of the free surface by $h(x, t)$, and assume that the initial level $h(x, 0) = \text{const}$.

Since the stratum has horizontal contact surfaces between the layers, the water is expected to move predominantly in the horizontal direction. A combined law of flow can, therefore, be set up for each layer to determine the quantity of flow corresponding to a definite hydraulic gradient. This process can be simplified by using a graphical representation of the relation

$$Q = f \left| \frac{\partial(\Delta h)}{\partial x} \right| \quad (7.8.1)$$

$$\Delta h = h(x, t) - h(x, 0) \quad (7.8.2)$$

where Q is the quantity of flow passing through each vertical line.

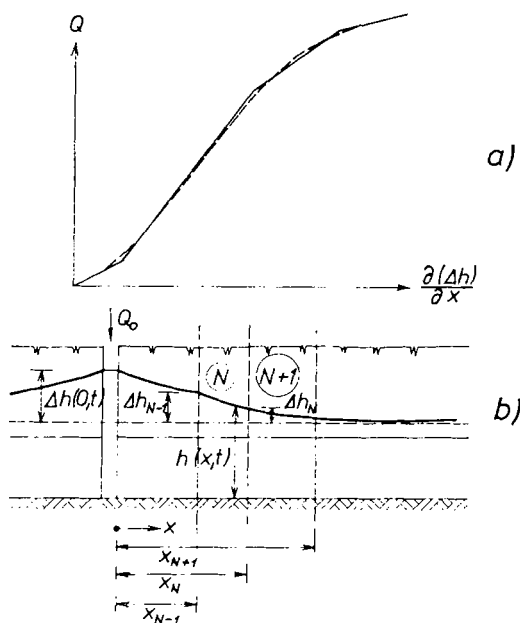


Fig. 7.16. Seepage of water from a narrow hydraulically perfect cut.

The law established in the above way is approximated by a set of straight lines linking onto one another. Within the range of each straight line, the law is linear (Fig. 7.16a). If the controlling layers are of equal thickness, the overall characteristic seepage coefficient evidently varies so as to make the ratio of the seepage activities in the various parts of the broken line assume a certain value which is constant within the range of one segment.

Within certain sections of the one-dimensional flow we can, therefore, assume the law of flow to be linear, viz.

$$\frac{\partial(\Delta h)}{\partial t} = a_N \frac{\partial^2(\Delta h)}{\partial x^2} \quad (7.8.3)$$

$$a_N = \frac{k_N T}{\mu} \quad (7.8.4)$$

$$k_N = \frac{Q}{T[\partial(\Delta h)/\partial x]_N} \quad (7.8.5)$$

where T is the characteristic thickness of the controlling layers below the water surface.

On the contact of two adjoining segments of one-dimensional flow, we have

$$\frac{k_N}{k_{N-1}} = \frac{1}{\kappa_N} \quad (7.8.6)$$

where κ_N is a constant.

As eqn. (7.8.5) implies, it must hold on the contact of two neighbouring layers that

$$\left. \frac{\partial}{\partial x} (\Delta h) \right|_{x=x_N} = \text{const.} \quad (7.8.7)$$

In keeping with the similarity theory, one of the parameters of the solution is the ratio

$$\left\{ \frac{l^2}{at} \right\} \quad (7.8.8)$$

where l is the length.

The boundary conditions are determined in a way to encourage the broadest possible intrinsic self-similarity. Consider, for example, that

$$\frac{\partial}{\partial x} (\Delta h) = F \left[\frac{x}{\sqrt{(at)}} \right] \quad (7.8.9)$$

Eqn. (7.8.7) should be valid on the contact of the segments independently of time, and this is possible under the assumption that

$$\frac{x_N}{\sqrt{t}} = c_N \quad (7.8.10)$$

where t is the time, and c_N a constant for the N -th contact of the adjoining segments.

Hence the problem being considered consists of finding a solution to eqn. (7.8.3) which satisfies the prescribed boundary and initial conditions as well as the condition (7.8.10). It is, of course, assumed that on the contact of the adjoining segments, the value Δh is independent of whether the vertical line is approached from the left or from the right.

To illustrate the procedure, consider the seepage into a narrow, hydraulically perfect cut incapable of storing water on its own (Fig. 7.16b). The water which is being supplied to it, seeps symmetrically to both sides. The quantity entering one side is given by

$$Q(0, t) = \frac{Q_0}{2} = \text{const.}$$

Hence the inlet profile clearly indicates the point on the curve shown in Fig. 7.16a, with which we are at present dealing. A time development of the water surface takes place in the earth massif, and a shift of the vertical lines between the segments in which a linear law of flow is assumed to apply, is expected to occur as a result.

The solution for the N -th segment, i.e. for $x_{N-1} \leq x \leq x_N$ is presented in the form

$$\Delta h = A_N \sqrt{(4a_N t)} \left\{ \text{Exp} \left(-\frac{x^2}{4a_N t} \right) - \frac{x \sqrt{\pi}}{\sqrt{(4a_N t)}} \text{Erfc} \left[\frac{x}{\sqrt{(4a_N t)}} \right] \right\} \quad (7.8.11)$$

$$\frac{\partial(\Delta h)}{\partial x} = A_N \sqrt{\pi} \text{Erfc} \left[\frac{x}{\sqrt{(4a_N t)}} \right] \quad (7.8.12)$$

where A_N is a constant valid in the N -th segment.

On the inlet vertical line, i.e. on the left-hand edge of the first segment we have

$$Q = k_1 T \frac{\partial}{\partial x} (\Delta h) \Big|_{x=0} = \frac{Q_0}{2} \quad (7.8.13)$$

For $x = 0$, eqn. (7.8.12) yields

$$A_1 = \frac{Q_0}{2k_1 T \sqrt{\pi}} \quad (7.8.14)$$

where k_1 is the characteristic value of the coefficient of permeability in the first segment, including the bounding inlet profile.

On the contact between the remaining segments, we obtain for the gradient from the left and from the right that

$$\frac{A_N \text{Erfc } c_N / \sqrt{(4a_N)}}{A_{N-1} \text{Erfc } c_N / \sqrt{(4a_{N-1})}} = \kappa_N \quad (7.8.15)$$

The condition of equality of Δh from the left and from the right leads to

$$\frac{A_N}{A_{N-1}} \sqrt{\frac{a_N}{a_{N-1}}} \frac{\text{Exp}(-c_N/4a_N) - c_N \sqrt{\pi/\sqrt{4a_N}} \text{Erfc}[c_N/\sqrt{4a_N}]}{\text{Exp}(-c_N/4a_{N-1}) - c_N \sqrt{\pi/\sqrt{4a_{N-1}}} \text{Erfc}(c_N/4a_{N-1})} = 1 \quad (7.8.16)$$

Expressions (7.8.15) and (7.8.16) represent a system of equations which enable us to calculate the values of the constants A_N , c_N .

The basic idea of the procedure just outlined is applicable to the solutions of various cases, including that of axi-symmetrical flow. Sometimes, however, no analytical solution can be obtained and the problem must be solved numerically. Naturally, this can be done only if the seepage characteristics of the materials and the fundamental laws of flow are known.

7.9. Inverse problems under predominantly horizontal axi-symmetrical motion

A geological and hydrogeological survey describes the structure of a permeable medium. The overall regional situation is made clearer by the results of probing and an analysis of the samples of minerals. The grain composition and other engineering properties of soils are studied in the laboratories. All these data are processed, frequently by means of statistical methods, to provide more detailed information needed in hydrotechnical calculations.

Sometimes these procedures fail, sometimes too many details are not warranted economically. What is worse still, is the fact that methodological errors are frequently committed, for example, by taking samples of soils from water, which do not contain fine fractions, etc.

This is the reason why the assortment of procedures employed in geological prospecting also includes methods which make it possible to investigate the properties of permeable media in large complexes. The parameters which are used to appraise these properties, instead of being related to the structure of the actual materials, are quantities which, though apparently possessing a physical meaning, in reality assess only some, more or less average properties of certain sections of the permeable space. They are usually established on the basis of analyses of field tests, arranged so that the mathematical description of the processes being observed is available beforehand. The parameters appearing in the equations are then obtained by proceeding in reverse.

Very convenient to analyze are the processes and phenomena which are amenable to a description by means of functions of the similarity indicators. These functions are determined analytically or numerically. The parameters which are of interest are best established by comparing the diagrams of the calculated relationship with those obtained by field tests.

Inverse problems are easier to solve when an analytical solution of the process being studied is already known. A method which has found widespread use, is the evaluation of pumping tests of wells, preferably hydraulically perfect wells. In these tests, water is pumped from the well, and observations are made of the motion of the water surface inside as well as outside the object. The endeavour to achieve harmony of a sort between the observed reality and the mathematical description of the pumping test, is facilitated by a suitable choice of the seepage characteristics.

The description of plane flow has so far been based on the theory of the G potential. The G potential is a function defined for stratified media for which the seepage characteristics of the layers and the position of the interfaces between the layers are known in advance. Such data are, however, not available before the beginning of the pumping tests. Since the information concerning the character of the stratification is likely to be meagre and incomplete, it is necessary to base the solution of inverse problems on certain schemes which, however, stem from the fundamental theory.

It may serve a useful purpose to discuss the subject at some length, although a brief explanation might do as well so far as a mere instruction of how to solve inverse problems is concerned.

Consider the flow into (or from) a well to be axially symmetrical. In the previous sections, we described such a flow by the equation

$$\frac{\partial(\Delta G)}{\partial t} = \frac{k_{n+1}h_s + \sum_{i=1}^n T_i(k_i - k_{i+1})}{\mu_{n+1}} \left[\frac{\partial^2(\Delta G)}{\partial r^2} + \frac{1}{r} \frac{\partial(\Delta G)}{\partial r} \right] \quad (7.9.1)$$

$$r_0 \leq r < \infty$$

$$a = \frac{k_{n+1}h_s + \sum_{i=1}^n T_i(k_i - k_{i+1})}{\mu_{n+1}} \quad (7.9.2)$$

$$\Delta G(r, t) = -k_{n+1} \frac{h^2(r, t) - h^2(r, 0)}{2} - [h(r, t) - h(r, 0)] \sum_{i=1}^n T_i(k_i - k_{i+1}) \quad (7.9.3)$$

where

- h_s — a certain mean value of the piezometric head,
- t — time,
- k_i, T_i — coefficient of permeability and the coordinate of the upper edge of the i -th layer,
- k_{n+1} — coefficient of permeability of the layer containing the free surface,
- $h(r, t)$ — piezometric head of a point on the water surface,
- $h(r, 0)$ — characterizes the initial level,
- μ_{n+1} — active porosity of the layer containing the free surface.

It is clear from the description that the motion is assumed to be unsteady, with the free surface moving only within the range of the $(n + 1)$ th layer. This layer is underlain by a single layer of thickness $\bar{T}$, which can be chosen at will, for example, as $\bar{T} = T_n$. The characteristic coefficient of permeability in the horizontal direction is denoted by $\bar{k}$; this coefficient enables us to define the permeability of the various layers as follows:

$$\begin{aligned} k_1 &= \bar{k}A_1 \\ k_2 &= \bar{k}A_2 \\ &\vdots \\ k_n &= \bar{k}A_n \\ k_{n+1} &= \bar{k}A_{n+1} \end{aligned} \quad (7.9.4)$$

The average seepage characteristics in the region underlying the layer in which the free surface exists, can be described by eqn. (1.4.46). Using this equation, we can simplify eqn. (7.9.3) as follows:

$$\Delta G(r, t) = -\bar{k} \left\{ A \frac{h^2(r, t) - h^2(r, 0)}{2} + \bar{T}(1 - A) [h(r, t) - h(r, 0)] \right\} \quad (7.9.5)$$

where

$$A = A_{n+1}$$

Writing

$$a = \frac{\bar{k}}{\mu} [Ah_s + \bar{T}(1 - A)] \quad (7.9.6)$$

where

$$\mu = \mu_{n+1}$$

we can simplify eqn. (7.9.2) in a similar way.

Let us introduce a new function, viz.

$$\Delta G_p(r, t) = \frac{\Delta G(r, t)}{\bar{k}\bar{T}(1 - A)} = - \left[B \frac{h^2(r, t) - h^2(r, 0)}{2} + h(r, t) - h(r, 0) \right] \quad (7.9.7)$$

where

$$B = \frac{\bar{k}A}{\bar{k}\bar{T}(1 - A)} = \frac{\bar{k}}{\bar{k}\bar{T}(1 - A)} \quad (7.9.8)$$

and $\bar{h}$ is the coefficient of permeability of the layer containing the free surface.

Using the symbols

$$\overline{\Delta h} = h(r, t) - h(r, 0) \quad (7.9.9)$$

$$\overline{\Delta h^2} = h^2(r, t) - h^2(r, 0) \quad (7.9.10)$$

we can write

$$\Delta G_p(r, t) = - \left(\overline{\Delta h} \frac{B}{2} + \overline{\Delta h} \right) \quad (7.9.11)$$

$$a = \frac{\overline{k} h_s + \overline{k} \overline{T}(1 - A)}{\mu} \quad (7.9.12)$$

Substituting in eqn. (7.9.1) we find that

$$\frac{\partial(\Delta G_p)}{\partial t} = a \left[\frac{\partial^2(\Delta G_p)}{\partial r^2} + \frac{1}{r} \frac{\partial(\Delta G_p)}{\partial r} \right] \quad (7.9.13)$$

and further,

$$\frac{1}{a} \frac{\partial}{\partial t} (\overline{\Delta h}) - \frac{\partial^2}{\partial r^2} (\overline{\Delta h}) - \frac{1}{r} \frac{\partial}{\partial r} (\overline{\Delta h}) = -C_1 \quad (7.9.14)$$

$$\frac{1}{a} \frac{\partial}{\partial t} (\overline{\Delta h}) - \frac{\partial}{\partial r^2} (\overline{\Delta h}) - \frac{1}{r} \frac{\partial}{\partial r} (\overline{\Delta h}) = C_2 \quad (7.9.15)$$

where C_1 is a constant, and C_2 is also a constant which depends on C_1 as follows

$$C_2 = \frac{2C_1}{B} = 2C_1 \frac{\overline{k} \overline{T}(1 - A)}{\overline{k}} \quad (7.9.16)$$

Eqns. (7.9.14) and (7.9.15) apply simultaneously and can be used to calculate the value of a , assuming that we know the variations of the functions $\overline{\Delta h}(r, t)$ and $\overline{\Delta h}(r, t)$. Usually only one of these functions suffices for the calculations, its choice depending on the geological conditions for which the functions were determined.

Assume, for example, that a pressure flow for which $\overline{k} \overline{T} / \overline{k} \rightarrow \infty$ predominates in the region being considered. Hence $B \rightarrow 0$, and only $\overline{\Delta h}$ will come into play in eqn. (7.9.11). Since eqn. (7.9.15) is to apply at the same time, C_1 must tend to zero according to eqn. (7.9.16). Therefore, we shall obtain the value of a by using eqn. (7.9.14) for $C_1 = 0$.

The quantity of water, Q , which passes through a cylinder of radius r per unit time is described by the equation

$$Q = 2\pi r \frac{\partial(\Delta G)}{\partial r} = 2\pi r \overline{k} \overline{T}(1 - A) \frac{\partial(\overline{\Delta h})}{\partial r} \quad (7.9.17)$$

Consider now the other extreme case, i.e. the predominant flow is that which takes place in the layer containing the free surface. Assume that $\overline{k} \overline{T} / \overline{k} \rightarrow 0$. By eqn. (7.9.16), $C_2 \rightarrow 0$ for any C_1 .

The quantity of water which passes through a cylinder of radius r per unit time is obtained from the equation

$$Q = 2\pi r \frac{\partial(\Delta G)}{\partial r} = 2\pi r \bar{k} \bar{T} (1 - A) \frac{\partial(\Delta G_p)}{\partial r} = \pi r \bar{k} \frac{\partial(\overline{\Delta h})}{\partial r} \quad (7.9.18)$$

Under certain conditions, the constant a can also be calculated on the basis of the results of observations of the phenomenon which is directly described by eqn. (7.9.13). To clarify this statement, we use eqn. (7.9.7) written down in the form

$$\begin{aligned} \Delta G_p(r, t) &= - \left\{ \frac{B}{2} [h(r, t) - h(r, 0)] [h(r, t) + h(r, 0)] + h(r, t) - h(r, 0) \right\} = \\ &= -\overline{\Delta h} \left(1 + B \frac{h(r, t) + h(r, 0)}{2} \right) \end{aligned} \quad (7.9.19)$$

The fraction in the parentheses on the right-hand side of eqn. (7.9.19) is replaced by the mean value h_s which is regarded as a constant in the whole region of flow. For an axi-symmetrical flow we choose it to be, for example,

$$h_s = h(r, 0) + \frac{1}{\bar{t}} \int_0^{\bar{t}} \left\{ \frac{1}{R_n^2 - r_s^2} \int_{r_s}^{R_n} [h(r, t) - h(r, 0)] r dr \right\} dt \quad (7.9.19a)$$

where

R_n is the action radius of the well,

r_s — the radius of the outer contour of the well,

$\bar{t}$ — the time between the beginning and the end of the pumping test.

Since a practical solution of eqn. (7.9.19a) is tedious, we choose h_s by means of a reasonable estimate, taking into account the conditions of the particular pumping test. What is essentially important is the fact that the substitution of the mean value $h = h_s$ leads to a simplified equation, viz.

$$\Delta G_p(r, t) = -\overline{\Delta h} (1 + B h_s) \quad (7.9.20)$$

On substituting it in eqn. (7.9.13), we find that we can use eqn. (7.9.14) for $C_1 = 0$.

The quantity of water which passes through a cylinder of radius r per unit time is obtained from the equation

$$\begin{aligned} Q &= 2\pi r \frac{\partial(\Delta G)}{\partial r} = 2\pi r [k \bar{T} (1 - A) + \bar{k} h_s] \frac{\partial(\overline{\Delta h})}{\partial r} = \\ &= 2\pi r \bar{k} [A h_s + \bar{T} (1 - A)] \frac{\partial(\overline{\Delta h})}{\partial r} \end{aligned} \quad (7.9.21)$$

As eqn. (7.9.21) implies, the flow is regarded as a common effect of two collocated streams, both having the character of a pressure flow. One takes into account the thickness $\bar{T}$, the other the thickness h_s .

The expression in the first square brackets on the right-hand side of eqn. (7.9.21) also appears in the numerator of the fraction in eqn. (7.9.12). It represents the product of a fictitious (characteristic) coefficient of permeability k , and the fictitious (characteristic) thickness of the saturated stratum, T . For $k = \bar{k}$, $T = Ah_s + \bar{T}(1 - A)$, we have

$$a = \frac{kT}{\mu} \quad (7.9.22)$$

The product kT has the dimension of m^2/s , and is called the coefficient of transmissivity. In a pressure-free flow, it is a quantity which depends to a greater or lesser degree on the boundary conditions.

In a clearly defined pressure flow, the product is a constant because $A \rightarrow 0$. Its determination is the main object of the solution of a number of inverse problems. The starting point of its calculation is the value a to which we referred above as quantity containing terms whose knowledge is presupposed. The active porosity of the layer in which the free surface moves, can, for example, be considered to be such a known parameter.

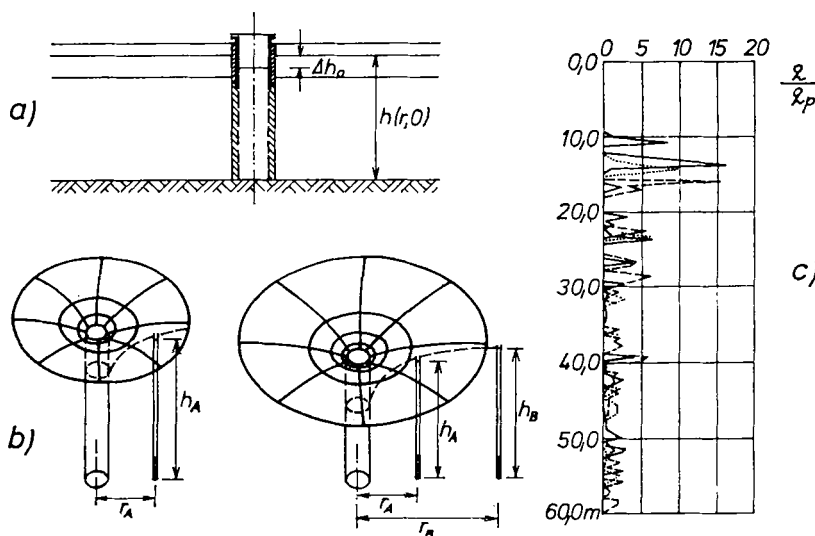


Fig. 7.17.

All three cases described in the foregoing, lead to the conclusion that an inverse problem can be solved by the application of an equation of the type

$$\frac{\partial(\Delta h)}{\partial t} = a \left[\frac{\partial^2(\Delta h)}{\partial r^2} + \frac{1}{r} \frac{\partial(\Delta h)}{\partial r} \right] \quad (7.9.23)$$

Depending on the nature of the problem at hand, we can write $\Delta h = \overline{\Delta h}$, or $\Delta h = \overline{\Delta h}$, or use a third alternative based on the concept of the mean value, h_s .

Eqn. (7.9.23) was derived without paying heed to the effect of inertia forces of the water flow either in the soil or in the well. It is, therefore, helpful to establish whether or not this condition is being satisfied during the field tests, especially those designed to study flow in coarse materials. An arrangement of an experiment directed towards that end, is shown schematically in Fig. 7.17a.

The well opening is covered with a lid, and compressed air is admitted into the space between the water surface and the lid. The water surface in the well drops by Δh_0 . After the water level has become stabilized, the air is suddenly released and observations are made of the time development of the drop in the water surface in the object, $\Delta h(t)$, in order to establish whether its character is periodic or aperiodic. The fluctuations of the water surface are represented graphically, and the record is used in the first preparatory stage of the solution of the inverse problem. The object of the solution is to establish the natural frequency ω_0 and the damping frequency ω_G .

The theory is founded on the solution described in Section 7.6.3, which assumes that eqn. (7.6.81) applies under the conditions

$$\Delta h(0) = \Delta h_0 \quad (7.9.24)$$

$$\left. \frac{\partial(\Delta h)}{\partial t} \right|_{t=0} = 0 \quad (7.9.25)$$

Since a description of the respective mathematical operations is of no essential importance from the point of view of subsequent exposition, the solution of the initial equation will not be given here. We shall sum up the results as follows:

In the presence of subcritical damping $\omega_G < \omega_0$, the vibration has a periodic character with a frequency $\omega'_0 = \sqrt{(\omega_0^2 - \omega_G^2)}$ and the time Δt for one period. The logarithmic decrement of damping is obtained from the record of the water surface fluctuations, i.e.

$$\vartheta = \ln \frac{\Delta h_1}{\Delta h_2} \quad (7.9.26)$$

where $\Delta h_1, \Delta h_2$ are the maximum deflections in two successive periods.

Since it holds at the same time that

$$\vartheta = \frac{2\pi\omega_G}{\omega'_0}, \quad \Delta t = \frac{2\pi}{\omega'_0} \quad (7.9.27)$$

we obtain the frequency of damping as

$$\omega_G = \frac{\vartheta}{\Delta t} \quad (7.9.28)$$

The natural frequencies are determined from the formula

$$\omega_0 = \frac{1}{\Delta t} \sqrt{(4\pi^2 + \vartheta^2)} \quad (7.9.29)$$

In the presence of supercritical damping, $\omega_G > \omega_0$ and the vibration is aperiodic. The characteristic frequency $\bar{\omega}_0$ is obtained from the relation

$$\bar{\omega}_0 = \sqrt{(\omega_G^2 - \omega_0^2)} \quad (7.9.30)$$

Assume that after the sudden release of compressed air, the water surface will return slowly to the original level, without overshooting it. The function $\Delta h(t)$ has a point of inflection at the time $t = t_{\text{infl}}$. To this time corresponds the ordinate $\Delta h = \Delta h_{\text{infl}}$. From the graphical representation of the function $\Delta h(t)$ we find the derivative $[\partial(\Delta h)/\partial t]_{\text{infl}}$, and denote the absolute value of the ratio

$$\left| \frac{\Delta h}{\partial(\Delta h)/\partial t} \right|_{\text{infl}} = C \quad (7.9.31)$$

by the letter C .

In addition, we also determine the absolute value of the ratio

$$\left| \frac{\Delta h}{\partial(\Delta h)/\partial t} \right| = D \quad (7.9.32)$$

at the time $t \gg t_{\text{infl}}$.

The damping frequency ω_G is then calculated using the formula

$$\omega_G = \frac{C}{2D(C - D)} \quad (7.9.33)$$

and the natural frequency ω_0 from the relation

$$\omega_0 = \sqrt{\frac{1}{D(C - D)}} \quad (7.9.34)$$

Since the case of critical damping when $\omega_G = \omega_0$, is very unlikely, it will not be discussed here.

For the determination of the coefficient a , which is the second and principal stage of the solution of the inverse problem, the compressed-air experiment is not suitable. A more straightforward way is an analysis of the motion which starts to take place as a result of the periodic fluctuations of the water surface in the well (Section 7.6.3). This motion can be induced by the application of a periodic external force with a frequency ω . The amplitude of this force is determined and a suitable frequency is chosen on the basis of the resonance curves shown in Fig. 7.13. It is clear from the diagrams that in the presence of subcritical damping, ω should be chosen in the proximity of the resonance region where the dynamic coefficient δ is at its maximum. For subcritical damping, the value of ω_G/ω_0 , which is one of the parameters of the graphical representation in Fig. 7.13, is obtained from the formula

$$\frac{\omega_G}{\omega_0} = \frac{1}{\sqrt{[(2\pi/9)^2 + 1]}} \quad (7.9.35)$$

For supercritical damping the formula becomes

$$\frac{\omega_G}{\omega_0} = \frac{C}{2\sqrt{[D(C-D)]}} \quad (7.9.36)$$

As Fig. 7.13 also suggests, the method of forced vibration is well suited for cases of

$$\frac{\omega_G}{\omega_0} < 5.0 \quad (7.9.37)$$

when no excessive force is needed to make the water surface vibrate.

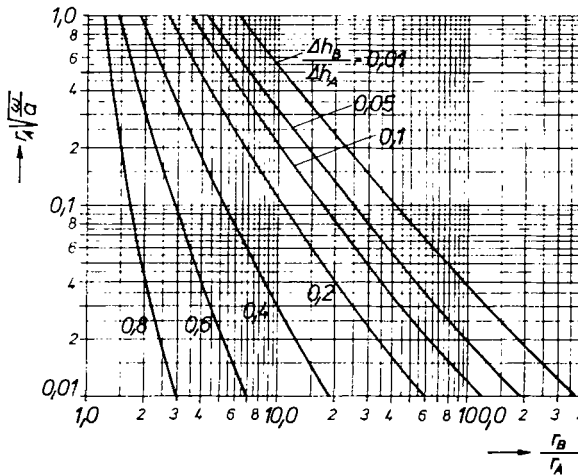


Fig. 7.18. Relation between the ratios of vibration amplitudes of the water surface in borings, and the frequency and position of boreholes.

In the experiment, the amplitude Δh_A of the water surface is observed by means of a boring located at $r = r_A$, and the amplitude Δh_B by means of another boring located at $r = r_B$. From the theory outlined in Section 7.6.3, follows the principle of the construction of a graph which will be found helpful in the evaluation of the results of observations. The diagram shown in Fig. 7.18 plots the ratios r_B/r_A on the axis of abscissas, and the parameter $r_A \sqrt{(\omega/a)}$ on the axis of ordinates. The curves correspond to the ratios $\Delta h_B/\Delta h_A$. If we know r_B/r_A as well as $\Delta h_B/\Delta h_A$, we can read $r_A \sqrt{(\omega/a)}$ from the diagram, from this value calculate the required a and, in turn, the transmissivity coefficient or other parameters.

It is advantageous that the method just outlined utilizes the effects of the inertia forces of the water flow inside the borehole. The appropriate experiments can be performed by comparatively simple means. We shall not discuss here either the instrumentation or the technique of measurement but shall stress the highly convenient fact that in the case being considered the choice of $h_s = h(r, 0)$, a quantity which is readily defined, is satisfactory from the hydraulic point of view, and that this fact considerably facilitates the determination of the characteristic coefficient of permeability of the saturated stratum.

It has already been suggested that for $\omega_0/\omega_o > 5$, the method is less straightforward because a heavy damping weakens the effects of the inertia forces. In the determination of the coefficient of transmissivity we prefer to use the results of experiments during which a certain quantity of water is actively taken off from the well, and the drop of the water surface is measured in the object as well as in neighbouring borings.

The method of water take-off is chosen so as to make the experiment simple to perform and easy to control. Thus, for example, the water surface in the well is decreased rather suddenly and kept permanently at this lowered level. A time variable quantity $Q_s(t)$ is then being withdrawn from a well of a radius $r = r_s$, and the values $\Delta h(r_B, t)$, $\Delta h(r_A, t)$ are determined in the borings which lie at distances $r = r_A$, $r = r_B$ (where $r_B > r_A$). The symbol Δh denotes the difference between the heights of the original and the resulting groundwater level.

In keeping with the theory explained in Section 7.6.1, an unsteady flow tends to become steady. The region of stationarity extends with time. Hence, it is possible in such a region to use the description which is based on the theory of steady flow. We shall start from the values of Δh_B , Δh_A and Q_s obtained by measurement at a given instant. The coefficient of transmissivity is calculated from the equation

$$kT = \frac{Q_s}{2\pi(\Delta h_B - \Delta h_A)} \ln \frac{r_B}{r_A} \quad (7.9.38)$$

where Q_s is taken to be positive when water is pumped into the well, and negative when it is pumped from the well. The condition which eqn. (7.9.38) must satisfy in order to be applicable to a pumping test, is

$$\frac{\Delta h_A}{\Delta h_B} > 4 \quad (7.3.39)$$

Formula (7.9.38) can also be used in the evaluation of pumping tests during which a constant quantity $Q_s = \text{const.}$ is permanently being taken at the time t from a well. In this case, the condition (7.9.39) is replaced by that stating that Δh_B should increase linearly with time.

The latter restrictive condition is based on the theory explained in Section 7.6.2, in which we showed that eqn. (7.6.23b) is valid in certain circumstances. Although the values of Δh increase with time and hence the water surface is in motion, at a certain instant, it can be described by means of equations derived for steady flow.

The assumption (7.6.23b) enables us to determine the value of a which contains the coefficient of transmissivity. Let us start from eqn. (7.6.24) rewritten in the form

$$\Delta h(r, t) = - \frac{Q_s}{4\pi kT} \ln \frac{2.25at}{r^2} \quad (7.9.40)$$

Jacob [18] proposed the following two variants of this equation:

$$\Delta h(r, t) = -\frac{2.3Q_s}{4\pi kT} \log \frac{2.25a}{r^2} - \frac{2.3Q_s}{4\pi kT} \log t \quad (7.9.41)$$

$$\Delta h(r, t) = -\frac{2.3Q_s}{4\pi kT} \log 2.25at + \frac{2.3Q_s}{2\pi kT} \log r \quad (7.9.42)$$

Both equations can be plotted on semilogarithmic paper. In either case the axis of ordinates shows the values of Δh while the axis of abscissas shows $\log t$ for a given r in the first case, and $\log r$ for a given t in the second case. The resulting figure is a straight line from the slope of which one can determine the required quantities.

Thus, for example, we find that in the first case the slope $\operatorname{tg} \alpha$ gives the ratio

$$\operatorname{tg} \alpha = -\frac{2.3Q_s}{4\pi kT} \quad (7.9.43)$$

so that

$$kT = -\frac{2.3Q_s}{4\pi \operatorname{tg} \alpha} \quad (7.9.44)$$

The value of a is then obtained from eqn. (7.9.41) in which eqn. (7.9.43) is substituted for a given r and a certain t .

Another calculation procedure follows from the fact that the straight line in the semilogarithmic representation intersects the axis of abscissas. It is found that in the first variant the point of intersection corresponds to the time $t = t'$. The corresponding $\Delta h = 0$. According to eqn. (7.9.40) it must hold in this case that

$$\frac{2.25at'}{r^2} = 1 \quad (7.9.45)$$

hence

$$a = \frac{r^2}{2.25t'} \quad (7.9.46)$$

For the second variant, eqn. (7.9.42), we again determine the slope of the straight line, $\operatorname{tg} \alpha$ and calculate the value of kT from the equation

$$\operatorname{tg} \alpha = -\frac{2.3Q_s}{2\pi kT} \quad (7.9.47)$$

which gives

$$kT = -\frac{2.3Q_s}{2\pi \operatorname{tg} \alpha} \quad (7.9.48)$$

The value of a is obtained from eqn. (7.9.42), in which eqn. (7.9.47) is substituted for a given t and a certain r .

Another calculation procedure according to the second variant is based on the fact that the straight line in the semilogarithmic representation intersects the axis of abscissas at the point $r = r'$. The corresponding value $\Delta h = 0$. In this case it must hold by eqn. (7.9.40) that

$$\frac{2.25at}{(r')^2} = 1 \quad (7.9.49)$$

hence

$$a = \frac{(r')^2}{2.25at} \quad (7.9.50)$$

The first variant is more amenable theoretically because it works with data obtained in one boring. Observing the water surface fluctuations during the entire pumping test makes it possible to choose that part of the $\Delta h(t)$ dependence which satisfies the conditions on which the theory is founded.

For practical reasons, the duration of the test during which a constant quantity of water, $Q_s = \text{const.}$, is being removed from the well, cannot be infinitely long. The pumping is terminated at the time $t = t_1$. The interruption of the take-off induces a rise of the water surface in the well as well as in its neighbourhood. This process is best described by means of the theory of collocated wells explained in Section 7.6.5. Under certain simplifying assumptions, the corresponding equation takes the form

$$\Delta h = - \frac{Q_s}{4\pi kT} \left[\ln \frac{2.25at}{r^2} - \ln \frac{2.25a(t - t_1)}{r^2} \right] = - \frac{2.3Q_s}{4\pi kT} \log \frac{t}{t - t_1} \quad (7.9.51)$$

For a given r , the function $\Delta h = f[\log t/(t - t_1)]$, $t > t_1$ drawn on semilogarithmic paper is found to be linear in certain sections. The slope of the straight line, $\text{tg } \alpha$, makes it possible to determine the coefficient of transmissivity using eqn. (7.9.44). The corresponding procedure which is used in the evaluation of so-called level raising tests was described by Wenzel [45].

The advantage of the methods proposed in the foregoing lies in the fact that, theoretically, they offer the possibility of determining a larger number of parameters than does a solution of inverse problems based on the description of pure steady flow. Thus, for example, if we know the coefficient of transmissivity kT and also the quantity a , it is not difficult to obtain the active porosity μ , for, as eqn. (7.9.22) implies,

$$\mu = \frac{kT}{a} \quad (7.9.52)$$

One of the questionable constants whose choice, particularly in a predominating free-surface flow, is sometimes likely to affect the calculation of the other parameters, is the mean thickness of the saturated stratum, h_s , which sometimes can be appro-

ximated by $h_s = h(r, 0)$. During the take-off from a well, a more exact value of h_s is somewhat lower but time dependent, for it is related to the position of the water surface in the well. Hence, strictly speaking, the quantity a cannot be a time-independent constant either. This is one of the reasons why the theoretical variations of the function $\Delta h(r, t)$ differ from reality and why an evaluation based on linearized equations shows signs of subjectivity. A comparison of the actual variations with the so-called standard curves, an example of which is shown in Fig. 7.7, gives a good idea of the situation.

In our opinion, more reliable results are obtained by means of pumping tests in the course of which the water surface in the well is suddenly lowered and the position of the level in the borings is observed only after it has become stabilized. In this case, the flow is described by Dupuit's equation which does not contain μ . The evaluation is based on the G potential written for an assumedly homogeneous layer. For a pressure flow, we have

$$k = - \frac{Q_s}{2\pi T(h_B - h_A)} \ln \frac{r_B}{r_A}, \quad r_B > r_A \quad (7.9.53)$$

and for a flow occurring predominantly in the layer which contains the free surface,

$$k = - \frac{Q_s}{\pi(h_B^2 - h_A^2)} \ln \frac{r_B}{r_A}, \quad r_B > r_A \quad (7.9.54)$$

Theoretically, we could choose $r_A = r_s$, i.e. the well radius, but such a choice is not recommended for practical reasons because, frequently, the value of r_s is not defined in sufficient detail. During the drilling the soil becomes disturbed, pumping is apt to cause washing-out of sand and non-linear laws of flow are likely to come into play in the vicinity of a well with large take-offs. All this is at variance with the assumptions on which the theoretical description is based.

Further difficulties arise whenever a surface of seepage forms on the well lining. It was shown in Chapter 6 that this happens because of the three-dimensional nature of the flow in the vicinity of the well. Thus the effect of anisotropic permeability which was described in Chapter 1 by the elevation of the flow region, becomes of special significance, for in materials characterized by a high coefficient of anisotropy, the water surface in the vicinity of the well is apt to decrease only slightly during the pumping tests.

For a constant take-off from the well, the effect of the surface of seepage grows stronger with time (Fig. 7.17b) due to the fact that inlet velocities on the well lining increase in due course.

It was concluded on the basis of Charny's proof (Section 6.1) that in steady state, Dupuit's equation is also well suited for a comparatively exact quantitative description in cases when a surface of seepage forms on the object. Since a solution of inverse problems should be based on representative data, it is recommended that

observation borings be provided with perforations only in that part of the permeable stratum which lies close to the base. This measure is expected to restrict the possibility of water moving inside the boring and in turn, the effect on the development in the neighbourhood.

Under unsteady conditions, a surface of seepage influences the quantitative distribution of the volumes in the radial direction, and this fact makes the theory of one-dimensional flow hard to apply. Consequently, the results of the solution of an inverse problem are burdened with greater errors whenever the thickness of the saturated layer is not large enough. It is best to place the borings at some distance from the well where the "disturbing" effects are already weaker. Since an increase of r_B naturally requires a longer time of pumping, it is better to use the older but time-honoured method of prolonged take-off under the conditions of constant lowering of the water surface in the well.

An indispensable condition for practical applicability of the results of the solutions of inverse problems is a detailed knowledge of the geological structure of the region. It is particularly important to know when a stratum can be regarded as an anisotropically permeable medium and when the hydraulic significance of some of the layers must not be ignored.

As an example, consider the record of the results of measurements made in three borings placed relatively close to one another. (Fig. 7.17c). The axis of ordinates shows the depths (in m), the axis of abscissas, the ratios between the inflow per linear metre of boring (letter q) and the average inflow per linear metre of boring (letter q_p). As the diagram shows, certain zones participate very actively in the seepage while no water flows from other sections of the borings. It may well happen that in three-dimensional flow the latter will display the leakage effect which can no longer be described by the elevation of the flow region. A pumping experiment arranged in the usual manner would fail to disclose this situation.

7.10. Application of the method of substitute lengths to one-dimensional flow. The boundary condition of the third kind

The partial differential equations which were derived in Chapter 1, are solved so as to satisfy the prescribed initial and boundary conditions. In the outline of the methods of analysis of unsteady flows presented so far, we considered the boundary condition of the first kind which prescribes on the boundary the functional dependence of the potential (or of the piezometric head h) on time (Dirichlet's condition), viz.

$$h(0, t) = f_1(t) \quad (7.10.1)$$

Alternatively, we had to know the boundary flow, i.e. the derivative of the potential

(or of the piezometric head h) on the boundary as a function of time (Neumann's condition)

$$\frac{\partial h}{\partial n} = f_2(t) \quad (7.10.2)$$

where n denotes that the partial derivative is taken in the direction of the normal to the boundary.

In the theory of partial differential equations, we also come across the boundary condition of the third kind (Fourier's condition) which, in addition to prescribing that we should know the derivative in the direction of the normal to the boundary, also demands that the sought function should satisfy a supplementary requirement, such as

$$\frac{\partial h}{\partial n} - Ah = f_3(t) \quad (7.10.3)$$

where A is a constant whose value can also depend on h .

As an example of a problem formulated in this way, consider the seepage from a channel which is not cut down as far as the base of the permeable subsoil (Fig. 7.19a). At the time $t = 0$, the water surface in the channel suddenly rose from the level $h(x, 0)$ to the level h_0 . The effect of the resistance induced by the curvature of the streamlines as well as the effect of other inlet resistances (for example, the effect of an underground wall) is expressed by means of the substitute length L_n (Section 4.4). In the range of this length, we expect a pressure flow without storage of water in the underground. In the range $0 \leq x \leq \infty$, a free surface develops. On the interface between the two zones, i.e. at $x = 0$, the equation of continuity applies. The quantity seeping from the left is

$$Q = kT \frac{h_0 - h(0, t)}{L_n} \quad (7.10.4)$$

while from the right, we have

$$Q = -kT \left. \frac{\partial h}{\partial x} \right|_{x=0} \quad (7.10.5)$$

Comparing eqn. (7.10.4) with eqn. (7.10.5), we obtain the condition

$$\frac{h_0 - h(0, t)}{L_n} = - \left. \frac{\partial h}{\partial x} \right|_{x=0} \quad (7.10.6)$$

On substituting

$$\Delta h = h(x, t) - h_0 \quad (7.10.7)$$

we obtain, in place of (7.10.6), the condition

$$\left. \frac{\partial(\Delta h)}{\partial x} \right|_{x=0} - \frac{1}{L_n} \Delta h(0, t) = 0 \quad (7.10.8)$$

Eqn. (7.10.8) is a boundary condition of the third kind which — assuming that $\Delta h(x, 0) = \text{const.}$ — co-determines the solution of the equation

$$\frac{\partial(\Delta h)}{\partial t} = a \frac{\partial^2(\Delta h)}{\partial x^2} \quad (7.10.9)$$

$$a = \frac{kT}{\mu}$$

where k, T, μ are quantities discussed in Section 7.9.

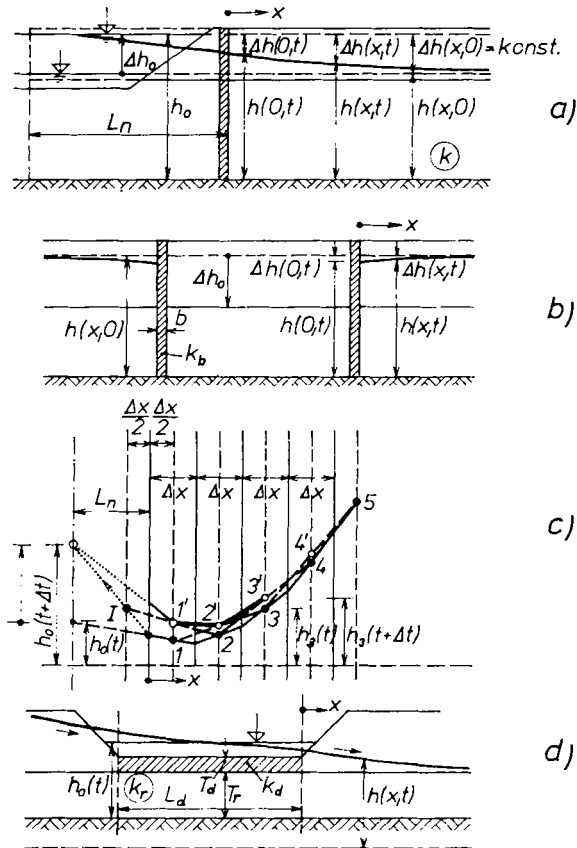


Fig. 7.19. Boundary condition of the third kind.

The boundary condition must be duly respected in an analysis of the similarity conditions. The similarity invariants are determined in the usual way as

$$\left\{ \frac{x^2}{at} \right\}, \quad \left\{ \frac{x}{L_n} \right\} \quad (7.10.10)$$

The solution of the parabolic equation (7.10.9) under the condition of the third kind is known from heat transfer theory. We can, therefore, simply write the result

$$\begin{aligned} \frac{\Delta h(x, t)}{\Delta h_0} = & \frac{1}{\sqrt{(4\pi at)}} \int_0^\infty \left\{ \text{Exp} \left[-\frac{(x - \xi)^2}{4at} \right] - \text{Exp} \left[-\frac{(x + \xi)^2}{4at} \right] \right\} d\xi + \\ & + \frac{1}{\sqrt{(\pi at)}} \int_0^\infty \text{Exp} \left\{ -\left[\frac{\xi}{L_n} + \frac{(x + \xi)^2}{4at} \right] \right\} d\xi \end{aligned} \quad (7.10.11)$$

On rearranging the integrand of the second integral on the right-hand side of eqn. (7.10.11) into the form

$$\begin{aligned} & \frac{1}{\sqrt{(\pi at)}} \int_0^\infty \text{Exp} \left\{ -\left[\frac{\xi}{L_n} + \frac{(x + \xi)^2}{4at} \right] \right\} d\xi = \\ & = \frac{1}{\sqrt{(\pi at)}} \int_0^\infty \text{Exp} \left(\frac{x}{L_n} + \frac{at}{L_n^2} \right) \text{Exp} \left[-\frac{[x + \xi + (2at/L_n)]^2}{4at} \right] d\xi \end{aligned}$$

and writing

$$\begin{aligned} & \frac{1}{\sqrt{(4\pi at)}} \int_0^\infty \text{Exp} \left[-\frac{(x - \xi)^2}{4at} \right] d\xi = \frac{1}{2} \left[1 + \text{Erf} \left(\frac{x}{\sqrt{(4at)}} \right) \right] \\ & \frac{1}{\sqrt{(4\pi at)}} \int_0^\infty \text{Exp} \left[-\frac{(x + \xi)^2}{4at} \right] d\xi = \frac{1}{2} \left[1 - \text{Erf} \left(\frac{x}{\sqrt{(4at)}} \right) \right] \\ & \frac{1}{\sqrt{(\pi at)}} \int_0^\infty \text{Exp} \left\{ -\frac{[x + \xi + (2at/L_n)]^2}{4at} \right\} d\xi = \\ & = \left\{ 1 - \text{Erf} \left[\frac{x}{\sqrt{(4at)}} + \frac{\sqrt{(at)}}{L_n} \right] \right\} \left\{ \text{Exp} \left[\frac{1}{L_n} \left(x + \frac{at}{L_n} \right) \right] \right\} \end{aligned}$$

we obtain

$$\Delta h(x, t) = \Delta h_0 \left[\text{Erf} \frac{x}{\sqrt{(4at)}} + \left\{ \text{Erfc} \left[\frac{x}{\sqrt{(4at)}} + \frac{\sqrt{(at)}}{L_n} \right] \right\} \left[\text{Exp} \frac{1}{L_n} \left(x + \frac{at}{L_n} \right) \right] \right] \quad (7.10.12)$$

For $x = 0$, i.e. on the interface between the two zones being considered, this becomes

$$\Delta h(0, t) = \Delta h_0 \left[\text{Erfc} \left(\frac{\sqrt{(at)}}{L_n} \right) \right] \left[\text{Exp} \left(\frac{at}{L_n^2} \right) \right] \quad (7.10.13)$$

As eqn. (7.10.13) implies, for

$$\frac{at}{L_n^2} > 3 \quad (7.10.14)$$

$\Delta h(0, t)$ approaches zero, which means that the water surface does not rise markedly.

This fact enables us to design the vertical element in a way which prevents the water surface from rising excessively during river floods that subside during the time $t = t_{\max}$. In order to achieve this end, we must make

$$L_n > \sqrt{\frac{at_{\max}}{3}} \quad (7.10.15)$$

The effect of the inlet resistance grows weaker during floods of relatively long durations, as can be demonstrated by the following quantitative analysis: the quantity of water which seeps through a unit length per unit time is

$$q = -kT \frac{\partial(\Delta h)}{\partial x} = -\frac{kT\Delta h_0}{L_n} \left\{ \operatorname{Erfc} \left[\frac{x}{\sqrt{4at}} + \frac{\sqrt{at}}{L_n} \right] \right\} \left[\operatorname{Exp} \frac{1}{L_n} \left(x + \frac{at}{L_n} \right) \right] \quad (7.10.16)$$

On the interface between the two zones being considered, i.e. at the point $x = 0$, this becomes

$$q|_{x=0} = -\frac{kT\Delta h_0}{L_n} \left[\operatorname{Erfc} \left(\frac{\sqrt{at}}{L_n} \right) \right] \left[\operatorname{Exp} \left(\frac{at}{L_n^2} \right) \right] \quad (7.10.17)$$

For comparatively large values of $\sqrt{at}/L_n$ we have, approximately,

$$\operatorname{Erfc} \frac{\sqrt{at}}{L_n} \rightarrow \frac{L_n}{\sqrt{(\pi at)}} \operatorname{Exp} \left(-\frac{at}{L_n^2} \right) \quad (7.10.18)$$

$$q|_{x=0} \rightarrow -\frac{kT\Delta h_0}{\sqrt{(\pi at)}} \quad (7.10.19)$$

Expression (7.10.19) is analogous to expression (7.4.14) which was derived on the assumption that an inlet resistance does not exist. The quantity of seepage decreases with time anyway, because for $t \rightarrow \infty$, $q \rightarrow 0$.

The boundary condition of the third kind is equally suitable for descriptions of cases involving, for example, a sudden fall of the water surface at an outlet resistance. We are required to solve the inflow into a rectangular construction pit bounded by a sheetpile wall of thickness b ; the coefficient of permeability of the material of the sheetpile wall (including the effects of possible defects of the interlocks) is k_b . Ignoring the influence of the curvature of the streamlines at the entrance to the pit, we have

$$L_n = b \frac{k}{k_b} \quad (7.10.20)$$

Assume that the initial water surface in the region continued to be horizontal during the construction of the wall and the excavation of the pit. At the time $t = 0$, it suddenly fell by Δh_0 .

We shall formulate the problem so as to reduce the solution to the determination of the particular integral for zero initial condition.

On setting $h_0 = h(x, 0)$ in eqn. 7.10.7 we obtain $\Delta h(x, 0) = 0$. According to Fig. 7.19b, the equation of continuity at the point $x = 0$ is of the form

$$\frac{\Delta h(0, t)}{L_n} = \frac{\partial(\Delta h)}{\partial x} \Big|_{x=0} \quad (7.10.21)$$

Referring to eqn. (7.10.7), the equation valid on the boundary turns out to be similar to eqn. (7.10.8), and the solution of the problem being considered is obtained by an obvious rearrangement of eqn. (7.10.12):

$$\Delta h(x, t) = \Delta h_0 \left[\operatorname{Erfc} \left[\frac{x}{\sqrt{(4at)}} \right] - \left\{ \operatorname{Erfc} \left[\frac{x}{\sqrt{(4at)}} \right] + \frac{\sqrt{(at)}}{L_n} \right\} \left[\operatorname{Exp} \frac{1}{L_n} \left(x + \frac{at}{L_n} \right) \right] \right] \quad (7.10.22)$$

At the point $x = 0$

$$\Delta h(0, t) = \Delta h_0 \left[1 - \left\{ \operatorname{Erfc} \left[\frac{\sqrt{(at)}}{L_n} \right] \right\} \left[\operatorname{Exp} \left(\frac{at}{L_n^2} \right) \right] \right] \quad (7.10.23)$$

$$\Delta h_0 = h(0, t) - h(x, 0) = \Delta h(0, t)$$

The quantity of water seeping per unit length of the wall

$$q|_{x=0} = \frac{kT\Delta h_0}{L_n} \left\{ \operatorname{Erfc} \left[\frac{\sqrt{(at)}}{L_n} \right] \right\} \left[\operatorname{Exp} \left(\frac{at}{L_n^2} \right) \right] \quad (7.10.24)$$

becomes

$$q|_{x=0} = \frac{kT\Delta h_0}{L_n} \quad (7.10.25)$$

at the time $t = 0$, i.e. at the instant of a sudden fall of the water surface in the pit.

With passing time, the seepage decreases and approaches the value

$$q|_{x=0} = \frac{kT\Delta h_0}{\sqrt{(\pi at)}}$$

The linearity of the initial equations and the nature of the boundary conditions of the third kind suggest the possibility that suitably formulated solutions may be superposed upon one another. Suppose, for example, that h_0 in Fig. 7.19a is a function of time, replace it by a sequence of rectangular pulses and carry out the superposition using the method explained in Section 7.4.5. A continuous function on the boundary can be obtained by making the pulses infinitely narrow. Some variations lend themselves more readily to an analytical description than others. Suppose, for example, that

$$h_0 = h(x, 0) + \Delta h_0 \sin \omega t \quad (7.10.26)$$

where ω is the angular frequency of a sinusoidal vibration of the water surface.

Ignoring the effect of the initial condition, we obtain an approximate solution in the form of the function

$$\Delta h(x, t) = \frac{\Delta h_0}{1 + L_n \sqrt{(\omega/2a)}} \left[\text{Exp} \left(-x \sqrt{\frac{\omega}{2a}} \right) \right] \left[\sin \left(\omega t - x \sqrt{\frac{\omega}{2a}} \right) \right] \quad (7.10.27)$$

The damping of the vibrations is influenced by the inlet resistance as well as by the frequency of fluctuations of the water surface in the river — higher frequencies produce heavier damping.

The solution in the form of eqn. (7.10.22) suggests the possibility of superposition of a step-wise character. Assume, for example, that the emptying of the pit shown in Fig. 7.19b proceeds gradually. At the time t_1 , the water surface will decrease by Δh_{01} , at the time t_2 , the decrease will reach Δh_{02} . After the n -th decrease, we have

$$\begin{aligned} \Delta h(x, t) = & \sum_{i=1}^n (\Delta h_{0,i} - \Delta h_{0,i-1}) \text{Erfc} \left[\frac{x}{4a(t - t_i)} \right] - \\ & - \sum_{i=1}^n (\Delta h_{0,i} - \Delta h_{0,i-1}) \left[\text{Erfc} \left\{ \frac{x}{\sqrt{[4a(t - t_i)]}} + \frac{\sqrt{[a(t - t_i)]}}{L_n} \right\} \right] \cdot \\ & \cdot \left\{ \text{Exp} \frac{1}{L_n} \left[x + \frac{a(t - t_i)}{L_n} \right] \right\} \end{aligned} \quad (7.10.28)$$

and at the point $x = 0$, this becomes

$$\Delta h(0, t) = \sum_{i=1}^n (\Delta h_{0,i} - \Delta h_{0,i-1}) \left[1 - \left\{ \text{Erfc} \frac{\sqrt{[a(t - t_i)]}}{L_n} \right\} \left[\text{Exp} \frac{a(t - t_i)}{L_n^2} \right] \right] \quad (7.10.29)$$

The quantity of water seeping per unit length of the wall into the pit is

$$q|_{x=0} = \frac{kT}{L_n} \sum_{i=1}^n (\Delta h_{0,i} - \Delta h_{0,i-1}) \left\{ \text{Erfc} \frac{\sqrt{[a(t - t_i)]}}{L_n} \right\} \left[\text{Exp} \frac{a(t - t_i)}{L_n^2} \right] \quad (7.10.30)$$

More general problems defined in the region $0 \leq x \leq L$ can be solved in a similar way. The resulting equations take the form of infinite series whose direct evaluation is quite tedious. We shall, therefore, present here a simple graphical method whose principles are known from the work of Schmidt [37], and which is based on an equation of the type

$$\frac{\partial h}{\partial t} = a \frac{\partial^2 h}{\partial x^2} \quad (7.10.31)$$

or, in terms of the finite differences,

$$\frac{\Delta h}{\Delta t} = a \frac{\Delta^2 h}{\Delta x^2} \quad (7.10.32)$$

Suppose that the region $0 \leq x \leq L$ is divided into a certain number of equidistant segments Δx , at the centre of which we know the value $h_{j,t}$ at the time t . By the rules of the difference calculus, we have from the right

$$\left. \frac{\Delta h}{\Delta x} \right|_+ = \frac{h_{j+1,t} - h_{j,t}}{\Delta x} \quad (7.10.33)$$

and from the left

$$\left. \frac{\Delta h}{\Delta x} \right|_- = \frac{h_{j,t} - h_{j-1,t}}{\Delta x} \quad (7.10.34)$$

$$\frac{\Delta^2 h}{\Delta x^2} = \frac{1}{\Delta x} \left[\left. \frac{\Delta h}{\Delta x} \right|_+ - \left. \frac{\Delta h}{\Delta x} \right|_- \right] = \frac{1}{\Delta x^2} (h_{j+1,t} + h_{j-1,t} - 2h_{j,t}) \quad (7.10.35)$$

$$\frac{\Delta h}{\Delta t} = \frac{h_{j,t+\Delta t} - h_{j,t}}{\Delta t} \quad (7.10.36)$$

Substituting in eqn. (7.10.34), we obtain

$$\frac{h_{j,t+\Delta t} - h_{j,t}}{\Delta t} = a \frac{h_{j+1,t} + h_{j-1,t} - 2h_{j,t}}{\Delta x^2} \quad (7.10.37)$$

and from there

$$h_{j,t+\Delta t} = \frac{2a \Delta t}{\Delta x^2} \left(\frac{h_{j+1,t} + h_{j-1,t}}{2} \right) + \left(1 - \frac{2a \Delta t}{\Delta x^2} \right) h_{j,t} \quad (7.10.38)$$

If the intervals Δx , Δt are chosen so that

$$\frac{2a \Delta t}{\Delta x^2} = 1 \quad (7.10.39)$$

we obtain

$$h_{j,t+\Delta t} = \frac{h_{j+1,t} + h_{j-1,t}}{2} \quad (7.10.40)$$

It is seen from the above expression that the value $h_{j,t+\Delta t}$ in the time $t + \Delta t$ is an arithmetic mean of the values of h in the time interval Δt . The solution which has a simple graphical interpretation, is capable of providing a straightforward description of the boundary conditions of the first and second kind. It is recommended for problems with a boundary condition of the third kind whenever the inlet resistance can be assumed to be considerable, i.e. whenever the substitute length is large enough, $L_n > \Delta x$. If $L_n \leq \Delta x$, it is more convenient to neglect L_n and to convert the solution to that of Dirichlet's problem. The error arising due to this conversion is commensurate with the error of the method.

The value L_n is calculated using the procedure outlined in Chapter 2. Provided that

$$\left. \frac{\partial h}{\partial x} \right|_{x=0, t=0} = \operatorname{tg} \alpha \neq 0,$$

it can be determined from the initial condition, and becomes

$$L_n = \frac{h(0, 0) - h_0(0)}{\operatorname{tg} \alpha} \quad (7.10.41)$$

where

- $h(0, 0)$ is the initial value of h on the boundary,
- $h_0(0)$ — the initial value of h corresponding to the water level in the river,
- $\operatorname{tg} \alpha$ — the slope of the tangent to the initial water surface on the boundary of the earth massif.

Fig. 7.19c shows the graphical procedure for the case when a considerable inlet resistance is present. The region $0 \leq x \leq L$ is divided into the segments Δx , and centrelines are drawn in the resulting vertical strips. At the time t , the specified water level intersects the centrelines at the points $1, 2, \dots$; in the graph it is replaced by a light solid broken line. The points $2', 3', \dots$ which correspond to the values at the time $t + \Delta t$ are obtained with the help of dashed connecting lines. The value of Δt is given by

$$\Delta t = \frac{\Delta x^2}{2a} \quad (7.10.42)$$

The point $1'$ which lies on the water surface corresponding to the time $t + \Delta t$ is established by means of an auxiliary graphical construction using the auxiliary point I . We have thus determined all the points of the required course of the function $h(t + \Delta t)$ (heavy solid broken line in Fig. 7.19c).

We assumed in Chapter 2 that the substitute length L_n is a constant independent of time, whose magnitude is related to the geometrical configuration of the flow region. In some cases this assumption is inadequate. Suppose, for example, that the river bed does not reach all the way to the highly permeable stratum but that its bottom is underlain by a layer of a relatively poor permeability, having a thickness T_d and a coefficient of permeability (in the vertical direction) k_d . The motion in the lower, more permeable stratum is determined by the position of the water surface in the river but equally so, by the position of the groundwater surface on both banks. Hence the substitute length $L_n(t)$ is a non-linear function of the time t . It may happen that the inlet resistance is so large that the river ceases to be a boundary condition, and a nearly undisturbed transport of water from one shoreline zone to another takes place under the river bottom. Such a situation can be expected whenever

$$\frac{T_d}{k_d} > \frac{10L_d^2}{k_r T_r} \quad (7.10.43)$$

where

L_d is the width of the river,

k_r, T_r — the coefficient of permeability (in the horizontal direction) and the thickness of the permeable layer underneath the river bottom, respectively (Fig. 7.19d).

The inequality (7.10.43) is quite important from an engineering point of view, for it can serve as a criterion for estimating the effect of artificial (as well as natural) choking of the river bottom. However, it is applicable only in the case when the layer of poor permeability which remains underneath the river bottom is stable against erosion of flowing river water as well as against the uplift which is apt to be produced during a decrease of the water surface in the river.

7.11. Three-dimensional axi-symmetrical flow in non-deforming soils

The one-dimensional flow on which we have concentrated so far, can be described by equations which can serve as a starting point for analyses of more complicated problems. One of the possible synthetic procedures is based on the idea that the function $h(x, y, z, t)$ can be formulated in terms of the product

$$h = U(t) X(x) Y(y) Z(z) \quad (7.11.1)$$

where $U(t)$ is a function of time and $X(x), Y(y), Z(z)$ are functions of the independent variables x, y, z .

The above solution satisfies the equation

$$\frac{\partial h}{\partial t} = a \left(\frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2} + \frac{\partial^2 h}{\partial z^2} \right) \quad (7.11.2)$$

which is obtained by the synthesis of the equations

$$\begin{aligned} \frac{\partial h}{\partial t} &= a \frac{\partial^2 h}{\partial x^2} \\ \frac{\partial h}{\partial t} &= a \frac{\partial^2 h}{\partial y^2} \\ \frac{\partial h}{\partial t} &= a \frac{\partial^2 h}{\partial z^2} \end{aligned} \quad (7.11.3)$$

each of which can be solved by separation of the variables (cf. Section 7.4). This

fact allows us to apply the known solutions of one-dimensional problems. Thus, for example, using the particular integral (7.4.7) we obtain the function

$$h = \left\{ \frac{c_1}{\sqrt{(4\pi at)}} \text{Exp} \left[-\frac{(x - \xi)^2}{4at} \right] \right\} \left\{ \frac{c_2}{\sqrt{(4\pi at)}} \text{Exp} \left[-\frac{(y - \eta)^2}{4at} \right] \right\} \cdot \left\{ \frac{c_3}{\sqrt{(4\pi at)}} \text{Exp} \left[-\frac{(z - \zeta)^2}{4at} \right] \right\} = \frac{c}{(4\pi at)^{3/2}} \text{Exp} \left[-\frac{(x - \xi)^2 + (y - \eta)^2 + (z - \zeta)^2}{4at} \right] \quad (7.11.4)$$

where

c_1, c_2, c_3, c are constants,

ξ, η, ζ are some definitely chosen Cartesian coordinates which make it possible to shift the origin of the system of coordinates x, y, z .

The integral

$$h = c \int_0^t \frac{1}{(4\pi a\tau)^{3/2}} \left\{ \text{Exp} \left[-\frac{(x - \xi)^2 + (y - \eta)^2 + (z - \zeta)^2}{4a\tau} \right] \right\} d\tau \quad (7.11.5)$$

must be another particular solution.

Assume that a source from which water flows along linear paths operates at the point ξ, η, ζ . The equipotentials are shaped like spherical surfaces of radius

$$r = \sqrt{[(x - \xi)^2 + (y - \eta)^2 + (z - \zeta)^2]} \quad (7.11.6)$$

Hence

$$h = c \int_0^t \frac{1}{(4\pi a\tau)^{3/2}} e^{-r^2/4a\tau} d\tau = \frac{c}{2ra\pi^{3/2}} \int_{r/\sqrt{4a\tau}}^{\infty} e^{-u^2} du = \frac{c}{4ra\pi} \text{Erfc} \left[\frac{r}{\sqrt{(4at)}} \right] \quad (7.11.7)$$

The quantity of water which flows through a sphere of radius r is

$$q = 4\pi r^2 k \frac{\partial h}{\partial r} = -\frac{ck}{a} \left\{ \frac{2\pi r}{\sqrt{(4\pi at)}} \text{Exp} \left(-\frac{r^2}{4at} \right) + \text{Erfc} \left[\frac{r}{\sqrt{(4at)}} \right] \right\} \quad (7.11.8)$$

where k is the coefficient of permeability.

The value of the constant c is determined from the condition that $q = q_0$ in the vicinity of the source where $r \rightarrow 0$. Eqn. (7.11.8) then gives

$$q_0 = -\frac{ck}{a} = \text{const.} \quad (7.11.9)$$

and hence

$$h = -\frac{q_0}{4\pi kr} \operatorname{Erfc} \left[\frac{r}{\sqrt{(4at)}} \right] \quad (7.11.10)$$

The velocity potential $\varphi = -kh$ then becomes

$$\varphi = \frac{q_0}{4\pi r} \operatorname{Erfc} \left[\frac{r}{\sqrt{(4at)}} \right] \quad (7.11.11)$$

Eqn. (7.11.10) assumes a permanently constant discharge from the source. The resulting expression which describes the distribution of the potential in the space, was obtained by integrating eqn. (7.11.4) with respect to time. The function in the integrand is of the form

$$\varphi = \frac{aq_0}{(4\pi a\tau)^{3/2}} \left[\operatorname{Exp} \left(-\frac{r^2}{4a\tau} \right) \right] d\tau \quad (7.11.12)$$

The product $q_0 d\tau$ represents the volume of water which enters the space in a brief moment. Hence eqn. (7.11.4) describes the distribution of the potential around a source which came into existence, worked with a strength q_0 and became extinct immediately afterwards.

The water which comes into the space collects in it, and it is interesting to find out where and how this storage takes place. We can obtain an answer to this question by analyzing a particular problem.

Assume, for example, that several sources are distributed at regular distances $2T$ along the z axis in the space. The situation is shown schematically in Fig. 7.20a. The flow is axi-symmetrical, formed by superposition which is formulated as follows:

$$\begin{aligned} \bar{\varphi} = \sum_{n=-\infty}^{+\infty} \int_0^t \frac{aq}{(4\pi a\tau)^{3/2}} \left\{ \operatorname{Exp} \left[-\frac{r^2 + (z + \zeta - 2nT)^2}{4a\tau} \right] + \right. \\ \left. + \operatorname{Exp} \left[-\frac{r^2 + (z - \zeta - 2nT)^2}{4a\tau} \right] \right\} d\tau \end{aligned} \quad (7.11.13)$$

where

q is the strength of an elementary source,
 $n = 1, 2, \dots$

Suppose further that a line segment of a length T_s lies on the z axis, and along this segment there is regularly distributed a system of sources of equal constant strength q defined as

$$q = \frac{Q}{T_s} d\zeta \quad (7.11.14)$$

where Q is the total constant take-off along the segment.

According to the principle of superposition we have

$$\bar{\varphi} = \frac{aQ}{T_s} \int_0^{T_s} \left[\sum_{n=-\infty}^{+\infty} \int_0^t \frac{1}{(4\pi a\tau)^{3/2}} \left\{ \text{Exp} \left[-\frac{r^2 + (z + \zeta - 2nT)^2}{4a\tau} \right] + \right. \right. \\ \left. \left. + \text{Exp} \left[-\frac{r^2 + (z - \zeta - 2nT)^2}{4a\tau} \right] \right\} d\tau \right] d\zeta$$

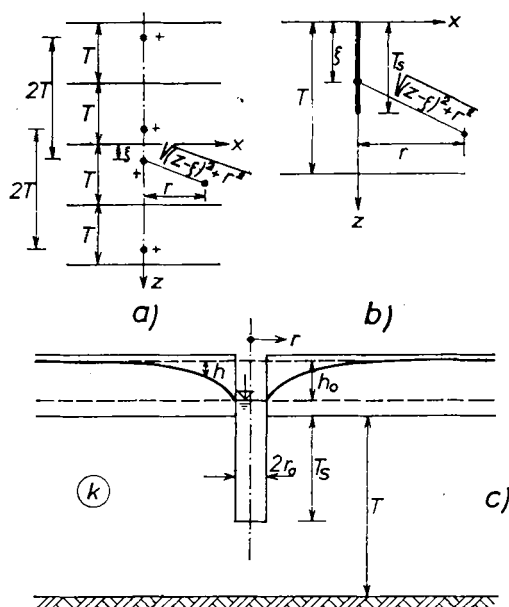


Fig. 7.20. Flow towards a hydraulically imperfect well.

Suppose that the scheme shown in Fig. 7.20a describes the effect of inflow to an hydraulically imperfect well of length T_s . The accuracy of the solution can be enhanced by the application of the mean value theorem (Chapter 6), viz.

$$\varphi = \frac{1}{T_s} \int_0^{T_s} \bar{\varphi} dz \quad (7.11.15)$$

The integrals were evaluated by Verigin [43] who arrived at the conclusion that in the x, y plane

$$\varphi = \frac{Q}{4\pi T} \left[-\text{Ei} \left(-\frac{r^2}{4at} \right) - \chi \right] \quad (7.11.16)$$

$$\chi = \frac{2}{\pi^2} \left(\frac{T}{T_s} \right)^2 \sum_{n=1}^{\infty} \frac{1}{n} (\sin n\pi) \frac{T_s}{T} \int_{r^2/4at}^{\infty} \left\{ \text{Exp} \left[-\left(u + \frac{1}{4u} \frac{n\pi r}{T} \right) \right] \right\} \frac{du}{u} \quad (7.11.17)$$

The square brackets of eqn. (7.11.16) enclose an integral logarithmic function

$-\text{Ei}(-x)$, across which we came when solving an unsteady inflow to an hydraulically perfect well. The function χ describes the effect of hydraulic imperfection, which varies with time. According to Kheyn [20] the effect of three-dimensionality increases with time so that χ approaches a constant value at a relatively fast rate. We are, therefore, of the opinion that in keeping with Section (7.6.2) the simplified formula

$$\varphi = \frac{Q}{4\pi T} \ln \frac{2.25at}{(\bar{r})^2}, \quad \bar{r} = \frac{r}{B}, \quad B = f\left(\frac{T_s}{T}, \frac{T}{r}\right) \quad (7.11.18)$$

will be good enough for practical purposes. In the above, B is a correction factor which can be taken from the diagram shown in Fig. 6.8 for specified T_s/T . On dividing by the factor B , the actual distance of a point in the x, y plane from the well axis, we obtain a reduced value $\bar{r}$ with which we can operate as though the well were hydraulically perfect.

With increasing r , the value of B approaches unity. The effect of the spatial nature of the flow grows weaker as the point of the x, y plane moves from the well axis. Clearly, the value of a has a similar meaning as the coefficient

$$a = \frac{kT}{\mu}$$

which we used in the analysis of the motion towards an hydraulically perfect well. The value of h relative to the reference plane which can be placed onto the level of the un-lowered water surface (Fig. 7.20c) is defined by

$$h = - \frac{Q}{4\pi kT} \ln \frac{2.25at}{(\bar{r})^2} \quad (7.11.19)$$

Right in the well, we have

$$h_0 = - \frac{Q}{4\pi kT} \ln \frac{2.25at}{(\bar{r}_0)^2} \quad (7.11.20)$$

where Q is the constant take-off from the well.

The effect of hydraulic imperfection is more noticeable in a medium whose permeability is higher in the horizontal than in the vertical direction.

For such media, the calculation of the reduction factor is based on the idea of the elevated region, the magnitude of the elevation depending on the value of the anisotropy coefficient λ . This property is put to good use in the solutions of inverse problems where not only a, kT but also B and in turn λ are determined from the known values of h .

The task is easier in cases when the results of these calculations can be compared with the results of field tests made on objects which pass through a saturated stratum to a greater or lesser degree.

7.12. Flow under one-dimensional elastic deformation

So far, we have assumed the water to be moving through an incompressible medium. In an unsteady motion, the water accumulated in the medium due to an increase of the flow region produced, for example, by the movement of the free surface. In a number of practical cases it is, however, no longer possible to ignore the deformations of the material induced by the action of an applied load. A theoretical description of such cases is naturally more complicated, and this is the reason why efforts have been and are still made to formulate the problem as simply as possible. Relatively most straightforward deformation conditions are obtained on the assumption that the material deforms elastically in one direction only, usually in the vertical direction, and that the deformations are so small as not to influence the seepage characteristics of the material.

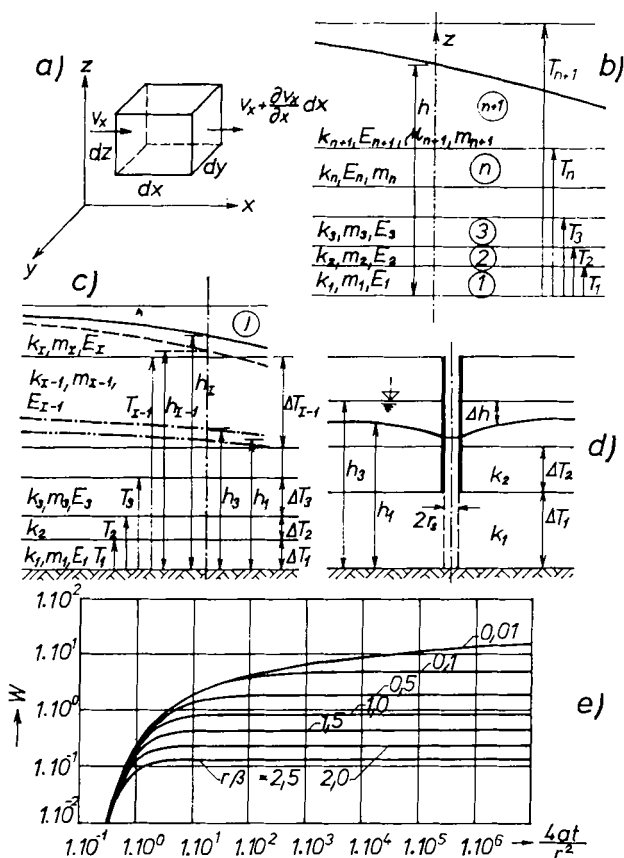


Fig. 7.21. Flow under one-dimensional elastic deformation.

The theory was worked out by Jacob and we shall illustrate its principles by analyzing the situation in a right-angle elementary prism (Fig. 7.21a), filled with soil and liquid. In unit time, a liquid of a specific mass ρ enters the prism along the x

axis with a seepage velocity v_x . The liquid which leaves the prism on the opposite side has a specific mass

$$\rho + \frac{\partial \rho}{\partial x} dx \quad (7.12.1)$$

and a seepage velocity

$$v_x + \frac{\partial v_x}{\partial x} dx \quad (7.12.2)$$

This means that in unit time the mass of the liquid inside the prism changed by

$$\rho v_x dy dz - \left(v_x + \frac{\partial v_x}{\partial x} dx \right) \left(\rho + \frac{\partial \rho}{\partial x} dx \right) dy dz = - \frac{\partial(\rho v_x)}{\partial x} dx dy dz$$

As a similar analysis of the flow in the directions of the y, z axes shows, in the time dt , the mass of the liquid inside the prism changed by a total of

$$- \left[\frac{\partial(\rho v_x)}{\partial x} + \frac{\partial(\rho v_y)}{\partial y} + \frac{\partial(\rho v_z)}{\partial z} \right] dx dy dz dt \quad (7.12.3)$$

Changing to the differences

$$- \left[\frac{\partial(\rho v_x)}{\partial x} + \frac{\partial(\rho v_y)}{\partial y} + \frac{\partial(\rho v_z)}{\partial z} \right] \Delta x \Delta y \Delta z dt \quad (7.12.4)$$

we can write

$$- \left[\rho \left(\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} \right) + \left(v_x \frac{\partial \rho}{\partial x} + v_y \frac{\partial \rho}{\partial y} + v_z \frac{\partial \rho}{\partial z} \right) \right] \Delta x \Delta y \Delta z dt$$

Assuming the flow velocities to be small, we can neglect the second term in the square brackets and write

$$- \rho \left(\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} \right) \Delta x \Delta y \Delta z dt \quad (7.12.5)$$

The mass of the liquid contained in an element with sides $\Delta x, \Delta y, \Delta z$ is

$$\Delta M = m \rho \Delta x \Delta y \Delta z \quad (7.12.6)$$

where m is the porosity

Assume that ΔM changes with time due to compression acting in the z direction:

$$\frac{\partial(\Delta M)}{\partial t} dt = \left[m \rho \frac{\partial(\Delta z)}{\partial t} + \rho \Delta z \frac{\partial m}{\partial t} + m \Delta z \frac{\partial \rho}{\partial t} \right] \Delta x \Delta y dt \quad (7.12.7)$$

The meaning of the various terms of eqn. (7.12.7) can be made clear by a thorough analysis of the conditions in the soil. As is well known, the total stress consists

of the effective stress σ_z and of the neutral stress p . When the soil is subjected to a load, the neutral stress which predominates in the initial stages, changes later to the effective stress, while, however,

$$\sigma_z + p = \text{const.} \quad (7.12.8)$$

Differentiating the above, we obtain

$$d\sigma_z = -dp \quad (7.12.9)$$

Assume that the relative compression depends linearly on the stress σ_z among the soil grains, i.e. that

$$d\sigma_z = -E_z \frac{d(\Delta z)}{\Delta z} \quad (7.12.10)$$

where E_z is the modulus of soil elasticity determined by uniaxial loading tests.

From the above, we obtain

$$\frac{d(\Delta z)}{dt} = -\frac{\Delta z}{E_z} \frac{\partial \sigma_z}{\partial t} = \frac{\Delta z}{E_z} \frac{\partial p}{\partial t} \quad (7.12.11)$$

Since the total volume o_z of the soil grains does not change during compression, we have

$$d(\Delta o_z) = d[(1 - m) \Delta x \Delta y \Delta z] = 0 \quad (7.12.12)$$

After deformation which takes place only in the z direction, we obtain

$$d(\Delta o_z) = d(1 - m) \Delta z + (1 - m) d(\Delta z) = 0 \quad (7.12.13)$$

and from there

$$dm = (1 - m) \frac{d(\Delta z)}{\Delta z} \quad (7.12.14)$$

$$\frac{\partial m}{\partial t} = \frac{1 - m}{\Delta z} \frac{\partial (\Delta z)}{\partial t} = \frac{1 - m}{E_z} \frac{\partial p}{\partial t} \quad (7.12.15)$$

To a liquid enclosed in the pores of an element, we can apply the law of conservation of mass

$$\varrho \Delta o_v = \text{const.} \quad (7.12.16)$$

where o_v is the volume of the liquid in the element.

Differentiating the above gives

$$\varrho d(\Delta o_v) + \Delta o_v d\varrho = 0 \quad (7.12.17)$$

Assume the following linear relationship between the relative compression and the

water pressure p :

$$\frac{d(\Delta o_v)}{\Delta o_v} = - \frac{dp}{E_v} \quad (7.12.18)$$

where E_v is the modulus of elasticity of water.

Substituting eqn. (7.12.18) in eqn. (7.12.17), gives

$$-\varrho \frac{\Delta o_v dp}{E_v} + \Delta o_v d\varrho = 0$$

From there we obtain

$$\frac{\varrho}{E_v} \frac{\partial p}{\partial t} = \frac{\partial \varrho}{\partial t} \quad (7.12.19)$$

Substituting eqns. (7.12.11), (7.12.14), (7.12.18) in eqn. (7.12.7) gives

$$\frac{\partial(\Delta M)}{\partial t} dt = \left[m\varrho \frac{\Delta z}{E_z} \frac{\partial p}{\partial t} + \varrho \Delta z \frac{1-m}{E_z} \frac{\partial p}{\partial t} + m \Delta z \frac{\varrho}{E_v} \frac{\partial p}{\partial t} \right] \Delta x \Delta y dt$$

and after a simple rearrangement, we have

$$\begin{aligned} \frac{\partial(\Delta M)}{\partial t} dt &= \left[\frac{m\varrho}{E_z} + \frac{\varrho(1-m)}{E_z} + \frac{m\varrho}{E_v} \right] \frac{\partial p}{\partial t} \Delta x \Delta y \Delta z dt = \\ &= \varrho \left(\frac{1}{E_z} + \frac{m}{E_v} \right) \frac{\partial p}{\partial t} \Delta x \Delta y \Delta z dt \end{aligned} \quad (7.12.20)$$

Since the equation of continuity applies to the element, eqn. (7.12.20) must be equal to eqn. (7.12.5), and accordingly

$$-\left(\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} \right) = \left(\frac{1}{E_z} + \frac{m}{E_v} \right) \frac{\partial p}{\partial t} \quad (7.12.21)$$

Recall that

$$p = \gamma h \quad (7.12.22)$$

where γ is the specific weight of water

$$-\left(\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} \right) = \gamma \left(\frac{1}{E_z} + \frac{m}{E_v} \right) \frac{\partial h}{\partial t} \quad (7.12.23)$$

Under a linear law of flow when the coefficient of permeability k is a function of position, we have

$$v_x = -k \frac{\partial h}{\partial x}, \quad v_y = -k \frac{\partial h}{\partial y}, \quad v_z = -k \frac{\partial h}{\partial z} \quad (7.12.24)$$

$$\frac{\partial}{\partial x} \left(k \frac{\partial h}{\partial x} \right) + \frac{\partial}{\partial y} \left(k \frac{\partial h}{\partial y} \right) + \frac{\partial}{\partial z} \left(k \frac{\partial h}{\partial z} \right) = \gamma \left(\frac{1}{E_z} + \frac{m}{E_v} \right) \frac{\partial h}{\partial t} \quad (7.12.25)$$

Eqn. (7.12.25) is valid in a saturated medium below the free surface, while the situation on the free surface is described by eqn. (1.4.43).

In a homogeneous material, k is regarded as a constant, so that

$$\frac{\partial h}{\partial t} = \frac{k}{\gamma[(1/E_z) + (m/E_v)]} \left(\frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2} + \frac{\partial^2 h}{\partial z^2} \right) \quad (7.12.26)$$

or

$$\frac{\partial p}{\partial t} = \frac{k}{\gamma \left(\frac{1}{E_z} + \frac{m}{E_v} \right)} \left(\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} + \frac{\partial^2 p}{\partial z^2} \right) \quad (7.12.27)$$

7.12.1. Plane seepage under one-dimensional elastic deformation

In plane seepage, the predominant effect is assumed to be exercised by the horizontal component of flow, while the main effect of deformation derives from its vertical component. The interaction between the flow of water and the deformation is of the kind which makes averaging possible.

In a stratified system (Fig. 7.21b) consisting of i layers ($i = 1, 2, \dots, n$) saturated with water, and of the $(n + 1)$ th layer containing the free surface, the average modulus of elasticity E_p is assumed to be defined by the equation

$$E_p = \frac{E_{n+1}(h - T_n) + \sum_{i=1}^n E_i(T_i - T_{i-1})}{h} = \frac{E_{n+1}h + \sum_{i=1}^n T_i(E_i - E_{i+1})}{h} \quad (7.12.28)$$

and the average porosity m_p , by the equation

$$m_p = \frac{m_{n+1}(h - T_n) + \sum_{i=1}^n m_i(T_i - T_{i-1})}{h} = \frac{m_{n+1}h + \sum_{i=1}^n T_i(m_i - m_{i+1})}{h} \quad (7.12.29)$$

Assuming that the theory of the G potential (Section 7.3) can be applied in this case, we have

$$- \int_0^h \left(\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} \right) dz = - \left(\frac{\partial^2 G}{\partial x^2} + \frac{\partial^2 G}{\partial y^2} \right) \quad (7.12.30)$$

The linear increment of velocity in the vertical direction is taken to be

$$- \int_0^h \frac{\partial v_z}{\partial t} dz = v_h \quad (7.12.31)$$

where v_h , the vertical component of velocity on the free surface is given by the equation

$$v_h = -\mu_{n+1} \frac{\partial h}{\partial t} \quad (7.12.32)$$

and μ_{n+1} is the active porosity of the layer containing the free surface.

The time variation of pressure is constant along the entire vertical line; hence

$$\int_0^h \left(\frac{1}{E_z} + \frac{m}{E_v} \right) \frac{\partial p}{\partial t} dz = \gamma \frac{\partial h}{\partial t} \int_0^h \left(\frac{1}{E_p} + \frac{m_p}{E_v} \right) dz = \gamma h \left(\frac{1}{E_p} + \frac{m_p}{E_v} \right) \frac{\partial h}{\partial t} \quad (7.12.33)$$

where γ is the specific weight of water.

By eqn. (7.3.2), we have

$$\frac{\partial G}{\partial t} = - \frac{k_{n+1}h + \sum_{i=1}^n T_i(k_i - k_{i+1})}{\mu_{n+1}} \frac{\partial h}{\partial t} \quad (7.12.34)$$

On integrating eqn. (7.12.23) between the limits 0 and h , and substituting eqn. (7.12.34), we obtain

$$\frac{\partial G}{\partial t} = a \left(\frac{\partial^2 G}{\partial x^2} + \frac{\partial^2 G}{\partial y^2} \right) \quad (7.12.35)$$

where

$$a = \frac{k_{n+1}h + \sum_{i=1}^n T_i(k_i - k_{i+1})}{\gamma h [(1/E_p) + (m_p/E_v)] + \mu_{n+1}} \quad (7.12.36)$$

Since the value of the coefficient a depends on the value of the dependent variable G , eqn. (7.12.35) is non-linear. A linearized relationship is obtained on setting the piezometric height h in eqn. (7.12.36) equal to a suitably chosen mean value h_s :

$$a = \frac{k_{n+1}h_s + \sum_{i=1}^n T_i(k_i - k_{i+1})}{\gamma h_s [(1/E_p) + (m_p/E_v)] + \mu_{n+1}} = \frac{kT}{\mu_p} \quad (7.12.37)$$

$$E_p = \frac{E_{n+1}h_s + \sum_{i=1}^n T_i(E_i - E_{i+1})}{h_s} \quad (7.12.38)$$

$$m_p = \frac{m_{n+1}h_s + \sum_{i=1}^n T_i(m_i - m_{i+1})}{h_s} \quad (7.12.39)$$

$$kT = k_{n+1}h_s + \sum_{i=1}^n T_i(k_i - k_{i+1}) \quad (7.12.40)$$

$$\mu_p = \gamma h_s \left(\frac{1}{E_p} + \frac{m_p}{E_v} \right) + \mu_{n+1} \quad (7.12.41)$$

In the English literature, the product kT is known as transmissivity. The value μ_p is called the storage coefficient by some authors, for it characterizes the storage capacity in a given vertical line.

In Section 7.9, we mentioned the conditions under which we can choose alternatively the values $h, \Delta h, \overline{\Delta h}$ in place of the function G . Such a choice makes the formulation of plane flow somewhat easier.

In the solution of inverse problems, we always have to keep in mind that both the transmissivity and the storage coefficient include the effects of a number of partial phenomena. We may find that the knowledge of actual deformations can help us in separating them.

It may perhaps serve a useful purpose to explain once more and in another way the physical meaning of the coefficient a as an entity. Consider a column of saturated soil with a base area equal to one. The effect of elastic deformation and the effect of the motion of free surface (neglecting the deformation of water) changes the head h by Δh . A change of the total volume of the column of the saturated soil, ΔV , is characterized by

$$\Delta V = \frac{\gamma h_s \Delta h}{E_p} + \mu_{n+1} \Delta h = h_s \left(\frac{\gamma}{E_p} + \frac{\mu}{h_s} \right) \Delta h \quad (4.12.42)$$

where h_s is the mean value of h which is considered to be constant through the entire field.

It can readily be shown that because of the initial value h , we have

$$\Delta G = -\Delta h \left[k_{n+1}h_s + \sum_{i=1}^n T_i(k_i - k_{i+1}) \right] = -h_s \Delta h \left[k_{n+1} + \frac{1}{h_s} \sum_{i=1}^n T_i(k_i - k_{i+1}) \right] \quad (7.12.43)$$

$$h_s \Delta h = \frac{\Delta G}{k_{n+1} + \frac{1}{h_s} \sum_{i=1}^n T_i(k_i - k_{i+1})} \quad (7.12.44)$$

and on substituting in eqn. (7.12.42)

$$\Delta V = - \frac{\Delta G}{a} \quad (7.12.45)$$

A change of the volume is in direct proportion to a change of G . The factor of proportionality is the reciprocal of the value of a . A mathematical description (in the linearized form) of the flow in a deforming medium clearly has a character similar

to that of the flow in a non-deforming medium. In plane flow, the methods of solution of the initial equations are the same in both cases, and we can, therefore, use the procedures which we have described in the foregoing sections.

7.12.2. The leakage effect under one-dimensional elastic deformation

Consider the following case: in a given space, the layers alternate in a way that, starting from the base, all the odd horizons are of good permeability, with water flowing through them mainly in the horizontal direction. All the even layers are less permeable and of comparatively small thickness, with water flowing through them predominantly in the vertical direction. Since they are thinner, they experience relatively small deformations under a static load.

In the odd layers, the lines $h_j = \text{const.}$ are taken to be vertical while in the even layers, the equipotentials are mostly horizontal. Hence, below the groundwater surface we have, according to Fig. 7.21c,

$$\int_{T_{j-1}}^{T_j} k_j \left(\frac{\partial^2 h_j}{\partial x^2} + \frac{\partial^2 h_j}{\partial y^2} \right) dz + \int_{T_{j-1}}^{T_j} k_j \frac{\partial^2 h_j}{\partial z^2} dz = \int_{T_{j-1}}^{T_j} \gamma \left(\frac{1}{E_j} + \frac{m_j}{E_v} \right) \frac{\partial h_j}{\partial t} dz \quad (7.12.46)$$

for

$$j = i \lfloor \sin(i\pi/2) \rfloor; \quad i = 1, 2, 3, \dots, I; \quad j > 0$$

where i denotes the position in the sequence of layers starting from the base.

Within the range of the odd layers, the vertical velocity component varies linearly; within the range of the even layers, it remains constant. The solution of the problem consists of finding the distribution of the function $h_j(x, y)$ for each odd layer. To this end, we use the systems of equations which are obtained by carrying out the integrations indicated in eqns. (7.12.46):

$$\frac{\partial h_j}{\partial t} = \frac{k_j \Delta T_j}{\mu_{pj}} \left(\frac{\partial^2 h_j}{\partial x^2} + \frac{\partial^2 h_j}{\partial y^2} \right) + \frac{k_{j-1}(h_{j-2} - h_j)}{\Delta T_{j-1} \mu_{pj}} + \frac{k_{j+1}(h_{j+2} - h_j)}{\Delta T_{j+1} \mu_{pj}} \quad (7.12.47)$$

where

$$\mu_{pj} = \gamma \Delta T_j \left(\frac{1}{E_j} + \frac{m_j}{E_v} \right) \quad (7.12.48)$$

If the free surface lies in one of the odd layers, we substitute

$$\Delta T_j = h_j - T_{j-1} \quad (7.12.49)$$

in the respective equation.

The last term on the right-hand side of eqn. (7.12.47) is replaced by

$$\frac{v_0}{\mu_{pj}} \quad (7.12.50)$$

$$\mu_{pj} = \gamma(h - T_{j-1}) \left(\frac{1}{E_j} + \frac{m_j}{E_v} \right) + \mu_j \quad (7.12.51)$$

where

v_0 is the velocity which is used to express the effect of an external inflow, for example, by infiltration or evaporation,

μ_{jp} — the active porosity of the last odd layer.

If the free surface lies in the even layer for which $i = I$, the expression

$$j = I - 1 \quad (7.12.52)$$

is substituted in the equation for the last odd layer.

The last term on the right-hand side of eqn. (7.12.47) is replaced by

$$\frac{1}{\mu_{pj}} \left[v_0 + \frac{k_{j+1}(h_I - h_j)}{h_I - T_{I-1}} \right] \quad (7.12.53)$$

where h_I is the ordinate of the free surface in the last even layer.

We also add the following supplementary equation

$$\frac{\partial h_{j+1}}{\partial t} = \frac{1}{\mu_{pI}} \left(k_{j+1} \frac{h_I - h_j}{h_I - T_{I-1}} + v_0 \right) \quad (7.12.54)$$

$$\mu_{pI} = \gamma(h_I - T_{I-1}) \left(\frac{1}{E_I} + \frac{m_I}{E_v} \right) + \mu_I \quad (7.12.55)$$

where μ_I is the active porosity of the last even layer.

A solution of the entire system of equations is tedious, and a closed-form solution is only obtained in fairly simple cases. As an example, consider the problem discussed by Hantush and Jacob [16]. The authors studied the performance of a well (Fig. 7.21d) whose inlet section is located in a permeable layer lying on the base, and which operates under a constant take-off. The less permeable material of the first even layer allows the water to flow over. The second odd layer consists of a highly permeable soil which contains the stationary free surface. Since this surface remains horizontal during the pumping process, we have

$$\frac{\partial h_3}{\partial t} = 0 \quad (7.12.56)$$

In the first odd layer above the base

$$\frac{\partial h_1}{\partial t} = \frac{k_1 \Delta T_1}{\mu_{p1}} \left(\frac{\partial^2 h_1}{\partial x^2} + \frac{\partial^2 h_1}{\partial y^2} \right) + \frac{k_2(h_3 - h_1)}{\Delta T_2 \mu_{p1}} \quad (7.12.57)$$

Introduce the substitution

$$\Delta h = h_3 - h_1 \quad (7.12.58)$$

and write eqn. (7.12.57) in cylindrical coordinates as follows:

$$\frac{\mu_{p1}}{k_1 \Delta T_1} \frac{\partial(\Delta h)}{\partial t} = \frac{\partial^2(\Delta h)}{\partial r^2} + \frac{1}{r} \frac{\partial(\Delta h)}{\partial r} - \frac{k_2 \Delta h}{k_1 \Delta T_1 \Delta T_2}$$

Using the notation

$$a = \frac{k_1 \Delta T_1}{\mu_{p1}}, \quad \beta = \sqrt{\frac{k_2}{k_1 \Delta T_1 \Delta T_2}}$$

and recalling that $\Delta h(r, 0) = 0$, and moreover, that

$$\lim_{r \rightarrow 0} 2\pi r \frac{\partial(\Delta h)}{\partial r} = Q_s = \text{const.}$$

where Q_s is the constant take-off from the well (a negative value), we obtain the solution in the form

$$\begin{aligned} \Delta h(r, t) &= - \frac{Q_s}{4\pi k_1 \Delta T_1} \int_{r^2/4at}^{\infty} \frac{\text{Exp} \{ -[\eta + (r^2 \beta^2 / 4\eta)] \}}{\eta} d\eta = \\ &= - \frac{Q_s}{4\pi k_1 \Delta T_1} W\left(\frac{r^2}{4at}, r\beta\right) \end{aligned}$$

The function $W(r^2/4at, r\beta)$ was evaluated numerically and the results were set out in tables suitable for practical applications. Fig. 7.21e shows the character of the corresponding curves.

7.13. Plane unsteady flow

The possibility of adequately formulating a description of a motion with a free surface in non-deforming media under a predominantly vertical deformation makes for a somewhat easier solution of general problems of plane groundwater flow. Several variants which have found fairly wide practical application are presented in the next sections.

7.22a; the scheme represents the flow in a system of wells operating in a region near the confluence of a river and its tributary. On the assumption of a right-angle junction of the two streams, a system of fictitious wells is considered in the region, and the sign of their strengths is varied in accordance with the required constant value of ΔG along the images of the shorelines. The absolute values of Q are the same in symmetri-

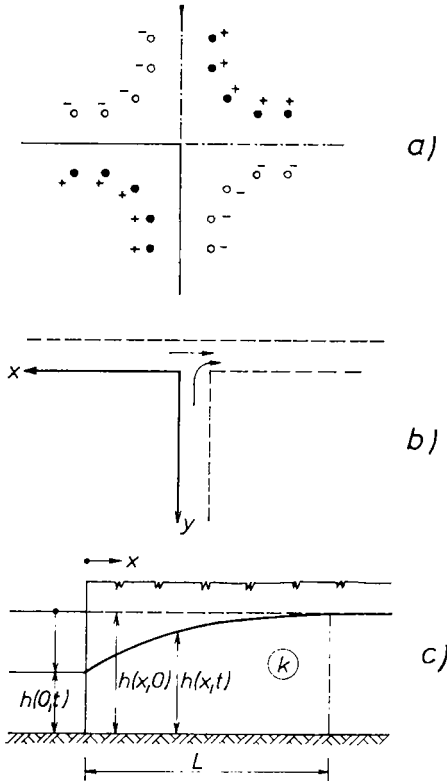


Fig. 7.22. Typical plane flows.

cally placed wells. In the case when M symmetrical wells correspond to each real well, the matrix of the coefficients of the corresponding system of linear algebraic equations takes the form

$$\begin{vmatrix}
 \sum_{i=1}^M F(\delta_{i,1}) & \sum_{i=M+1}^{2M} F(\delta_{i,1}) & \dots & \sum_{i=I-M+1}^I F(\delta_{i,1}) \\
 \sum_{i=1}^M F(\delta_{i,M+1}) & \sum_{i=M+1}^{2M} F(\delta_{i,M+1}) & \dots & \sum_{i=I-M+1}^I F(\delta_{i,M+1}) \\
 \dots & \dots & \dots & \dots \\
 \sum_{i=1}^M F(\delta_{i,I-M+1}) & \sum_{i=M+1}^{2M} F(\delta_{i,I-M+1}) & \dots & \sum_{i=I-M+1}^I F(\delta_{i,I-M+1})
 \end{vmatrix} \quad (7.13.5)$$

A correct sign must, of course, always be assigned to the terms in the summations. The solution of the above system of equations yields the required values of Q_i and can be carried out on a computer. Since the task consists of finding solutions for several time intervals spaced closely together, the extent of the computational work is considerable. The system of equations must, in fact, be solved repeatedly for each time t .

7.13.2. Solution of plane flow by means of the method of the product of partial results

As indicated in the foregoing, the solution of plane seepage can be supposed to have the form of a synthesis of the results obtained by an analysis of partial one-dimensional flows. This method is particularly convenient in cases when the boundary of the region is of a polygonal shape. In a region bounded by a rectangular polygon, we can manage with just two partial solutions each of which contains, besides the time t , only one independent variable. The resulting function $G(x, y, t)$ can be regarded as the product of two partial functions $G_1(x, t)$, $G_2(y, t)$, viz.

$$G(x, y, t) = G_1(x, t) G_2(y, t) \quad (7.13.6)$$

Eqn. (7.13.6) implies that

$$\frac{\partial G}{\partial t} = G_1 \frac{\partial G_2}{\partial t} + G_2 \frac{\partial G_1}{\partial t} \quad \frac{\partial G}{\partial x} = G_2 \frac{\partial G_1}{\partial x}, \quad \frac{\partial G}{\partial y} = G_1 \frac{\partial G_2}{\partial y}$$

$$\frac{\partial^2 G}{\partial x^2} = G_2 \frac{\partial^2 G_1}{\partial x^2}, \quad \frac{\partial^2 G}{\partial y^2} = G_1 \frac{\partial^2 G_2}{\partial y^2}$$

Substituting in eqn. (7.13.6), we obtain the equation

$$G_2 \left(\frac{\partial G_1}{\partial t} - a \frac{\partial^2 G_1}{\partial x^2} \right) + G_1 \left(\frac{\partial G_2}{\partial t} - a \frac{\partial^2 G_2}{\partial y^2} \right) = 0$$

which is satisfied for

$$\frac{\partial G_1}{\partial t} = a \frac{\partial^2 G_1}{\partial x^2} \quad (7.13.7)$$

$$\frac{\partial G_2}{\partial t} = a \frac{\partial^2 G_2}{\partial y^2} \quad (7.13.8)$$

It is, of course, necessary that the boundary and initial conditions of the two partial solutions should also be satisfied.

Consider the right-angle junction of a river and its tributary shown in Fig. 7.22b. The levels of the initial water surface in both streams and the values of $G(x, y) = \text{const.}$ corresponding to them are supposed to be the same. A sudden fall of the water surface in both rivers takes place at the time $t = 0$. The new position of the water surface is characterized by $G(t) = \text{const.}$

Write

$$G(x, y, t) = G(t) + \Delta G(x, y, t)$$

with

$$\Delta G(x, y, t) = G(x, y, t) - G(t)$$

and

$$\Delta G(x, y, 0) = \Delta G_0 = \text{const.}$$

$$\Delta G = 0$$

along the boundaries.

Rewrite eqns. (7.8.2) and (7.8.3) as follows:

$$\frac{\partial(\Delta G_1)}{\partial t} = a \frac{\partial^2(\Delta G_1)}{\partial x^2}$$

$$\Delta G_1(x, 0) = \Delta G_0, \quad \Delta G_1(0, t) = 0$$

$$\frac{\partial(\Delta G_2)}{\partial t} = a \frac{\partial^2(\Delta G_2)}{\partial y^2}$$

$$\Delta G_2(y, 0) = \Delta G_0, \quad \Delta G_2(0, t) = 0$$

In keeping with the foregoing discussion, we can write the solution in the form

$$\Delta G_1(x, t) = \Delta G_0 \operatorname{Erf} \left[\frac{x}{\sqrt{(4at)}} \right]$$

$$\Delta G_2(y, t) = \Delta G_0 \operatorname{Erf} \left[\frac{y}{\sqrt{(4at)}} \right]$$

In the sense of eqn. (7.13.6), the resulting solution has the form of the product

$$\Delta G(x, y, t) = \Delta G_0 \operatorname{Erf} \left[\frac{x}{\sqrt{(4at)}} \right] \operatorname{Erf} \left[\frac{y}{\sqrt{(4at)}} \right] \quad (7.13.9)$$

For $x = 0$, we obtain

$$\Delta G(x, y, t) = 0$$

for $y = 0$,

$$\Delta G(x, y, t) = 0$$

and for $t = 0$,

$$\Delta G(x, y, 0) = \Delta G_0$$

The solution (7.13.9) clearly satisfies the prescribed boundary and initial conditions.

7.13.3. Methods of successive variations of steady states.

Variational procedure

The method of successive variations of steady states, which belongs among the approximate procedures for solving problems of unsteady flow is one of the oldest forms of study based on numerous theoretical assumptions. It regards a given state at each instant as a steady state, and derives the time relationships on the basis of a total balance of inflow and outflow. This fact makes it directly connected with methods frequently employed in hydrology.

The principle of the method can best be explained by using a simple example discussed by Aravin and Numerov [1]. Fig. 7.22c shows an earth massif in which the initial water surface is characterized by $h(x, 0) = \text{const}$. The fluctuation of the water surface on the boundary is described by the function $h(0, t)$. The response inside the soil block is determined by the distribution of the function $h = h(x, t)$. It is assumed that the effect of the fluctuation on the boundary at a given instant will not manifest itself closer than at the point $x = L$. At $x > L$, the initial state experiences no changes. The flow per unit time through each vertical line is determined using the relation (Chapter 1)

$$q = -kh \frac{\partial h}{\partial x} \quad (7.13.10)$$

The solution to this equation is Dupuit's equation, valid for the steady state, viz.

$$qx = -\frac{k}{2} [h^2 - h^2(0, t)] \quad (7.13.11)$$

Hence, on the left-hand boundary, we have

$$q = -\frac{k}{2L} [h^2(x, 0) - h^2(0, t)] \quad (7.13.12)$$

Assume that at a certain instant the situation looks like that shown in Fig. 7.22c. The volume which became empty relative to the initial state, is described by

$$V = \mu \left[Lh(x, 0) - \int_0^L h \, dx \right] \quad (7.13.13)$$

where μ is the active porosity.

Defining the function h by means of eqn. (7.13.11), we obtain eqn. (7.13.13) in the form

$$V = \frac{\mu k}{6q} [h^3(x, 0) - 3h^2(0, t) h(x, 0) + 2h^3(0, t)] \quad (7.13.14)$$

The emptying of the soil takes place through the left-hand boundary. Hence the

throughflow at the point $x = 0$ is given by

$$q = \frac{dV}{dt} = \frac{\mu k}{6} \frac{d\{[h^3(x, 0) - 3h^2(0, t)h(x, 0) + 2h^3(0, t)]/q\}}{dt} \quad (7.13.15)$$

On multiplying both sides of eqn. (7.13.15) by the expression

$$\frac{h^3(x, 0) - 3h^2(0, t)h(x, 0) + 2h^3(0, t)}{q} \quad (7.13.16)$$

and applying eqn. (7.13.12), we obtain, in place of eqn. (7.13.15) the equation

$$\begin{aligned} & h^3(x, 0) - 3h^2(0, t)h(x, 0) + 2h^3(0, t) = \\ & = \frac{\mu L^2}{3k} \frac{d\left[\frac{h^3(x, 0) - 3h^2(0, t)h(x, 0) + 2h^3(0, t)}{h^2(x, 0) - h^2(0, t)}\right]^2}{dt} \end{aligned} \quad (7.13.17)$$

Integrating eqn. (7.13.17) on the condition that $L = 0$ at the time $t = 0$

$$\begin{aligned} & \int_0^t [h^3(x, 0) - 3h^2(0, t)h(x, 0) + 2h^3(0, t)] dt = \\ & = \frac{\mu L^2}{3k} \left[\frac{h^3(x, 0) - 3h^2(0, t)h(x, 0) + 2h^3(0, t)}{h^2(x, 0) - h^2(0, t)} \right]^2 \end{aligned}$$

we can write

$$\begin{aligned} L = & \frac{h^2(x, 0) - h^2(0, t)}{h^3(x, 0) - 3h^2(0, t)h(x, 0) + 2h^3(0, t)} \cdot \\ & \cdot \sqrt{\left\{ \frac{3k}{\mu} \int_0^t [h^3(x, 0) - 3h^2(0, t)h(x, 0) + 2h^3(0, t)] dt \right\}} \end{aligned}$$

For $h(x, 0) = \text{const.}$, $h(0, t) = \text{const.}$

$$\begin{aligned} L = & \frac{h^2(x, 0) - h^2(0, t)}{\sqrt{[h^3(x, 0) - 3h^2(0, t)h(x, 0) + 2h^3(0, t)]}} \sqrt{\frac{3kt}{\mu}} = \\ = & \frac{h(x, 0) + h(0, t)}{\sqrt{[h(x, 0) + 2h(0, t)]}} \sqrt{\frac{3kt}{\mu}} \end{aligned} \quad (7.13.18)$$

Eqn. (7.13.18) characterizes the unknown L , i.e. the range of the effect of unsteady flow. By eqns. (7.13.11) and (7.13.12) we have, at each instant,

$$h = \sqrt{\left\{ h^2(0, t) + [h^2(x, 0) - h^2(0, t)] \frac{x}{L} \right\}}, \quad x \leq L \quad (7.13.19)$$

Thus, we also know the curve of the free surface. The solution is based on the assumption of equal flow through each vertical line selected in the interval $0 < x < L$, and this is at variance with physical experience. Hence the results of calculations made by means of the method of successive steady states are acceptable only after a thorough analysis of the character and definition of the problem at hand.

Results which are more correct theoretically, are obtained by means of a modification of the method of successive steady states, which pays heed to the details of the development of the free surface at a given point in the region being considered. As an example, we shall discuss the case of plane seepage taking place in the region S . The motion is described by the equation

$$\mu \frac{\partial h}{\partial t} = \frac{\partial}{\partial x} \left(kT \frac{\partial h}{\partial x} \right) + \frac{\partial}{\partial y} \left(kT \frac{\partial h}{\partial y} \right) + v_0 \quad (7.13.20)$$

where

kT is the transmissivity whose value generally depends on h . We assume, however, that this is not so in the case being considered,
 v_0 is the velocity which describes the effect of external factors, for example, climatic.

Eqn. (7.13.20) is written out as follows:

$$\mu \frac{\partial h}{\partial t} = kT \left(\frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2} \right) + \frac{\partial h}{\partial x} \frac{\partial(kT)}{\partial x} + \frac{\partial h}{\partial y} \frac{\partial(kT)}{\partial y} + v_0 \quad (7.13.21)$$

Assume that the unsteady process is divided into a sequence of steady states immediately following one another. The transition from one state to another occurs by a jump. Each steady state exists during a time interval Δt . At the beginning of this interval, at the time $t - \Delta t$, we know the value of the function $h(x, y, t - \Delta t)$ and are required to find the value $h(x, y, t) = h_t$ at the end of the interval, i.e. at the time t . To that end, eqn. (7.13.21) is rearranged as follows:

$$\frac{\partial^2 h_t}{\partial x^2} + \frac{\partial^2 h_t}{\partial y^2} = \frac{\mu h_t}{kT \Delta t} - \frac{\mu h_{t-\Delta t}}{kT \Delta t} - \frac{1}{kT} \left[\frac{\partial(h_{t-\Delta t})}{\partial x} \frac{\partial(kT)}{\partial x} + \frac{\partial(h_{t-\Delta t})}{\partial y} \frac{\partial(kT)}{\partial y} + v_0 \right] \quad (7.13.22)$$

Using the notation

$$f_{t-\Delta t}(x, y) = - \frac{1}{kT} \left[\mu \frac{\partial(h_{t-\Delta t})}{\Delta t} + \frac{\partial(h_{t-\Delta t})}{\partial x} \frac{\partial(kT)}{\partial x} + \frac{\partial(h_{t-\Delta t})}{\partial y} \frac{\partial(kT)}{\partial y} + v_0 \right] \quad (7.13.23)$$

we see that the function $f_{t-\Delta t}$ covers various effects, including that of a change of kT in the region:

$$\frac{\partial^2 h_t}{\partial x^2} + \frac{\partial^2 h_t}{\partial y^2} = \frac{\mu h_t}{kT \Delta t} + f_{t-\Delta t}(x, y) \quad (7.13.24)$$

Eqn. (7.13.24) can be solved by applying the principles explained in Section 4.9. The solution of eqn. (7.13.24) is equivalent to the minimization of the functional

$$I = \int_S \left[\left(\frac{\partial h_t}{\partial x} \right)^2 + \left(\frac{\partial h_t}{\partial y} \right)^2 + \frac{\mu h_t^2}{kT \Delta t} + 2h_t f_{t-\Delta t}(x, y) \right] dS \quad (7.13.25)$$

This variational problem is usually solved numerically. The solution is easier in cases when the steady state can be supposed to last throughout two successive time intervals Δt . The function $f_{t-\Delta t}$ is then written in the form

$$f_{t-\Delta t}(x, y) = \frac{\mu}{kT} \frac{h_{t-\Delta t} - h_{t-2\Delta t}}{\Delta t} - \frac{1}{kT} \left[\frac{\partial(h_{t-\Delta t})}{\partial x} \frac{\partial(kT)}{\partial x} + \frac{\partial(h_{t-\Delta t})}{\partial y} \frac{\partial(kT)}{\partial y} + v_0 \right] \quad (7.13.26)$$

because the value $h(x, y, t - 2\Delta t) = h_{t-2\Delta t}$ as well as the value $h(x, y, t - \Delta t) = h_{t-\Delta t}$ are assumed to be known. The required value h_t is obtained from the equation

$$\frac{\partial^2 h_t}{\partial x^2} + \frac{\partial^2 h_t}{\partial y^2} = f_{t-\Delta t}(x, y) \quad (7.13.27)$$

In the Dirichlet problem, consider the region S with a boundary L on which the function $h_t(L) = h_{tL}$ is prescribed. Assume that we know an arbitrary particular solution of eqn. (7.13.27) in the form of the function h_{t0} , and that we also know several particular solutions of the equation

$$\frac{\partial^2 h_t}{\partial x^2} + \frac{\partial^2 h_t}{\partial y^2} = 0 \quad (7.13.28)$$

These particular solutions are denoted by symbols h_{tj} , $j = 1, 2, \dots, m$. It is easy to show that the function

$$\bar{h}_t = h_{t0} + \sum_{j=1}^m b_j h_{tj} \quad (7.13.29)$$

is also a solution of eqn. (7.13.27). This means that the solution of eqn. (7.13.27) is approximated by a number of partial functions. The problem is solved if the solution satisfies the specified boundary condition. Since there are several partial functions, this specification can only be met approximately. We are, of course, interested in achieving as high an accuracy as possible. This can be done by obtaining the minimum value of the integral

$$I = \int_L (\bar{h}_t - h_t)^2 dL = \int_L (\bar{h}_t - h_{tL})^2 dL \quad (7.13.30)$$

The solution of eqn. (7.13.27) is equivalent to a variational problem involving mini-

mization of eqn. (7.13.30). This requirement leads to the condition

$$\frac{\partial I}{\partial b_i} = 2 \int_L (\bar{h}_i - h_{iL}) h_{ti} dL = 0, \quad i = 1, 2, \dots, m \quad (7.13.31)$$

On substituting eqn. (7.13.29), we obtain the following system of equations

$$\sum_{j=1}^m b_j \int_L h_{tj} h_{ti} dL = \int_L (h_{iL} - h_{i0}) h_{ti} dL \quad (7.13.32)$$

which enables us to calculate the values of the coefficients b_j .

The method just outlined was worked out by Trefftz who made use of the fact that h_{ti} is a solution of the Laplace equation (7.13.28). This makes it possible to apply Green's formula (Sections 6.8 and 6.9) and to utilize to advantage the derivative of the function L in the direction of the normal. The system of equations is expressed in the form

$$\sum_{j=1}^m b_j \int_L h_{tj} \frac{\partial h_{ti}}{\partial n} dL = \int_L (h_{iL} - h_{i0}) \frac{\partial h_{ti}}{\partial n} dL \quad (7.13.33)$$

where n denotes that the derivative is taken in the direction of the normal.

The advantage of this method lies in the fact that the integration is carried out along the curve L rather than on the surface S . A variational problem defined under similar conditions by means of the Galerkin (Ritz) method leads to the minimization of the integral

$$I = \int_S \left[\left(\frac{\partial h_t}{\partial x} \right)^2 + \left(\frac{\partial h_t}{\partial y} \right)^2 + 2h_t f_{t-\Delta t}(x, y) \right] dS \quad (7.13.34)$$

7.13.4. Influence function

The method of successive variations of steady states was based on the assumption that the derivative of the dependent variable function $h(x, y, t)$ with respect to time can be replaced by the differences

$$\frac{\partial h}{\partial t} = \frac{h_t - h_{t-\Delta t}}{\Delta t} \quad (7.13.35)$$

or

$$\frac{\partial h}{\partial t} = \frac{h_{t-\Delta t} - h_{t-2\Delta t}}{\Delta t} \quad (7.13.36)$$

The two equations represent the so-called first difference backwards. It is, of course, equally possible to solve the equations of unsteady flow with the help of the

first difference forwards

$$\frac{\partial h}{\partial t} = \frac{h_{t+\Delta t} - h_t}{\Delta t} \quad (7.13.37)$$

This variant is essentially the basis for writing a partial differential equation in the form of differences; the definition of the process rests on the expansion in terms of Taylor's series [14]. The procedure will not be given here, for the problem involved is predominantly one of numerical mathematics.

The shortcoming of the variational and the difference methods lies in the fact that a continuous derivative of the function h with respect to time is replaced by a certain value which approaches, to a greater or lesser degree, the average in the interval Δt . The error introduced by this substitution can only be estimated by an analysis of the problem being considered, for it depends on the specification of the flow region and on the character of the boundary conditions as well as on the choice of the interval Δt . This handicap is absent in the method which the literature sometimes calls the influence function method.

In the application to plane flow, the method assumes a point source at the point x, y inside the region S , which comes into existence under zero initial conditions and becomes extinct immediately afterwards. The corresponding flow is described by the function $\bar{G}(x, y, t)$ which becomes zero on the boundary L of S , viz.

$$\bar{G}(x_L, y_L, t) = 0 \quad (7.13.38)$$

where t is the time.

The initial equation which describes the distribution of the function $\bar{G}$ inside S was discussed in Section 7.11. In plane flow, the equation takes the form

$$\frac{\partial \bar{G}}{\partial t} = a \left(\frac{\partial^2 \bar{G}}{\partial x^2} + \frac{\partial^2 \bar{G}}{\partial y^2} \right) \quad (7.13.39)$$

We have also shown in Section 7.12 that a change of the volume V of the liquid in the vicinity of a given point is given by

$$V = - \frac{\bar{G}}{a} \quad (7.13.40)$$

The point of location of the source and its vicinity merits special attention. At the instant when the liquid penetrates into the region, the source is a singularity (the value of $\bar{G}$ is infinitely large). In an element of the ground area which contains a singularity, the value of $\bar{G}$ is variable; we shall, however, consider an average $\bar{G}$. The value of V determines the magnitude of the increment in volume of the liquid which has collected above the elementary area towards the end of the short period during which the source had been at work. It is assumed that during this short period the liquid was incapable of flowing to the outside of the considered element. Hence the total increment of volume in the region S is equal to the increment of volume in

the element. Consequently,

$$V = -\frac{1}{a} \int_S \bar{G} \, dS \quad (7.13.41)$$

Next to the flow being considered, we shall tackle another unsteady process which is described in the region S with a boundary L by the equation

$$\frac{\partial G}{\partial t} = a \left(\frac{\partial^2 G}{\partial x^2} + \frac{\partial^2 G}{\partial y^2} \right) \quad (7.13.42)$$

Suppose that the initial condition is zero

$$G(x, y, 0) = 0 \quad (7.13.43)$$

and the boundary condition is defined by

$$G_L(x_L, y_L, t) = \text{const.} \quad (7.13.44)$$

Assume that the two flows are studied at the same time t , and that the source starts working at the time $\tau < t$. In the interest of a unified description, write

$$-\frac{\partial \bar{G}}{\partial \tau} = a \left(\frac{\partial^2 \bar{G}}{\partial x^2} + \frac{\partial^2 \bar{G}}{\partial y^2} \right), \quad \frac{\partial G}{\partial \tau} = a \left(\frac{\partial^2 G}{\partial x^2} + \frac{\partial^2 G}{\partial y^2} \right) \quad (7.13.45)$$

Form the product $\bar{G}G$ and differentiate it with respect to time:

$$\frac{\partial}{\partial \tau} (\bar{G}G) = \bar{G} \frac{\partial G}{\partial \tau} - G \frac{\partial \bar{G}}{\partial \tau} = -a \left[G \left(\frac{\partial^2 \bar{G}}{\partial x^2} + \frac{\partial^2 \bar{G}}{\partial y^2} \right) - \bar{G} \left(\frac{\partial^2 G}{\partial x^2} + \frac{\partial^2 G}{\partial y^2} \right) \right] \quad (7.13.46)$$

Integrate both the left-hand and the right-hand sides for $\varepsilon < t$, where ε is a small positive quantity:

$$\int_0^{t-\varepsilon} \frac{\partial(\bar{G}G)}{\partial \tau} \, d\tau = -a \int_0^{t-\varepsilon} \left[G \left(\frac{\partial^2 \bar{G}}{\partial x^2} + \frac{\partial^2 \bar{G}}{\partial y^2} \right) - \bar{G} \left(\frac{\partial^2 G}{\partial x^2} + \frac{\partial^2 G}{\partial y^2} \right) \right] d\tau \quad (7.13.47)$$

The expression on the left-hand side of eqn. (7.13.47) is

$$\bar{G}G|_{\tau=t-\varepsilon} - \bar{G}G|_{\tau=0} \quad (7.13.48)$$

At the time $\tau = 0$, $\bar{G} = 0$, so that integration inside S yields

$$\int_S \bar{G}G|_{\tau=t-\varepsilon} \, dS = -a \int_S \left\{ \int_0^{t-\varepsilon} \left[G \left(\frac{\partial^2 \bar{G}}{\partial x^2} + \frac{\partial^2 \bar{G}}{\partial y^2} \right) - \bar{G} \left(\frac{\partial^2 G}{\partial x^2} + \frac{\partial^2 G}{\partial y^2} \right) \right] d\tau \right\} dS \quad (7.13.49)$$

The effect of the inflow from the source is most marked in the element in which the source is contained, and at the time immediately after the source has come into existence. During the source operation, $\bar{G} = 0$ throughout the region. In the small element being considered, G can also be regarded as a mean value, i.e. as a constant. By eqn. (7.13.42), we then have on the left-hand side of eqn. (7.13.49)

$$\int_S \bar{G} G|_{\tau=t-\varepsilon} dS = G \int_S \bar{G} dS = -a V G \quad (7.13.50)$$

where V is the volume of liquid which has penetrated into the given region from the source.

Assume that we are free to change the order of integration on the right-hand side of eqn. (7.13.49). Then, by Green's formula, we obtain

$$\begin{aligned} -a \int_0^{t-\varepsilon} \left\{ \int_S \left[G \left(\frac{\partial^2 \bar{G}}{\partial x^2} + \frac{\partial^2 \bar{G}}{\partial y^2} \right) - \bar{G} \left(\frac{\partial^2 G}{\partial x^2} + \frac{\partial^2 G}{\partial y^2} \right) \right] dS \right\} d\tau = \\ = -a \int_0^{t-\varepsilon} \left[\int_L \left(G_L \frac{\partial \bar{G}}{\partial n} - \bar{G}_L \frac{\partial G}{\partial n} \right) dL \right] d\tau \end{aligned} \quad (7.13.52)$$

where n denotes that the derivative is in the direction of the normal to the boundary L . G_L , $\bar{G}_L$ are the values of G , $\bar{G}$ on the boundary L , where we have assumed that $\bar{G}_L = 0$, $G_L = \text{const}$. Hence for $\varepsilon \rightarrow 0$, the right-hand side of eqn. (7.13.51) becomes

$$-a G_L \int_0^{t-\varepsilon} \left(\int_L \frac{\partial \bar{G}}{\partial n} dL \right) d\tau = -a G_L \int_0^{t-\varepsilon} Q_L d\tau = -a G_L V_L(t) \quad (7.13.52)$$

where

Q_L is the outflow per unit time across the boundary L , defined in terms of the G potential,

$V_L(t)$ — the volume which has passed across the boundary L during the time from 0 to t as a result of the operation of a sink placed at the point x, y .

We thus have

$$-a V G(x, y, t) = -a G_L V_L(t) \quad (7.13.53)$$

$$G(x, y, t) = G_L \frac{V_L(t)}{V} \quad (7.13.54)$$

The function $V_L(t)/V$ which co-determines the value $G(x, y, t)$ is called the influence function. Its form depends on the geometry of the flow region. The sink theory enables us to establish it for a plane, a half-plane, a circle, etc.

The influence function can also receive another form. Assume, for example, that at a given point (x, y) there operates successively a series of sinks of equal strengths, each of which comes into existence at a certain time and becomes extinct immediately

afterwards. This process results in a flow produced around the sink from which the liquid permanently flows out. During the time from 0 to t , we also have

$$G(x, y, t) = \frac{G_L}{Vt} \int_0^t \frac{\partial V_{t-t_v}}{\partial t} dt_v = G_L \frac{W_L(t)}{W} \quad (7.13.55)$$

where

$W_L(t)$ — the total volume of liquid which moved across the boundary L in the time t ,

W — the total volume of liquid which has flown out of the sink.

The function $W_L(t)/W$ can also be regarded as an influence function and be defined for a given region of flow. In an engineering interpretation we can use a well of a constant strength, located at the point x, y in the region S with a boundary L , and consider the function $W_L(t)/W$ to be the ratio of the total volume of water which passed in the time t across the boundary L , to the amount of water which was pumped out of the well in the same time.

Since the theory of sinks and the theory of wells were both dealt with in the foregoing sections, it is not necessary to dwell any longer on the problem of setting up particular forms of the influence function.

The fundamental theory of the influence function is based on the assumption that the function $G(x, y, t)$ has a zero value at the time $t = 0$. Cases in which this is not so can be simplified by eliminating the effect of the stationary-conceived initial condition by means of a suitable substitution.

More complicated boundary conditions can be interpreted as a result of the substitution of simpler solutions. The effect of a non-uniform distribution of the values of the function G on the boundary, which can also be prescribed sometimes, can be diminished by dividing the boundary into segments and applying to them the principle of superposition. Successive solutions and the superposition of such solutions are possible. One such possibility is evident from the equation

$$G(x, y, t) = \frac{1}{W} \sum_{i=1}^n \sum_{j=1}^m G_{Li}(W_{i,t-t_j} - W_{i,t-t_{j+1}}) \quad (7.13.56)$$

where

$i = 1, 2, \dots, n$ is the number of segments on the boundary L ,

$W_{i,t-t_j}$ — the total volume of liquid which passed through the i -th segment in the time $t > t_j$. The symbol $t - t_j$ denotes the time during which the boundary condition prescribed along the i -th segment, is not zero,

G_{Li} — the boundary condition which has a constant value in a given time interval.

By superposing solutions according to eqn. (7.13.55) we can obtain the result for any, arbitrarily defined, non-constant value of G on the boundary.

7.13.5. Method of superposition of reciprocal effects

Numerical solutions of plane seepage problems are harder to carry out in cases when the boundary conditions are only defined in small sections of the plane and have the character of singularities. The effect of such conditions can be described by approximations.

Whenever one can base the solution on the initial linear equations, one can apply a procedure called the superposition of reciprocal effects.

Suppose, for example, that in a given region bounded by the curve L we are required to find the distribution of the values of $G(x, y, t)$ when a system of wells from which a quantity Q_i is pumped per second, is situated inside the curve L . The specification prescribes the distribution of the values of $G(L, t)$ on the boundary of the region.

In keeping with the theory described in Section 7.13.3, it is possible to calculate separately the value of $\Delta G(x, y, t)$ which characterizes the decrease of the water surface in an unbounded plane, produced merely by the effect of the wells. We find that a certain $\Delta G(L, t)$ exists at the points of the curve L .

The initial equation is eqn. (7.6.3), which we shall also use to solve the field inside the curve L , but without the wells. In this solution we assume that $\overline{\Delta G}(L, t) = -\Delta G(L, t)$ on the curve L . In this way, we obtain $\overline{\Delta G}(x, y, t)$ inside L . The resulting decrease of the water surface produced by the effect of the wells, taking into account the existence of the curve L , is obtained by the superposition of the two reciprocal solutions proposed above. The resulting value is

$$\overline{\overline{\Delta G}}(x, y, t) = \Delta G(x, y, t) + \overline{\Delta G}(x, y, t)$$

Hence $\overline{\overline{\Delta G}}(L, t) = 0$ on the boundary L .

The initial problem was defined by the distribution of the values of $G(L, t)$ on the boundary L . The solution inside the boundary L which ignores the effect of the wells, is denoted by the symbol $\overline{G}(x, y, t)$. On superposing the latter onto the previous solution, we finally have

$$G(x, y, t) = \overline{G}(x, y, t) + \overline{\overline{\Delta G}}(x, y, t)$$

The procedure outlined above enables us to calculate numerically the flow in the region L without wells. The effect of the wells is accounted for analytically, in a separate process. The two solutions are then superposed onto one another. In this way one can solve systems of wells situated in a region with a curved boundary, with consideration also given to the time dependent boundary values prescribed on the limit-line.

7.14. Hydraulic aspects of soil consolidation

The theory of flow under unidirectional elastic deformation counts on the predominant effect of the vertical component of deformation. To this is also adapted the mathematical description of groundwater flow which is retroactively connected with the deformation.

In more general cases, the factors of importance are the stress distribution at each point of the soil and the direction of the applied forces. The soil permeability depends on the magnitude of deformation. Since the concept of deformation also contains the premise of motion of soil particles, we can say that not only the water but the soil as well „flows” in a sense. Complete saturation of soil with water cannot be assumed in every case; consequently, compressibility and the motion of gases in pores are apt to have a substantial effect. The various factors are analyzed in soil mechanics, and its literature is replete with theories which describe them and their complex effect to a greater or lesser degree of accuracy.

The simplest of all is a mathematical description which, though acknowledging the three-dimensional nature of the effect of the applied forces, ignores their influence on the magnitude of the coefficient of permeability and neglects the action of possible dilatancy, contractancy.

7.14.1. Consolidation without the retroactive effect on seepage characteristics

Let us start from eqn. (7.12.5) which describes the state in a soil element of sides $\Delta x, \Delta y, \Delta z$. The mass of the liquid contained in this element is

$$\Delta M = m\rho \Delta x \Delta y \Delta z = m\rho \Delta o_z \quad (7.14.1)$$

Assuming a change with time, we obtain

$$\frac{\partial(\Delta M)}{\partial t} dt = \left[m\rho \frac{\partial(\Delta o_z)}{\partial t} + \rho \Delta o_z \frac{\partial m}{\partial t} + m \Delta o_z \frac{\partial \rho}{\partial t} \right] dt \quad (7.14.2)$$

In an anisotropic soil, due attention must be paid to the different values of the moduli of elasticity in the directions of the coordinate axes, E_x, E_y, E_z , and to the different values of Poisson ratios in the directions of the coordinate axes ν_x, ν_y, ν_z . According to the well-known laws of linear elasticity, the stresses $\sigma_x, \sigma_y, \sigma_z$ in the directions of the coordinate axes are related as follows:

$$\frac{d(\Delta x)}{\Delta x} = \frac{1}{E_x} \left[\sigma_x - \frac{1}{\nu_x} (\sigma_y + \sigma_z) \right]$$

$$\frac{d(\Delta y)}{\Delta y} = \frac{1}{E_y} \left[\sigma_y - \frac{1}{\nu_y} (\sigma_x + \sigma_z) \right]$$

$$\frac{d(\Delta z)}{\Delta z} = \frac{1}{E_z} \left[\sigma_z - \frac{1}{\nu_z} (\sigma_x + \sigma_y) \right]$$

Hence in the case being considered

$$-\frac{d(\Delta o_z)}{\Delta o_z} = \sigma_x \left(\frac{1}{E_x} - \frac{1}{\nu_y E_y} - \frac{1}{\nu_z E_z} \right) + \sigma_y \left(\frac{1}{E_y} - \frac{1}{\nu_x E_x} - \frac{1}{\nu_z E_z} \right) + \sigma_z \left(\frac{1}{E_z} - \frac{1}{\nu_x E_x} - \frac{1}{\nu_y E_y} \right) \quad (7.14.3)$$

Since elasticity theory frequently operates with the mean of the principal stresses (the so-called hydrostatic view), we can formally take it that

$$-\frac{d(\Delta o_z)}{\Delta o_z} = \frac{\sigma}{E} \quad (7.14.4)$$

where

σ — the mean of the principal stresses,

E — a value which includes all the remaining factors in a way that makes the right-hand side of eqn. (7.14.4) equal to the right-hand side of eqn. (7.14.3).

The omnidirectional water pressure p acting in the pores is given by

$$\frac{\partial(\Delta o_z)}{\partial t} = -\frac{\Delta o_z}{E} \frac{\partial \sigma}{\partial t} = \frac{\Delta o_z}{E} \frac{\partial p}{\partial t} \quad (7.14.5)$$

Since in compression the volume of the soil grains experiences no changes, we have, analogously to Section 7.12

$$d(\Delta o_z) = d(1 - m) \Delta o_z = 0$$

and from there

$$\frac{\partial m}{\partial t} = \frac{1 - m}{E} \frac{\partial p}{\partial t} \quad (7.14.6)$$

Just as in Section 7.12, we write

$$\frac{\partial q}{\partial t} = \frac{q}{E_v} \frac{\partial p}{\partial t} \quad (7.14.7)$$

where E_v is the modulus of elasticity of water.

Substituting eqns. (7.14.5), (7.14.6) and (7.14.7) in eqn. (7.14.2) we obtain

$$\frac{\partial(\Delta M)}{\partial t} dt = \left[m q \frac{\Delta o_z}{E} \frac{\partial p}{\partial t} + \frac{q \Delta o_z}{E} (1 - m) \frac{\partial p}{\partial t} + m \Delta o_z \frac{q}{E_v} \frac{\partial p}{\partial t} \right] dt \quad (7.14.8)$$

Since, at the same time,

$$h = \frac{p}{\gamma} + z, \quad \frac{\partial p}{\partial t} = \gamma \frac{\partial h}{\partial t}$$

the above becomes

$$\frac{\partial(\Delta M)}{\partial t} dt = \Delta o_z \gamma \varrho \left(\frac{1}{E} + \frac{m}{E_v} \right) \frac{\partial h}{\partial t} \quad (7.14.9)$$

According to eqn. (7.12.5) and in keeping with Darcy's linear law of flow, we can write

$$\frac{\partial(\Delta M)}{\partial t} dt = \Delta o_z \varrho \left[\frac{\partial}{\partial x} \left(k_x \frac{\partial h}{\partial x} \right) + \frac{\partial}{\partial y} \left(k_y \frac{\partial h}{\partial y} \right) + \frac{\partial}{\partial z} \left(k_z \frac{\partial h}{\partial z} \right) \right] dt \quad (7.14.10)$$

Comparison of eqn. (7.14.9) with eqn. (7.14.10) yields

$$\frac{\partial}{\partial x} \left(k_x \frac{\partial h}{\partial x} \right) + \frac{\partial}{\partial y} \left(k_y \frac{\partial h}{\partial y} \right) + \frac{\partial}{\partial z} \left(k_z \frac{\partial h}{\partial z} \right) = \gamma \left(\frac{1}{E} + \frac{m}{E_v} \right) \frac{\partial h}{\partial t} \quad (7.14.11)$$

In a permeable soil, the principal direction of anisotropy is assumed to be horizontal, the coefficient of anisotropy being

$$A = \frac{k_x}{k_z} = \frac{k_y}{k_z}, \quad k_x = k_y = k$$

After introducing the substitutions

$$x = \bar{x}, \quad y = \bar{y}, \quad z = \frac{\bar{z}}{\sqrt{A}}, \quad h = \frac{\bar{h}}{\sqrt{A}}, \quad k_z = \frac{k}{A}$$

in eqn. (7.4.11), we obtain

$$\frac{\partial \bar{h}}{\partial t} = \frac{k}{\gamma_v [(1/E) + (m/E_v)]} \left(\frac{\partial^2 \bar{h}}{\partial \bar{x}^2} + \frac{\partial^2 \bar{h}}{\partial \bar{y}^2} + \frac{\partial^2 \bar{h}}{\partial \bar{z}^2} \right) \quad (7.14.12)$$

Eqn. (7.14.12) clearly applies to the elevated region in which the mean coefficient of permeability is given by

$$k_s = \sqrt{(k_x k_z)} = \sqrt{(k_y k_z)}$$

On the strength of this fact, eqn. (7.14.12) turns out to be

$$\frac{\partial \bar{h}}{\partial t} = \frac{k_s \sqrt{A}}{\gamma [(1/E) + (m/E_v)]} \left(\frac{\partial^2 \bar{h}}{\partial \bar{x}^2} + \frac{\partial^2 \bar{h}}{\partial \bar{y}^2} + \frac{\partial^2 \bar{h}}{\partial \bar{z}^2} \right)$$

or

$$\frac{\partial h}{\partial t} = \frac{k_s \sqrt{A}}{\gamma [(1/E) + (m/E_v)]} \left(\frac{\partial^2 h}{\partial \bar{x}^2} + \frac{\partial^2 h}{\partial \bar{y}^2} + \frac{\partial^2 h}{\partial \bar{z}^2} \right)$$

Writing

$$C = \frac{k_s \sqrt{A}}{\gamma[(1/E) + (m/E_v)]} \quad (7.14.13)$$

$$\frac{\partial p}{\partial t} = C \left(\frac{\partial^2 p}{\partial \bar{x}^2} + \frac{\partial^2 p}{\partial \bar{y}^2} + \frac{\partial^2 p}{\partial \bar{z}^2} \right) \quad (7.14.14)$$

we note that the value of the coefficient C covers the effect of a number of factors which influence the process of consolidation. It is a very difficult matter indeed to distinguish between the partial effects from an engineering point of view. An overall estimate is somewhat facilitated by an analysis of material tests, particularly if such tests are made under the conditions which are representative of the bearing situation and the states of loading. Thus, for example, the construction of earth dams is preceded by extensive experiments designed to verify the technology of placing the material, especially that of watertight cores. An analysis of the character of the test provides valuable information which can be put to good use in theoretical solutions.

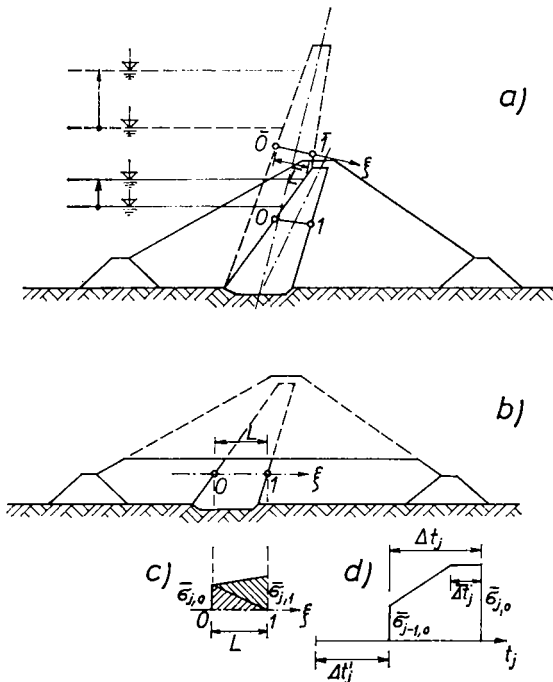


Fig. 7.23. Consolidation in the core of an earth dam.

The complexity of the problem usually forces the engineer to turn to experimental or numerical procedures. However, in many cases an analytical solution is quite adequate for the purpose at hand. As an example, consider the time development of pore pressures in a relatively slender core of an earth dam. Assume that the consolidation processes induced by the weight of the dam and flooding the reservoir

to a certain level came to an end after the construction had been terminated. At a certain instant which is taken to be the beginning of a further consolidation process, the water surface undergoes a linear rise from the initial level to the final position. Denote the velocity of the rise by the letter v_h . Both the protective and the supporting sections of the dam are considered to be of relatively high permeability, and the fluctuations of the water surface in the reservoir are supposed to be transmitted directly to the upstream side of the core (Fig. 7.23a). Since the seepage through the core is quantitatively small, the atmospheric pressure is assumed to act on the downstream side of the core.

For technological reasons the core material is permeable anisotropically, the principal direction of anisotropy being horizontal. The flow through the core is studied in the elevated region. It will be enough for the purpose of our discussion to calculate the pressure in the section $\overline{OI}$ perpendicular to the elevated axis of the core. The position of the section is actually characterized by the segment OI . At the point O , the pressure is p_0 in the initial, and p_k in the final position of the water surface. To each point of the line segment $\overline{OI}$ corresponds a point of the line segment OI . Hence the distribution of the pressure $p(\xi, 0)$ at the beginning of the process is known. During the course of consolidation, we determine the increment of pressure

$$\Delta p(\xi, t) = p(\xi, t) - p(\xi, 0) \quad (7.14.15)$$

and solve the equation

$$\frac{\partial(\Delta p)}{\partial t} = C \frac{\partial^2(\Delta p)}{\partial \xi^2} \quad (7.14.16)$$

under the conditions

$$\left. \frac{\partial(\Delta p)}{\partial t} \right|_{\xi=0} = \gamma v_h, \quad \left. \frac{\partial(\Delta p)}{\partial t} \right|_{\xi=L} = 0 \quad (7.14.17)$$

$$\Delta p(\xi, 0) = 0 \quad (7.14.18)$$

where γ is the specific weight of water.

Assume first that the rise of the water surface in the reservoir is unlimited, and base the solution on the principle of superposition. The specified boundary function is a sequence of sudden jumps. The response inside the field is defined by the function $f(\xi, t)$. Assuming that $\Delta t = t_{i+1} - t_i$, we therefore have

$$\Delta p(\xi, t) \sum_{i=1}^{\infty} [\Delta p(0, t - t_{i+1}) - \Delta p(0, t - t_i)] f(\xi, t - t_i)$$

or, in the differential form

$$\Delta p(\xi, t) = \int_0^t f(\xi, t - \tau) \frac{\partial[\Delta p(0, \tau)]}{\partial \tau} d\tau$$

The function $f(\xi, t - \tau)$ in the integrand is replaced by eqn. (7.4.26). Noting that the

partial derivative in the integrand is actually the prescribed boundary condition (7.14.17), we obtain

$$\begin{aligned}\Delta p(\xi, t) &= \gamma v_h \int_0^t \left\{ 1 - \frac{\xi}{L} - \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \sin \frac{n\pi\xi}{L} \exp \left[-\frac{n^2\pi^2 C}{L^2} (t - \tau) \right] \right\} d\tau = \\ &= \gamma v_h \left\{ \left(1 - \frac{\xi}{L} \right) t - \frac{2L^2}{C\pi^3} \sum_{n=1}^{\infty} \frac{1}{n^3} \sin \frac{n\pi\xi}{L} \left[1 - \exp \left(-\frac{n^2\pi^2 C t}{L^2} \right) \right] \right\}\end{aligned}\quad (7.14.19)$$

The final position of the water surface in the reservoir is reached at the time

$$t_k = \frac{p_k - p_0}{\gamma v_h} = \frac{\Delta p_h}{\gamma v_h} \quad (7.14.20)$$

According to the specification of the problem, the final position of the water surface is to be stationary in the times $t > t_k$. We shall, therefore, assume that the water surface, though continuously rising with the velocity v_k , falls off with equal velocity starting at the time t_k .

For the times $t > t_k$, superposition yields

$$\begin{aligned}\Delta p(\xi, t) &= \gamma v_h \left[\left(1 - \frac{\xi}{L} \right) t_k - \right. \\ &\quad \left. - \frac{2L^2}{C\pi^3} \sum_{n=1}^{\infty} \frac{1}{n^3} \sin \frac{n\pi\xi}{L} \left\{ \exp \left[-\frac{n^2\pi^2 C}{L^2} (t - t_k) \right] - \exp \left(-\frac{n^2\pi^2 C t}{L^2} \right) \right\} \right]\end{aligned}$$

As eqn. (7.14) implies, after a long time when $t \gg t_k$, the above becomes

$$\Delta p(\xi, t) \rightarrow \gamma v_h \left(1 - \frac{\xi}{L} \right) t_k = \Delta p_k \left(1 - \frac{\xi}{L} \right) \quad (7.14.21)$$

The approximate solution carried out for a chosen cross-section clearly shows that after a sufficient lapse of time from the beginning of the process, the distribution of the pressure becomes linear along the segment $\overline{OI}$ and hence also along the segment $0I$.

7.14.2. Migration of time

The theory which ignores the effect of elastic deformations on seepage characteristics, is comparatively simple and fairly easy to apply to a broad range of practical problems. Sometimes, however, it is not adequate to the task at hand because consolidation can hardly be regarded as a process in which readily separable actions of static loading are superimposed onto corresponding hydraulic effects. Recent theories take due account of the effect of plastic deformations and, at the same time,

acknowledge the existence of non-linear laws of water motion. In our opinion, the most comprehensive of such theories is that evolved by Florin [8] who assumes the validity of Puzyrevski's notion (Chapter 1) in the pre-linear regime. A detailed elaboration of Florin's theory and a number of examples of its application to practice, can be found in a book by Zarecki [46].

As recent findings suggest, the phenomenon is not less complicated from the hydraulic point of view, and the formulation of the fundamental laws might well take notice of time. Additional and more perfect theories are being developed continuously with a view to achieving this end.

A characteristic feature of recent theories is the non-linearity of partial differential equations which describe the process of consolidation. A solution is usually obtained by means of experimental methods and numerical procedures. In practice, it is sometimes convenient to start a solution by using simple and clear procedures which enable us to gain an idea of the character of the process involved. Let us start, for example, from eqn. (7.14.11) and accept all of the assumptions on the basis of which it was derived. Moreover, recall that $k = k(\kappa)$

$$\kappa = \frac{m}{1 - m} \quad (7.14.22)$$

where

κ — the porosity number,
 m — the porosity.

Just as in the foregoing discussion, we assume that the total stress consists of the effective stress $\bar{\sigma}$ and the neutral stress p (the pressure of water in the pores). In the space, however, it holds, more generally, that

$$\bar{\sigma} + p = \text{const.} \quad (7.14.23)$$

$$\bar{\sigma} = A\sigma \quad (7.14.24)$$

where

σ — the mean of the principal stresses (the effective part),
 A — a coefficient whose value depends on the kind of soil being considered (it can be determined by tests in a triaxial apparatus. It is clearly related to the coefficient of lateral pressure but also includes the effect of other factors, for example, the air pressure).

Assume further the linear law of deformation

$$d\bar{\sigma} = -E_m \frac{d(\Delta V)}{\Delta V} \quad (7.14.25)$$

where

ΔV — the volume of soil being examined,
 E_m — the bulk modulus of elasticity.

Hence

$$dm = (1 - m) \frac{1}{AE_m} dp \quad (7.14.26)$$

Our considerations are thus essentially based on fundamental relations which are formally similar to those used in previous sections. On carrying out analogous operations, neglecting the effect of the compressibility of water, we obtain the equation

$$\begin{aligned} \frac{\partial p}{\partial t} = & k_x \left(\frac{\partial p}{\partial x} \right)^2 + k_y \left(\frac{\partial p}{\partial y} \right)^2 + k_z \left(\frac{\partial p}{\partial z} \right)^2 + \\ & + \frac{1}{\gamma AE_m} \left[\frac{\partial}{\partial x} \left(k_x \frac{\partial p}{\partial x} \right) + \frac{\partial}{\partial y} \left(k_y \frac{\partial p}{\partial y} \right) + \frac{\partial}{\partial z} \left(k_z \frac{\partial p}{\partial z} \right) \right] \end{aligned} \quad (7.14.27)$$

where

γ — the specific weight of water.

For small values of pressure gradients, we can neglect the first three terms of eqn. (7.14.27) and write

$$\frac{\partial p}{\partial t} = \frac{AE_m}{\gamma} \left[\frac{\partial}{\partial x} \left(k_x \frac{\partial p}{\partial x} \right) + \frac{\partial}{\partial y} \left(k_y \frac{\partial p}{\partial y} \right) + \frac{\partial}{\partial z} \left(k_z \frac{\partial p}{\partial z} \right) \right] \quad (7.14.28)$$

According to Chapter 1, the coefficient of permeability is a quantity which, in the region of linear law of flow, depends mainly on the porosity number κ (and thus also on σ). Hence

$$\frac{\partial p}{\partial t} = \frac{AE_m}{\gamma} \left(\frac{\partial k_x}{\partial \kappa} \frac{\partial \kappa}{\partial x} \frac{\partial p}{\partial x} + \frac{\partial k_y}{\partial \kappa} \frac{\partial \kappa}{\partial y} \frac{\partial p}{\partial y} + \frac{\partial k_z}{\partial \kappa} \frac{\partial \kappa}{\partial z} \frac{\partial p}{\partial z} + k_x \frac{\partial^2 p}{\partial x^2} + k_y \frac{\partial^2 p}{\partial y^2} + k_z \frac{\partial^2 p}{\partial z^2} \right) \quad (7.14.29)$$

The first three terms in the parentheses on the right-hand side of eqn. (7.14.29) are the products of the derivatives of the functions k , κ , p . In a number of cases we can assume that the effect of these products is negligible, and thus obtain the simple relation

$$\frac{\partial p}{\partial t} = \frac{AE_m}{\gamma} \left(k_x \frac{\partial^2 p}{\partial x^2} + k_y \frac{\partial^2 p}{\partial y^2} + k_z \frac{\partial^2 p}{\partial z^2} \right) \quad (7.14.30)$$

For an anisotropically permeable region in which the constant values of k in the x , y , plane differ from the value in the z direction, eqn. (7.14.30) can be reduced to the yet simpler form of eqn. (7.14.14) where

$$C = \frac{k_s AE_m \sqrt{A}}{\gamma} \quad (7.14.31)$$

The difference between eqn. (7.14.14) and eqn. (7.14.31) lies in the fact that in the latter, the quantity C is not regarded as a constant.

We accepted the assumption concerning the linearity of the law of deformation. The effect of the porosity number will mainly manifest itself in the value of the coefficient of permeability k_s . The nature of some of the relationships was analyzed in Chapter 1. Depending on the soil in question and the required accuracy of the description, the functions $k_s = f(\kappa)$ can be approximated in different ways (linear functions, quadratic functions). Thus, for example, the relation

$$k_s = M \text{Exp } N\kappa, \quad \kappa = F(\bar{\sigma}) \quad (7.14.32)$$

where M, N are constants which can be determined experimentally, has been found suitable for clayey materials.

Although the function $k_s = f(\kappa)$ is considered to be continuous, it is replaced, for practical reasons, by a sequence of constant values of k_s , each of which applies to a certain finite interval. In this way, a corresponding value of C is also obtained for each interval.

In the flow field, the phenomenon is studied as an entity. The similarity indicator has a simple form, viz.

$$\left\{ \frac{\lambda_l^2}{\lambda_C \lambda_t} \right\} = 1 \quad (7.14.33)$$

$\lambda_l, \lambda_C, \lambda_t$ are the scales of lengths l , of the constant C and of the time t , respectively.

At a certain stage of development, a definite state defined by the distribution of the pressure function is found to exist in a definite field. According to the principle of auto-modelling which also relates to the boundary conditions, it is immaterial at which time this state was actually attained, because for a definite λ_l the product $\lambda_C \lambda_t$ must be a constant. What matters is the value of the product of the scale of the constant C and the scale of the time t (but not the values of these scales individually). It can be supposed that the state being considered originated at a certain C to which corresponds a certain preceding time interval. The same state could have originated at a different C , in a different preceding time interval. Hence in pre-history, various time intervals must be considered, and their length must be adapted to the choice of C . Considerations of this kind are embraced in the concept of migration of time.

The property outlined above can be used in solutions of many interesting practical problems. In order to illustrate, the subsequent discussion is concerned with an approximate solution of consolidation of a relatively slender core at the constructional stage. As Tölke [42] has shown, the distribution of the total stress in a horizontal plane section of relatively long earth dams without backwater is approximately proportional to the weight of the overburden. The inner structure of the core is highly stratified (as a result of placing the material in layers). During construction, water is squeezed out of the core, mainly in the horizontal direction. We can, therefore, assume that in the section $\overline{OI}$ (Fig. 7.23b) the water flows predominantly in the

horizontal direction, and that $k_s = k$ in eqn. (7.4.31) (k is the coefficient of permeability of the core material in the horizontal direction).

Studying the development in the section $\overline{OI}$ we must admit that, from the point of view of the total state of stress, gradual placing of the material has a character of gradual loading by two triangular massifs (Fig. 23c). We are required to find the neutral part of the total stress in the given section. Deformation is a result of the action of the effective part of the stress. As to the determination of the constant C : we shall start from the average value of the stress in the given section and at a given time. A certain value C_j applies to a certain j -th time interval Δt_j . As to an increase in the total stress: we assume that at the beginning of the j -th time interval, there existed the stress $\bar{\sigma}_{j-1}$ which increased linearly until the time $\Delta t_j - \bar{\Delta t}_j$. This applies to the triangular load characterized by the total stress with a maximum at the point O , as well as to the triangular load with a maximum at the point I . Fig. 7.23d shows the time development of the total stress at the point O .

The load acting on the boundaries of the section $\overline{OI}$ increases with a velocity v_j . For the point O , denote this velocity by

$$v_{j,0} = \frac{\bar{\sigma}_{j,0} - \bar{\sigma}_{j-1,0}}{\Delta t_j - \bar{\Delta t}_j} \quad (7.14.34)$$

where

$\bar{\sigma}_{j,0}$, $\bar{\sigma}_{j-1,0}$ is the total stress at the point O on the boundary at the beginning and the end, respectively, of the j -th time interval.

Similarly, at the point I ,

$$v_{j,1} = \frac{\bar{\sigma}_{j,1} - \bar{\sigma}_{j-1,1}}{\Delta t_j - \bar{\Delta t}_j} \quad (7.14.35)$$

The dam core adjoins highly permeable material. Under the stated conditions, a solution of eqn. (7.14.14) can be obtained by the superposition of the actions of a sequence of sudden load increments. On integrating with respect to time the term under the summation sign in eqn. (7.14.19) and performing some manipulation, we obtain, for the j -th time interval, that

$$\begin{aligned} p(\xi) = & \frac{2v_{j,0}L^2}{C_j\pi^3} \left[\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^3} \left\{ 1 - \text{Exp} \left[-\frac{n^2\pi^2 C_j (\Delta t_j + \Delta t'_{j,0})}{L^2} \right] \right\} \sin \frac{n\pi(L-\xi)}{L} - \right. \\ & - \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^3} \left[1 - \text{Exp} \left(-\frac{n^2\pi^2 C_j \bar{\Delta t}_{j,0}}{L^2} \right) \right] \sin \frac{n\pi(L-\xi)}{L} \Bigg] + \\ & + \frac{2v_{j,1}L^2}{C_j\pi^3} \left[\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^3} \left\{ 1 - \text{Exp} \left[-\frac{n^2\pi^2 C_j (\Delta t_j + \Delta t'_{j,1})}{L^2} \right] \right\} \sin \frac{n\pi\xi}{L} - \right. \\ & - \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^3} \left[1 - \text{Exp} \left(-\frac{n^2\pi^2 C_j \bar{\Delta t}_{j,1}}{L^2} \right) \right] \sin \frac{n\pi\xi}{L} \Bigg] \quad (7.14.36) \end{aligned}$$

For illustration, we calculated the function

$$f(\xi, \tau) = \frac{2}{\pi^3} \sum_{n=1}^{n=20} \frac{(-1)^{n+1}}{n^3} [1 - \text{Exp}(-n^2 \pi^2 \tau)] \sin \left[n\pi \left(1 - \frac{\xi}{L} \right) \right] \quad (7.14.37)$$

and show its diagram in Fig. 7.24, top. The relation to eqn. (7.14.36) is self-evident.

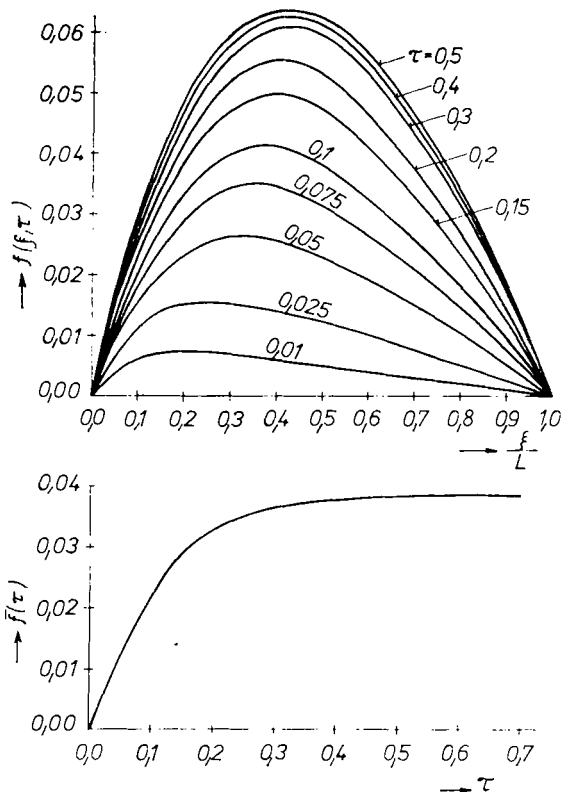


Fig. 7.24. Diagrams for calculating the development of pore pressure in the core of a dam, produced by the dam's own weight.

We can define τ , a dimensionless quantity, for example, as follows:

$$\tau = \frac{C_j(\Delta t_j + \Delta t'_{j,0})}{L^2}, \quad \tau = \frac{C_j(\Delta t_j + \Delta t'_{j,1})}{L^2} \quad (7.14.38)$$

or, alternatively, as

$$\tau = \frac{C_j \bar{\Delta t}_{j,0}}{L^2}, \quad \tau = \frac{C_j \bar{\Delta t}_{j,1}}{L^2} \quad (7.14.39)$$

In the first case, the value of τ includes the time interval $\Delta t'_{j,0}$ or $\Delta t'_{j,1}$ which characterizes the pre-history of consolidation before the j -th time interval (migrated time). $\Delta t'_j$ is determined using the relation for the average value $p = \bar{p}$, which applies

to the whole segment $\overline{OI}$. For a triangular load with a maximum at the point 0, we obtain

$$\bar{p}_{j,0} = \frac{4v_{j,0}L^2}{C_j\pi^4} \left[\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^4} \left\{ 1 - \exp \left[-\frac{n^2\pi^2 C_j (\Delta t_j + \Delta t'_{j,0})}{L^2} \right] \right\} - \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^4} \left[1 - \exp \left(-\frac{n^2\pi^2 C_j \Delta t_{j,0}}{L^2} \right) \right] \right] \quad (7.14.40)$$

and for a triangular load with a maximum at the point 1,

$$\bar{p}_{j,1} = \frac{4v_{j,1}L^2}{C_j\pi^4} \left[\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^4} \left\{ 1 - \exp \left[-\frac{n^2\pi^2 C_j (\Delta t_j + \Delta t'_{j,1})}{L^2} \right] \right\} - \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^4} \left[1 - \exp \left(-\frac{n^2\pi^2 C_j \Delta t_{j,1}}{L^2} \right) \right] \right] \quad (7.14.41)$$

where

$\bar{p}_{j,0}$, $\bar{p}_{j,1}$ are the average values for the corresponding triangular loads.

Assuming that the values of $\bar{p}_{j,0}$, $\bar{p}_{j,1}$ at the end of the preceding ($j-1$)th time interval are known, the values of $\Delta t'_{j,0}$, $\Delta t'_{j,1}$ are obtained from the equations

$$\bar{p}_{j-1,0} = \frac{4v_{j,0}L^2}{C_j\pi^4} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^4} \left[1 - \exp \left(-\frac{n^2\pi^2 C_j \Delta t'_{j,0}}{L^2} \right) \right] \quad (7.14.42)$$

$$\bar{p}_{j-1,1} = \frac{4v_{j,1}L^2}{C_j\pi^4} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^4} \left[1 - \exp \left(-\frac{n^2\pi^2 C_j \Delta t'_{j,1}}{L^2} \right) \right] \quad (7.14.43)$$

Fig. 7.24, bottom, shows a diagram of the function

$$\bar{f}(\tau) = \frac{4}{\pi^4} \sum_{n=1}^{n=20} \frac{(-1)^{n+1}}{n^4} [1 - \exp(-n^2\pi^2\tau)] \quad (7.14.44)$$

where

$$\tau = \frac{C_j \Delta t'_{j,0}}{L^2} \quad \text{or} \quad \tau = \frac{C_j \Delta t'_{j,1}}{L^2} \quad (7.14.45)$$

If the total average $\bar{p}$ is known, the total average effective stress $\bar{\sigma}$ can be determined for the beginning of the calculated interval. The value of $\bar{\sigma}$ is used to calculate k_s from eqn. (7.14.32). The value of C_j in the j -th interval is obtained using eqn. (7.14.31).

Although the procedure for calculating the pore pressure which is based on a number of assumptions, can be applied in practice to various preliminary studies, the main object of our discussion was to clarify the meaning and character of the concept of the migration of time rather than to propose a method for dealing with actual problems.

7.15. Seepage through fissured media

The theory of water motion through media penetrated by a system of fissures which divide the medium into relatively impervious blocks, was presented in Chapter 1. In this connection, we put forward the fundamental principles of methods for solving practical problems by means of numerical procedures. An analytical solution becomes somewhat more straightforward whenever the flow region can be defined in general terms and the hydraulic properties of the individual paths or their complex can be described without regard to particular details (continualized form) .

Efforts aimed at simplifying or unifying the description of a fissured medium, are based on the idea that the blocks of material, though separated by fissures, are in contact with one another on surfaces which transmit the effective stress. If the material is not crushed at the points of stress concentration, one can assume that the width of a fissure can be changed by the deformation of its walls, i.e. by the deformation of the blocks of material. Such deformations are obstructed (or supported) by the water pressure acting on the fissure walls. Since the material of the rock is permeable to a greater or lesser degree, deformations also take place inside the blocks.

Suppose that the network of fissures is dense enough to allow us to introduce the concept of statistically average properties. A part of the deformation takes place at the price of changing the width of the fissures (porosity). From a general point of view it appears as a component part of the total volume compression. The coefficient of volume compressibility is defined as follows:

$$\beta_1 = \frac{dm_1}{dp_1} \quad (7.15.1)$$

where

- m_1 is a part of the total porosity related to the network of fissures,
- β_1 — the coefficient of compressibility defined relative to the fissures,
- p_1 — the pressure in the fissures.

The other, major part of the deformation is considered to be a result of compression of the blocks of material. The corresponding coefficient of volume compressibility is defined by

$$\beta_2 = \frac{dm_2}{dp_2} \quad (7.15.2)$$

where

- m_2 is a part of the total porosity related to the blocks of material,
- β_2 — the coefficient of compressibility defined relative to the blocks,
- p_2 — the pressure in the blocks.

Suppose further that the system of fissures and the system of blocks exist next to one another in the same location. The coefficients of compressibility are then

defined by the relations

$$\frac{dm_1}{dp_1} = \frac{\partial m_1}{\partial p_1} dp_1 + \frac{\partial m_1}{\partial p_2} dp_2 \approx \beta_1 \quad (7.15.3)$$

$$\frac{dm_2}{dp_2} = \frac{\partial m_2}{\partial p_1} dp_1 + \frac{\partial m_2}{\partial p_2} dp_2 \approx \beta_2 \quad (7.15.4)$$

The deformation of the fissures takes place jointly with the deformation of the blocks. The liquid which is squeezed out of the fissures penetrates into the blocks, and the liquid which is squeezed out of the blocks penetrates into the fissures. Since the two systems are collocated, the transfer of the liquid from one system to another will mainly be a function of the difference in pressures. For $h = p/\gamma + \bar{z}$, we have

$$V_q = \frac{A}{\gamma} (p_2 - p_1) = A(h_2 - h_1) \quad (7.15.5)$$

where

γ is the unit weight of water,

V_q — the volume of water which is transferred per unit time in a unit total volume of the whole from one system to the other,

A — a coefficient whose dimension is $[L^{-1}T^{-1}]$ and whose value primarily depends on the permeability of the blocks, the density of the fissure network and the viscosity of water.

Realizing that the equation of continuity of motion of water which is to be relatively incompressible, applies to both systems, we write

$$\frac{\partial m_1}{\partial t} + \frac{\partial v_{1x}}{\partial \bar{x}} + \frac{\partial v_{1y}}{\partial \bar{y}} + \frac{\partial v_{1z}}{\partial \bar{z}} - V_q = 0 \quad (7.15.6)$$

$$\frac{\partial m_2}{\partial t} + \frac{\partial v_{2x}}{\partial \bar{x}} + \frac{\partial v_{2y}}{\partial \bar{y}} + \frac{\partial v_{2z}}{\partial \bar{z}} + V_q = 0 \quad (7.15.7)$$

Assuming, to begin with, the validity of the linear law of flow in both continualized media, we have

$$v_{1x} = -k_{1x} \frac{\partial h_1}{\partial \bar{x}}, \quad v_{1y} = -k_{1y} \frac{\partial h_1}{\partial \bar{y}}, \quad v_{1z} = -k_{1z} \frac{\partial h_1}{\partial \bar{z}} \quad (7.15.8)$$

$$v_{2x} = -k_{2x} \frac{\partial h_2}{\partial \bar{x}}, \quad v_{2y} = -k_{2y} \frac{\partial h_2}{\partial \bar{y}}, \quad v_{2z} = -k_{2z} \frac{\partial h_2}{\partial \bar{z}} \quad (7.15.9)$$

where

k_{1x}, k_{1y}, k_{1z} are the coefficients of permeability of the system of fissures in the directions $\bar{x}, \bar{y}, \bar{z}$,

k_{2x}, k_{2y}, k_{2z} — the coefficients of permeability of the system of blocks in the directions $\bar{x}, \bar{y}, \bar{z}$,
 h_1, h_2 — the piezometric heads in the system of fissures and the system of blocks.

The equations of continuity are now rearranged as follows:

$$\gamma\beta_1 \frac{\partial h_1}{\partial t} - \left[\frac{\partial}{\partial \bar{x}} \left(k_{1x} \frac{\partial h_1}{\partial \bar{x}} \right) + \frac{\partial}{\partial \bar{y}} \left(k_{1y} \frac{\partial h_1}{\partial \bar{y}} \right) + \frac{\partial}{\partial \bar{z}} \left(k_{1z} \frac{\partial h_1}{\partial \bar{z}} \right) \right] - A(h_2 - h_1) = 0 \quad (7.15.10)$$

$$\gamma\beta_2 \frac{\partial h_2}{\partial t} - \left[\frac{\partial}{\partial \bar{x}} \left(k_{2x} \frac{\partial h_2}{\partial \bar{x}} \right) + \frac{\partial}{\partial \bar{y}} \left(k_{2y} \frac{\partial h_2}{\partial \bar{y}} \right) + \frac{\partial}{\partial \bar{z}} \left(k_{2z} \frac{\partial h_2}{\partial \bar{z}} \right) \right] + A(h_2 - h_1) = 0 \quad (7.15.11)$$

Since each of them applies under the prescribed boundary conditions, generally different h_1, h_2 must be specified on the boundaries.

The fundamental theory of motion evolved by Zheltov and elaborated by Barenblatt, is comprehensively described in a monograph by Romm [36]. Practical solutions of the systems (7.15.10) and (7.15.11) are, of course, difficult and have so far been obtained for a very restricted number of problems only. Simplified forms which rely on various assumptions and exceptions, seem more promising for practical application.

Suppose, for example, that the material of the blocks is of poor permeability and that the regime of water moving through it, is predominantly pre-linear. Under the condition of usual gradients of the functions h , we can accept Puzyrevski's idea and set $k_{2x} = k_{2y} = k_{2z} = 0$.

The permeability of a rocky medium is usually strongly anisotropic. Assume that there are three principal directions of flow which are mutually perpendicular and do not change orientation throughout the entire space being studied. Identify the directions $\bar{x}, \bar{y}, \bar{z}$ with these principal directions, and introduce the substitution

$$x = \frac{k}{\sqrt{k_{1x}}} \bar{x}, \quad y = \frac{k}{\sqrt{k_{1y}}} \bar{y}, \quad z = \frac{k}{\sqrt{k_{1z}}} \bar{z} \quad (7.15.12)$$

where

k is the mean value of the coefficient of permeability of the continualized system of fissures. In the space

$$k = \sqrt[3]{(k_{1x}k_{1y}k_{1z})} \quad (7.15.13)$$

Rearrange the equations of continuity (7.15.6) and (7.15.7) as follows:

$$\frac{\partial m_1}{\partial t} - k \left(\frac{\partial^2 h_1}{\partial x^2} + \frac{\partial^2 h_1}{\partial y^2} + \frac{\partial^2 h_1}{\partial z^2} \right) - V_q = 0 \quad (7.15.14)$$

$$\frac{\partial m_2}{\partial t} + V_q = 0 \quad (7.15.15)$$

Substituting eqn. (7.15.5) and also

$$\frac{\partial m_2}{\partial t} = \beta_2 \frac{\partial p_2}{\partial t} = \gamma \beta_2 \frac{\partial h_2}{\partial t} \quad (7.15.16)$$

in eqn. (7.15.15), yields the equation which characterizes the relation between the pressures in the system of fissures and the system of blocks:

$$\gamma \beta_2 \frac{\partial h_2}{\partial t} + A(h_2 - h_1) = 0 \quad (7.15.17)$$

$$h_1 = h_2 + \frac{\gamma \beta_2}{A} \frac{\partial h_2}{\partial t} \quad (7.15.18)$$

Differentiating eqn. (7.15.18) with respect to t, x, y, z , gives

$$\frac{\partial h_1}{\partial t} = \frac{\partial h_2}{\partial t} + \frac{\gamma \beta_2}{A} \frac{\partial^2 h_2}{\partial t^2} \quad (7.15.19)$$

$$\frac{\partial^2 h_1}{\partial x^2} = \frac{\partial^2 h_2}{\partial x^2} + \frac{\gamma \beta_2}{A} \frac{\partial^2}{\partial x^2} \left(\frac{\partial h_2}{\partial t} \right) = \frac{\partial^2 h_2}{\partial x^2} + \frac{\gamma \beta_2}{A} \frac{\partial}{\partial t} \left(\frac{\partial^2 h_2}{\partial x^2} \right) \quad (7.15.20)$$

$$\frac{\partial^2 h_1}{\partial y^2} = \frac{\partial^2 h_2}{\partial y^2} + \frac{\gamma \beta_2}{A} \frac{\partial}{\partial t} \left(\frac{\partial^2 h_2}{\partial y^2} \right) \quad (7.15.21)$$

$$\frac{\partial^2 h_1}{\partial z^2} = \frac{\partial^2 h_2}{\partial z^2} + \frac{\gamma \beta_2}{A} \frac{\partial}{\partial t} \left(\frac{\partial^2 h_2}{\partial z^2} \right) \quad (7.15.22)$$

Using, moreover, the relations

$$\frac{\partial m_1}{\partial t} = \beta_1 \frac{\partial p_1}{\partial t} = \gamma \beta_1 \frac{\partial h_1}{\partial t}, \quad A(h_2 - h_1) = \gamma \beta_2 \frac{\partial h_2}{\partial t} \quad (7.15.23)$$

we obtain in eqn. (7.15.14) that

$$\begin{aligned} & \gamma \beta_1 \left(\frac{\partial h_2}{\partial t} + \frac{\gamma \beta_2}{A} \frac{\partial^2 h_2}{\partial t^2} \right) - k \left[\left(\frac{\partial^2 h_2}{\partial x^2} + \frac{\partial^2 h_2}{\partial y^2} + \frac{\partial^2 h_2}{\partial z^2} \right) + \right. \\ & \left. + \frac{\gamma \beta_2}{A} \frac{\partial}{\partial t} \left(\frac{\partial^2 h_2}{\partial x^2} + \frac{\partial^2 h_2}{\partial y^2} + \frac{\partial^2 h_2}{\partial z^2} \right) \right] + \gamma \beta_2 \frac{\partial h_2}{\partial t} = 0 \end{aligned} \quad (7.15.24)$$

and from there

$$\begin{aligned} & \frac{\beta_1}{\beta_2} \left(\frac{\partial h_2}{\partial t} + \frac{\gamma \beta_2}{A} \frac{\partial^2 h_2}{\partial t^2} \right) - \frac{k}{\gamma \beta_2} \left(\frac{\partial^2 h_2}{\partial x^2} + \frac{\partial^2 h_2}{\partial y^2} + \frac{\partial^2 h_2}{\partial z^2} \right) - \\ & - \frac{k}{A} \frac{\partial}{\partial t} \left(\frac{\partial^2 h_2}{\partial x^2} + \frac{\partial^2 h_2}{\partial y^2} + \frac{\partial^2 h_2}{\partial z^2} \right) + \frac{\partial h_2}{\partial t} = 0 \end{aligned} \quad (7.15.25)$$

Eqn. (7.15.25) represents the basic relation describing the motion in a rocky medium. Since it can usually be assumed that the major part of deformations will be taken up by the blocks, we can, for $\beta_1 \rightarrow 0$, $h_2 = h$, write the simplified equation

$$\frac{\partial h}{\partial t} - C \left(\frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2} + \frac{\partial^2 h}{\partial z^2} \right) - \kappa \frac{\partial}{\partial t} \left(\frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2} + \frac{\partial^2 h}{\partial z^2} \right) = 0 \quad (7.15.26)$$

where

$$C = \frac{k}{\gamma\beta_2}, \quad \kappa = \frac{k}{A} \quad (7.15.27)$$

or for $p_2 = p$, obtain

$$\frac{\partial p}{\partial t} = C \left(\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} + \frac{\partial^2 p}{\partial z^2} \right) - \kappa \frac{\partial}{\partial t} \left(\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} + \frac{\partial^2 p}{\partial z^2} \right) = 0 \quad (7.15.28)$$

Returning to the equations of continuity we see that in a transformed continualized medium it holds, approximately, that

$$v = -k \left[\frac{\partial h}{\partial s} + \frac{\gamma\beta_2}{A} \frac{\partial}{\partial t} \left(\frac{\partial h}{\partial s} \right) \right] \quad (7.15.29)$$

where

ds is an element of the path,

v is the velocity of seepage.

The velocity v increases (decreases) with time, depending on the character of the deformation and on the kind of load applied to the massif as a whole, as well as on changes in internal loading produced by flowing water. It is generally characteristic for seepage through a fissured medium that the development is retarded compared with that in a grainy, porous medium. In general, however, processes which continue for a time t longer than $\gamma\beta_2/A$ can be analyzed — without gross error — by means of the methods of the classical theory of consolidation. Essentially, a distinction should be made between the initial approaches of the two theories. The main subject of studies of fissured media is the effect of deformations of the blocks (grains) while the classical theory of consolidation primarily considers that of the volumes of pores (fissures). The agreement between the mathematical descriptions at $\kappa \rightarrow 0$, though of formal significance, makes it sometimes possible to employ simpler mathematical solutions, using, for example, the concept of the migration of time, even in non-linear problems. The choice of a particular solution should be a matter of an analysis of the actual specification of the problem. This is the reason why, rather than presenting a worked-out example, we refer the reader to the relevant literature.

7.16. Unsteady seepage through unsaturated media

In Chapter 1 we outlined the possibility of a mathematical description of flow through media which are not fully saturated with water. In heavy materials, the motion is mainly determined by the distribution of the forces of surface tension. The partial differential equations applicable to such cases are obtained by simplifying eqn. (1.6.66) to

$$\frac{\partial w}{\partial t} = \frac{\partial}{\partial x} \left[D(w) \frac{\partial w}{\partial x} \right] + \frac{\partial}{\partial y} \left[D(w) \frac{\partial w}{\partial y} \right] + \frac{\partial}{\partial z} \left[D(w) \frac{\partial w}{\partial z} \right] \quad (7.16.1)$$

where w is the „bulk” moisture,

$D(w)$ — the coefficient of diffusivity.

Since the coefficient of diffusivity is a function of moisture, eqn. (7.16.1) is non-linear. In simple cases it can be solved analytically, by means of the method of small parameter or by successive iterations [33].

The equation which is obtained from eqn. (7.16.1) after introducing the concept of saturated conductivity and saturated velocity potential W is somewhat simpler, viz.

$$\frac{\partial W}{\partial t} = C^{1/(n-1)} [(n-1) W]^{(n-2)/(n-1)} \left[\frac{\partial^2 W}{\partial x^2} + \frac{\partial^2 W}{\partial y^2} + \frac{\partial^2 W}{\partial z^2} \right] \quad (7.16.2)$$

where n is a number whose value is chosen in the range of 3 to 4. Writing

$$a = C^{1/(n-1)} [(n-1) W]^{(n-2)/(n-1)} \quad (7.16.3)$$

eqn. (7.16.2) becomes

$$\frac{\partial W}{\partial t} = a \left[\frac{\partial^2 W}{\partial x^2} + \frac{\partial^2 W}{\partial y^2} + \frac{\partial^2 W}{\partial z^2} \right] \quad (7.16.4)$$

Since the coefficient a is not a constant, eqn. (7.16.4) can only be solved in fairly simple cases of one-dimensional or axi-symmetrical flow, for instance, using the method of the migration of time.

If a certain mean value valid in the entire region being studied, is assigned to the coefficient a , then eqn. (7.16.4) changes to the Fourier equation which can be solved by means of the methods outlined in the previous sections. Problems in which the moisture and hence also the function on the boundaries (the Dirichlet problem) are assumed to be known, are among those defined most simply.

In anisotropically permeable materials in which the principal directions of anisotropy are mutually perpendicular, use can be made of the transformation of the region of flow and of eqn. (1.6.77) which is formally identical with eqn. (7.16.4), except that

$$a = C_{\max}^{1/(n-1)} [(n-1) W]^{(n-2)/(n-1)}$$

where

$C_{\max}$ is the maximum value of the coefficient of saturated conductivity (in the principal directions of anisotropy which we placed along the x, y axes in Chapter 1).

The problem is solved in the elevated region. Since the shape of the boundaries becomes deformed by this procedure, the solution is liable to be quite difficult. In the text that follows we shall describe a procedure which may be of interest from the methodological point of view; although it is approximate, it is good enough for practical purposes.

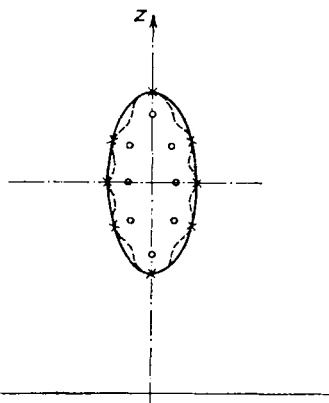


Fig. 7.25. Example of mathematical shaping of a boundary.

We start from the fact that the boundaries have a geometrical shape and, at the same time, are the loci of certain prescribed values of W . Using this relation, we shall try to shape them with the help of singularities (sources or sinks) which we choose so as to obtain the prescribed values of W at some suitable points of the space. The points are considered to be distributed along the boundaries of the region, the singularities outside the region.

As an example, consider a region with a circular gallery whose boundary, after the elevation, appears to be shaped like an ellipse. (Fig. 7.25). On this ellipse-shaped boundary, choose the points marked with crosses and assume that the values of W at these points are known. To each point corresponds a source (sink) marked with a circle. There are $j = 1, 2, \dots, J$ points and $i = 1, 2, 3, \dots, I$ singularities ($I = J$) in the region. According to Section 7.13, the solution of the linearized equation is of the form

$$W_j(t) = \sum_{i=1}^I B_i \left[-\text{Ei} \left(-\frac{r_{ij}^2}{4a(t-t_i)} \right) \right] = \sum_{i=1}^I B_i F(\delta_{i,j})$$

The time interval being studied is divided into short lengths for whose beginnings we determine the values of the constants B_i , using the principle of collocation. It is assumed that the sources (sinks) start to operate at the times t_i . For each time we

calculate the system of equations

$$\begin{aligned} B_{1,t} F(\delta_{1,1}) + B_{2,t} F(\delta_{2,1}) + \dots + B_{I,t} F(\delta_{I,1}) &= W_{1,t} \\ B_{1,t} F(\delta_{1,2}) + B_{2,t} F(\delta_{2,2}) + \dots + B_{I,t} F(\delta_{I,2}) &= W_{2,t} \\ \vdots &\vdots \\ B_{1,t} F(\delta_{1,J}) + B_{2,t} F(\delta_{2,J}) + \dots + B_{I,t} F(\delta_{I,J}) &= W_{J,t} \end{aligned}$$

The knowledge of the constants B_i is a prerequisite for the calculation of the distribution of the function W in each of the times being considered. At a general point of the region we have

$$W(x, z, t) = \sum_{i=1}^I B_i \left[-\text{Ei} \left(-\frac{r_i^2}{4a(t - t_i)} \right) \right]$$

where r_i is the distance of the point x, y from the i -th singularity.

As Fig. 7.25 shows, the approximation of the boundary is the more exact, the greater is the chosen number of singularities. A dense system of singularities naturally leads to the necessity of solving an extensive system of linear equations; this can be done most readily by a computer.

With the above reference to the possibility of boundary shaping we intend to conclude our discussion of the most fundamental methods and principles used in groundwater hydraulics. It is obvious that a single book cannot encompass everything that has been done in the field. Our aims have been more modest: to show some of the ways and means of solving diverse problems that may confront a practising engineer. As the authors have tried to show throughout the book, no set pattern exists that can be applied to all cases. Man's creative activity continues to be indispensable even in the era of vigorous development of digital techniques.

Understandably enough, every work begins to grow old the moment it comes into being. We expect the future to bring many methodological improvements and to extend the range of soluble problems. We shall be satisfied if the reader finds this book, in broad outline at least, to be a fairly representative description of the **present** state of the art.

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